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Traces of Menelaus' *Sphaerica* in Greek Scholia to the *Almagest*

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1. Introduction

The Greek mathematician Menelaus lived two generations before Ptolemy; his *Sphaerica* was the first monograph on "intrinsic" geometry on the surface of a sphere. The treatise, organized in three books, is lost in Greek but has survived in Arabic, as well as in Hebrew and Latin translations therefrom. The author of the Latin version was the celebrated translator Gerard of Cremona.[1] The Arabic tradition is quite complex:[2] during the 8th and 9th centuries, two or even three independent translations were completed, one of which probably through a Syriac intermediary; they were subsequently revised by a number of scholars. These revisions soon started to interact, to an extent that it is often impossible to assess, both with one another and with a tradition stemming from Thābit ibn Qurra's treatise on the so–called "Sector Theorem", a crucial result also attested in the Arabic *Sphaerica*. In Arabic, we only have access to manuscripts of some such revisions; a critical edition of Abū Naṣr's revision of Isḥāq ibn Ḥunayn's translation was published by M. Krause in 1936;[3] no printed edition, either critical or otherwise, is available of the others,[4] with the only exception being Naṣīr al–Dīn al–Ṭūsī's. Gerard of Cremona's translation from Arabic has not been critically edited, either, but the material relevant to this article has been published, as we shall see in due course. A Latin translation based on a Hebrew version and, to a lesser extent, on a series of Arabic sources was provided by Edmund Halley and published posthumously in 1758; the Hebrew text belongs in the same branch of the tradition as Gerard's; both are thought to be quite faithful to one of the original Arabic translations.

[1] For a first orientation of Gerard's life and work see Lemay 1974. See also, more recently, Burnett 2001 and the references therein.
[2] The basic data are conveniently summarized in Sidoli 2006, 48–51, relying on Krause 1936, and, as for the "Sector Theorem", on the very clear exposition in Lorch 2001, 327–35. I shall use interchangeably the terms "revision" and "recension".
[3] The numbering of the propositions of the *Sphaerica* used in this article is that of Abū Naṣr' recension according to Krause's edition. A concordance table of the proposition numbers in the different recensions is set out in Krause 1936, 6–9.
[4] Al–Māhānī's recension can only be recovered by means of the Hebrew and Latin translations, with the complication of a further intermediary revision that amalgamated it, to an extent that it is impossible to determine, with Isḥāq ibn Ḥunayn's translation. On al–Harawī's recension, extant in four manuscripts and in its turn also depending on al–Māhānī's, see Krause 1936, 1–2 and 32–42, and, most recently, Sidoli, Kusuba 2014. Al–Ṭūsī's revision was based on al–Harawī's and Abū Naṣr's.

As we shall see in the next section, it happens that two Greek authors preserve fragments from Menelaus' *Sphaerica*; in particular, we can read the entire text of six propositions. These authors are the 4th–century mathematicians Pappus and Theon, who operated in Alexandria and who also wrote extensive commentaries on Ptolemy's *Almagest*. The aim of the present article is simply to add three items to the list of traces of Menelaus' *Sphaerica* in Greek sources: these are two definitions and a proof sketch of a particular case of the Sector Theorem. These items are preserved in three scholia to Ptolemy's *Almagest*: the two definitions are contained in a single scholium; the proof sketch is distributed between two further scholia. It must be stressed that only the first scholium explicitly refers to Menelaus' *Sphaerica*.

The new evidence bears on parts of the *Sphaerica* that underwent major changes and revisions in the course of the transmission: therefore it is in principle a non–trivial task to compare the Greek text with the Medieval tradition. Still, the case of the two definitions will prove relatively easy to assess. As for the Sector Theorem, it must be borne in mind that our Greek sources do not even justify the hypothesis that it was included in the "original" *Sphaerica*: both Ptolemy and Theon provide, as we shall see, very detailed and *almost* complete proofs of the same result but do not mention Menelaus in this connection—still, Ptolemy (who lived just about fifty years later) reports two astronomical observations of his and calls him "the geometer";[5] Theon quotes two entire propositions from the *Sphaerica*.

A few words must also be said about the origins of the collection of scholia in which those edited in the present article are included. Heiberg knew of 36 manuscripts containing the *Almagest* (henceforth *Alm.*) in its entirety; he organized them into three families, whose best (and oldest) representatives are

- Par. gr. 2389 (in majuscule, beginning 9th century, *Alm.*);
- Vat. gr. 1594 (2nd half of the 9th century, *Prolegomena to the Almagest*, incomplete, Ptolemy, *Alm.*, *Phaseis*, *De judicandi facultate et animi principatu*, *De hypothesibus planetarum* I); Marc. gr. 313 (end 9th– beginning 10th century, *Prolegomena*, *Alm.*);
- Vat. gr. 180 (10th century, *Alm.*) and Vat. gr. 184 (2nd half of the 13th century, *varia arithmetica et astronomica*, *Prolegomena*, *scholia ad Alm.*, *Alm.*).

The first two families, of which Par. gr. 2389 on one side and Vat. gr. 1594 and Marc. gr. 313 on the other are also the prototypes, are linked by a series of conjunctive variants and thus give rise to a super–family. Heiberg notes that the tradition represented by the third

[5] The epithet is employed at *Alm.* VII.3, *POO* I.2, 30.18. The two observations are also cited in *Alm.* VII.3 (occultation of Spica by the Moon and alignment of notable points on the lunar disc with some fixed stars in Scorpius: *ibid.*, 30.18–31.2 and 33.3–10); they are dated to 98 CE, January 10/11 and 13/14, respectively, and were made in Rome. A papyrus containing a fragment of a planetary theory quite likely comes from a treatise of Menelaus; it contains an observation dated to 104 December 31/105 January 1, maybe also made in Rome (*POxy*. 4133, cf. Jones 1999 I, 69–80; II, 2–5).

family, although less correct and often interpolated, allows very old textual layers to be reached.[6] Overall, the structure of the *stemma* proposed by Heiberg makes it possible to go very far back in the tradition of *Alm*.

As for the scholia, the situation can be summarized as follows.[7]

a) Par. gr. 2389 is a *de luxe* exemplar and has no *scholia vetera*.
b) A large number of scholia transcribed by the main copyists can be found in Vat. gr. 1594 and Marc. gr. 313. The sets of scholia contained in these two codices are almost identical but they do not coincide, nor is the one a subset of the other. As a consequence, the two manuscripts are independent witnesses of a single collection assembled in Late Antiquity—almost surely within the 6th–century Alexandrine Neoplatonic school led by Ammonius—in the same way as they are independent witnesses of *Alm*. itself. An obvious lower bound to the date of composition of this collection can be set, since they plunder Theon's commentary *in Alm*., redacted about 360 CE.
c) Vat. gr. 184 is an apograph of Vat. gr. 1594 as for the *Prolegomena*. The earliest scholia in the margins of *Alm.* were transcribed by the main copyists themselves. Their text shows strict affinities with the readings of Marc. gr. 313, and I take it as certain that a model of Vat. gr. 184 is an apograph of the Venice codex as far as the marginal scholia are concerned.[8] A further, select collection of scholia was transcribed in Vat. gr. 184, before *Alm.* itself, at ff. 25r–80v. This collection was surely drawn from Vat. gr. 1594 since it also includes many annotations in a very active hand of the 12th century that were added to that codex. Hence, we sometimes find that the same annotation is found twice in Vat. gr. 184, both in the margins of *Alm*. and in the liminar collection, copied from different sources.
d) Vat. gr. 180 contains infrequent scholia in the hand of the main copyists, and a very rich and multi–layered apparatus of later annotations. Most of these were copied from Vat. gr. 1594.

As for the scholia edited in the present paper, the first is only contained in Vat. gr. 1594 and Marc. gr. 313, the second and the third are also present in the liminar collection of Vat. gr. 184.

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[6] In this order, partial *stemmata* are given at *POO* II, LIII, LXXVI, CXXXVI. See also the remarks in Toomer 1984, 3–5.

[7] This summary mentions results first presented in my forthcoming edition of the *scholia vetera* to the *Almagest*: Acerbi 2017.

[8] On the fact that the copyists of Vat. gr. 184 surely had access to Vat. gr. 1594, see Heiberg at *POO* II, XXXII–XXXIII and CXVII–CXXI. As for Marc. gr. 313, Heiberg already surmised that the model of Vat. gr. 184 was collated with it (*POO* II, CXXI). Only a model of Vat. gr. 184 can be involved since Marc. gr. 313 was in the West since the middle 12th century.

2. Known Traces of Menelaus' *Sphaerica* in Greek Sources

The list of known traces of Menelaus' *Sphaerica* in Greek sources includes the following items.

- Quotation of the entire prop. I.5, including a general enunciation, in Pappus, *Coll.* VI.2, 474.15–476.17 Hultsch: if three arcs of a great circle intersect, the sum of any two of the arcs cut off by the intersections[9] is greater than the remaining one. The result is also valid in plane geometry and its counterpart is proved in *El*. I.20. The proof of the spherical case in Pappus' text is an immediate application of *El*. XI.20; it differs considerably from those attested in Gerard's translation and Abū Naṣr's revision, that are identical[10] to one another and use a result that we read as Theodosius, *Sph*. III.1.
- Quotation of the entire prop. I.6, including a general enunciation, in Pappus, *Coll.* VI.3, 476.18–31 Hultsch: the sum of any two arcs issued from the extremes of the base of a spherical triangle [called "trilateral"] and intersecting within it is less than the sum of the other two sides of the trilateral. The result is also valid in plane geometry and its counterpart is proved in *El*. I.21; *mutatis mutandis*, the two proofs use the same idea, namely, repeated application of the preceding proposition (*Sph*. I.5 and *El*. I.20, respectively). The Arabo–Latin sources have the same proof as Pappus.
- Quotation of the entire prop. I.13, including a general enunciation, in Theon, *in Alm. VI.11*, p. 342 of the Basel edition: two trilaterals are equal if they have two sides and any of the angles not contained by them respectively equal, provided that the remaining angles not contained by the selected sides do not sum to two right angles.[11] This and the subsequent proposition are explicitly assigned to Menelaus' *Sphaerica* by Theon. The result is also valid in plane geometry; it has no counterpart in the *Elements*, even if Menelaus' proof would also apply to triangles.[12] The idea of the proof is the same in Theon and in some representatives of the Arabo–Latin tradition, but the formulation of the steps of the deduction may vary considerably. The other representatives of the Arabo–Latin tradition, among them Gerard's translation, have a differ-

[9] As we shall see, it is always assumed that the arcs of a great circle that are the sides of a spherical triangle are less than a semicircle.

[10] In comparing Gerard's Latin translation and the Arabic text of Abū Naṣr's revision of the same proposition, "identical" usually means that the deductive steps of the former are a subset of those of the latter, the additional steps being intended to make an argument that was perceived as too concise clearer (a "revision" quite frequently amounts to adding such steps). In the case of I.5, Abū Naṣr also adds an alternative proof by *reductio*. For Gerard's translation, I always have directly checked the readings on the manuscript Par. lat. 9335.

[11] The Basel edition can be found online at http://dx.doi.org/10.3931/e–rara–13800. Rome's edition *iA* only contains Pappus *in Alm*. V–VI and Theon *in Alm*. I–IV.

[12] It corresponds to the "missing case" of equality of triangles: two triangles are equal if they have two sides and any of the angles not contained by them respectively equal, provided that the remaining angles not contained by the selected sides do not sum to two right angles. Proclus, *in Eucl.*, 350.14–351.1, expounds a counterexample to the unrestricted validity of the theorem, ascribing it to Porphyry.

ent proof.[13] A part of the text of this and of the subsequent proposition is printed in Björnbo 1902, 22–6, in order to compare it with Gerard's translation; since the Basel edition contains a Byzantine recension of Theon's treatise, I provide in the Appendix a complete critical edition from the main manuscript witnesses of Theon, *in Alm. VI*.

- Quotation of the entire prop. I.14, including a general enunciation, in Theon, *in Alm. VI.11*, pp. 342–3 of the Basel edition: two trilaterals are equal if they have one side and the two angles adjacent to it respectively equal. Theon also applies this result earlier in his commentary, again ascribing it to Menelaus: ὡς Μενέλαος ἐν σφαιρικοῖς "as Menelaus in the *Spherics*" (*in Alm. I.16*, *iA*, 599.8–9). The result is also valid in plane geometry and its counterpart is proved as the first part of *El*. I.26; since Menelaus did not resort to indirect arguments in his treatise, the two proofs are necessarily different. The idea of the proof is the same in Theon and in the Arabo–Latin tradition, but the formulation of the steps of the deduction may vary considerably.[14]
- Quotation of a part of prop. I.34, not preceded by a general enunciation, in Pappus, *Coll.* VI.4, 478.1–21 Hultsch: if three arcs of a great circle less than a quadrant and issuing from the same point fall on a great circle, and if they cut off equal arcs on this great circle, the sum of the external arcs among the three falling on the great circle is greater than twice the inner arc. The result is also valid in plane geometry; it has no counterpart in the *Elements*. The (quite simple) idea of the proof is the same in Pappus and in the several representatives of the Arabo–Latin tradition, but the formulation of the steps of the deduction varies considerably. *Sph*. I.5 (= *Coll.* VI.2) is applied and its general enunciation is explicitly quoted.
- Quotation of a part of prop. I.37, not preceded by a general enunciation, in Pappus, *Coll.* VI.5, 478.22–480.6 Hultsch. The result (a generalization of the previous one to four arcs of a great circle issuing from the same point) is also valid in plane geometry; it has no counterpart in the *Elements*. The (quite simple) idea of the proof is the same in Pappus and in the several representatives of the Arabo–Latin tradition, but the formulation of the steps of the deduction varies considerably. Moreover, *Sph*. I.37 proves two results with the same proof, whereas Pappus states and proves only one of them, thereby destructuring the proof. *Sph*. I.6 (= *Coll.* VI.3) is applied and its enunciation is quoted in instantiated form; *Sph*. I.34 (= *Coll.* VI.4) is tacitly applied.
- Application, without quotation, of prop. I.4 (first part) in Pappus, *in Alm*. *VI.9*, *iA*, 275.16; the proposition is explicitly assigned to Menelaus' *Sphaerica*: ὡς ἔστιν Μενέλαος σφαιρικοῖς "as Menelaus in the *Spherics* has it". The result (two trilaterals are equal if their sides are respectively equal) is also valid in plane geometry and its counterpart is proved in *El*. I.8; since Menelaus did not resort to indirect arguments in his treatise, the two proofs are necessarily different.

[13] See Krause 1936, 132–3 and n. 4; Björnbo 1902, 22–3.
[14] See Krause 1936, 133–5 and n. 2; Björnbo 1902, 23–5.

- Application, without quotation, of prop. I.4 (first part) in Theon, *in Alm. II.7*, *iA*, 680.16; the proposition is explicitly assigned to Menelaus' *Sphaerica*: ὡς Μενέλαος ἐν τοῖς σφαιρικοῖς "as Menelaus in the *Spherics*". The result is also valid in plane geometry and its counterpart is proved in *El*. I.8.
- Application, without quotation, of prop. I.9[15] in Pappus, *in Alm. VI.9*, *iA*, 276.1–2; the proposition is explicitly assigned to Menelaus' *Sphaerica*: ὡς Μενέλαος σφαιρικοῖς "as Menelaus in the *Spherics*". The result (the greater side of a trilateral subtends the greater angle) is also valid in plane geometry and its counterpart is proved in *El*. I.18; the two proofs are necessarily different, for that of the *Elements* employs I.16, a result which is not valid in spherical geometry.
- Application, without quotation, of prop. I.17 in Theon, *in Alm. IV.2*, *iA*, 973.3; the proposition is explicitly assigned to Menelaus' *Sphaerica*: ὡς Μενέλαος ἐν σφαιρικοῖς "as Menelaus in the *Spherics*". The result (two trilaterals are equal if they have one side and two angles, one of which not adjacent to the side, respectively equal) is also valid in plane geometry and its counterpart is proved as the second part of *El*. I.26; since Menelaus did not resort to indirect arguments in his treatise, the two proofs are necessarily different.

To sum up: *Sph*. I.5 (Pappus), 6 (P), 13 (Theon), 14 (T), 34 (P), 37 (P) are quoted in full or in part, *Sph*. I.4 (PT), 9 (P), 17 (T) are only applied.

Pappus' and Theon's purposes for quoting in full the above–mentioned propositions are quite different. After an initial section devised to boast about the virtues of his own teaching of the "small astronomical corpus",[16] Pappus begins *ex abrupto* his exposition by stating and proving, in *Coll*. VI.2–5, the four theorems listed above; only with *Coll*. VI.6–7 we learn that these results are preliminary to provide an alternative proof and a completion of Theodosius, *Sph.* III.5. As a matter of fact, only the results proved in *Coll*. VI.4–5 are applied in *Coll*. VI.6–7, respectively; as we have seen, *Coll*. VI.2–3 simply provide the key steps to the proofs of *Coll*. VI.4–5, respectively. One remarkable feature of Pappus' exposition is that he expressly asserts at the end of *Coll*. VI.2 that the word employed by Menelaus to denote a spherical triangle was τρίπλευρον "trilateral <figure>"; only this sentence in Pappus' exposition refers to Menelaus' *Sphaerica*.[17]

Theon's intent is different: the two theorems he quotes will allow an accurate calculation of the προσνεύσεις "directions" or "inclinations", namely, the point on the horizon towards which the straight line joining the centres of the Sun and the Moon at eclipses

[15] But the formulation is bewildering, since it also specifies that the angles are the one obtuse and the other acute. Maybe for this reason, Rome (*iA*, 275 n. 2) asserts that he does not find the proposition to which the clause is alluding.

[16] The "small astronomical corpus" is better known as the "little astronomy". Theodosius' *Sphaerica* was included in it, Menelaus' was not.

[17] At 476.16–17 Hultsch: καλεῖ δὲ τὸ τοιοῦτο σχῆμα Μενέλαος ἐν τοῖς σφαιρικοῖς τρίπλευρον "In the *Spherics*, Menelaus calls such a figure "trilateral"". The word is attested with this meaning in Ptolemy, *Alm*. II.3, II.10, II.11, II.12 (*bis*) (at *POO* I.1, 96.24, 148.3, 155.3, 161.19, 163.19).

points. We shall not enter into details on this (see Rome 1948, Neugebauer 1975, 141–4). One notable feature of Theon's proof of *Sph*. I.14 is that it initially introduces a partition into three cases according to two angles being equal, greater, or less than two right angles, but then only the first case is proved, the others corresponding to what we read as *Sph*. I.15. Moreover, *Sph*. I.15 subsumes these two cases into the single case "angles not equal to two right angles". This might suggest that Theon is really quoting a segment of text from the *Sphaerica*, without even editing it, and that the Arabic text is the result of a revision.

The information provided above on the propositions quoted in full by Pappus or Theon shows that in some cases it may prove difficult to retrieve Menelaus' text, and that the Greek sources do not necessarily preserve a text which is more likely to be near to the "original". On the one hand, both Pappus and Theon might have had reasons to change the whole line of proof or merely some details of it (and Pappus almost surely did so with *Sph*. I.37). On the other hand, the Arabic sources present in some cases proofs that are fairly different from one another.

The entire scientific production of Menelaus is lost in Greek. A few splinters related to writings other than the *Sphaerica* are preserved in Greek and Arabic sources.[18]

- Menelaus wrote a treatise on *Geometrical elements* in three books, now lost. Fragments of it, and even its very title, can be only found in Arabic sources (Hogendijk 2000), and amount to the following. Al–Bīrūnī mentions a problem solved in prop. 2 of book III: to inscribe in a given semicircle an inflected straight line of given length. Al–Sijzī asserts that, at the beginning of his work, Menelaus proved, albeit non completely, "the property of equality <that results> from drawing, in an equilateral triangle, the perpendiculars as far as the perimeter". The property alluded to is that the sum of the distances from the sides of any point inside an equilateral triangle is costant (and therefore equal to the heigth of the triangle); al–Sijzī's text also presents a generalization to the case in which the point is external to the triangle, which might also be assigned with some plausibility to Menelaus.
- Commenting on *El*. I.25, Proclus ascribes to Menelaus an alternative proof of it; this proof is different from any of its counterparts in the Arabic tradition of the *Sphaerica*. Actually, the "proof" of such a counterpart in Abū Naṣr's revision is just a short sentence appended to the proof of the inverse—namely, *Sph*. I.8—claiming that the inverse can be proved by *reductio*.[19] Maybe the proof transcribed by Proclus was contained in the lost *Geometrical elements*, or maybe in a *Book of the Triangles* the *Fihrist* also ascribes to Menelaus along with the *Sphaerica* (Flügel 1872 I , 267).

[18] For general orientations on Menelaus and his legacy in the Arabic world see Bulmer–Thomas 1974; Sezgin 1974, 158–64, Fuentes González 2005 (to be used with caution: it is a compilation of ill–digested previous surveys; it contains a number of gross mistakes and does not even offer a complete bibliography).
[19] Proclus, *in Eucl.*, 345.13–346.13. For the proof attested in the other Arabic revisions, see Krause, 27–8. See also Björnbo 1902, 45–6, for a discussion.

- In a passage of the *Collectio* dealing with special curves, Pappus, *Coll*. IV.58, 270.24–6 Hultsch, has this passing remark: καί τινες αὐτῶν ὑπὸ τῶν νεωτέρων ἠξιώθησαν λόγου πλείονος, μία δέ τις ἐξ αὐτῶν ἐστιν ἡ καὶ παράδοξος ὑπὸ τοῦ Μενελάου κληθεῖσα γραμμή "some of them were regarded by the moderns worthy of a substantial treatment; one of them is the line also called "surprising" by Menelaus". Pappus refers to special curves that such otherwise totally unknown mathematicians as Demetrius of Alexandria and Philo of Tyana derived from the so–called "loci on surfaces". Some of these curves retained the attention of the "moderns", among them Menelaus. We simply do not have any grounds to guess what line his "surprising" curve could look like.[20]
- In a passage of the *Collectio* dealing with the rising and setting times of the zodiacal signs, Pappus, *Coll*. VI.110, 600.26–602.2 Hultsch, asserts that περὶ δὲ δύσεως αὐτῶν οὐθὲν λέγει· ὁ γὰρ λόγος τῆς ἀποδείξεως ἐμπίπτει εἰς τοὺς ἀνατολικοὺς διορισμούς, καὶ ἔστιν ἤδη πραγματεία περὶ τούτου γεγραμμένη Μενελάῳ τῷ Ἀλεξανδρεῖ, περὶ ἧς ὕστερον ἐπισκεψόμεθα "about their setting he [*scil*. Hipparchus] does not say anything: for the argument of the proof falls in the rising determinations, and there is even an exposition about this, written by Menelaus of Alexandria, about which we shall inquire later". Pappus did not keep his promise. No modern study exists as to what the "rising determinations" (already mentioned at *Coll*. VI.108, 600.6–7 Hultsch) might be that apparently set limitations on the general validity of Hipparchus' "proof" alluded to by Pappus.
- At the beginning of his exposition on Ptolemy's table of chords, Theon, *in Alm. I.10*, *iA*, 451.4–5, asserts: δέδεικται μὲν οὖν καὶ Ἱππάρχῳ πραγματείᾳ τῶν ἐν κύκλῳ εὐθειῶν ἐν ιβ βιβλίοις, ἔτι τε καὶ Μενελάῳ ἐν ς "it is also proved by Hipparchus in his exposition on the chords[21] in twelve books, as well as by Menelaus, in six <books>". To such an exposition might refer the citation at the end of the non–spurious part of *Sph*. III.14. A likely structure of Hipparchus' chord table is discussed by Toomer, who also suggested (1973, 19–20) that the numbers "twelve" and "six" in the quoted sentence refer in fact to the number of sections of the complete table, not to the number of books of the treatises. Against the possibility that expositions of Hipparchus (and Menelaus) contained a chord table, see Rome 1933a.
- P.Fouad inv. 267, verso line 5, probably mentions Menelaus, likely as the author of a table of ascensions.[22] The text is too fragmentary to allow giving consideration to any hypothesis.

[20] But see Tannery 1883–4, 16–18 of the reprint, for a guess. In this connection, one must also record the fact that, in the *Verba filiorum*, the Banū Mūsā report a solution of the problem of doubling the cube that they ascribe to Menelaus: "Et hec quidem operatio quam narramus est viri ex antiquis qui dicitur Mileus, cui est liber in geometria"; in fact, the method coincides with Archytas' (Clagett 1964, 336–40, quote from Gerard's Latin translation at 336).
[21] Here as elsewhere, the noun chords translates εὐθεῖα ἐν κύκλῳ, litt. "straight line in a circle".
[22] See Fournet, Tihon 2014, 24–5 (text) and 49–51 (discussion); see also the discussion in Jones 2016.

- A further treatise of Menelaus, in one Latin manuscript entitled *Liber de quantitate et distinctione corporum mixtorum*, is mentioned by the *Fihrist*; it is also transmitted only in Arabic translation and Latin version therefrom (German version in Würschmidt 1925). The dedicatee of the treatise is the Roman emperor Domitian (ruled 81–96 CE).

Finally, one must not forget that the initial segment of book I of the *Sphaerica* can quite obviously be read as a rewriting of the corresponding theorems of the *Elements*: the choice of using only direct proofs entails major changes in the deductive order.[23] This attests to Menelaus' foundational interests.

3. New Traces of Menelaus' *Sphaerica* in Greek Scholia to the *Almagest*

3.1 Definitions

In chapter II.10 of *Alm.*, Ptolemy sets out to calculate the angles between the ecliptic and some important great circles: the meridian, the horizon, the altitude circles. He starts his exposition by providing a definition of an angle between two great circles: "We must first make clear that we define an angle between <two> great circles as follows: we say that <two> great circles form a right angle when a circle having as pole the intersection of the great circles and as radius any distance whatever has <exactly> a quadrant intercepted between the segments of the great circles forming the angle; in general, whatever ratio the intercepted arc of a circle described in the above manner bears to the whole circle is the same as the ratio of the angle between the planes <of the two great circles> to 4 right angles. Thus, since we set the circumference of the circle as 360º, the angle subtending the intercepted arc will contain the same number of degrees as the arc, in the system where one right angle contains 90º".[24]

Thus, Ptolemy actually defines how to *measure* such an angle, namely, by measuring the arc of a circle, having as pole the intersection of the great circles and as radius any distance whatever, intercepted between the segments of the great circles forming the angle, but his definition can be immediately restated so as to say that "the angle between two great circles is the one subtending the arc of a circle, having as pole etc.".

The first scholium edited in the present article provides a definition alternative to that (implicitly) stated by Ptolemy, as well as a definition of a "trilateral figure". The scholiast asserts that both of them are drawn from Menelaus' *Sphaerica*. The scholium is found in Vat. gr. 1594, f. 42v marg. int., and Marc. gr. 313, f. 73v marg. ext. In the Vatican ma-

[23] See also Björnbo 1902, 32–45, for a discussion.
[24] Cf. *POO* I.1, 145.17.23–146.8. Unless otherwise stated, the translations of passages from the *Almagest* are those of Toomer 1984 (here from page 105).

nuscript, it is located in such a way that its end is just by the side of the title of *Alm*. II.10; in the Venice manuscript, the beginning of the scholium also has such a position with respect to the main text. Since Ptolemy's definition quoted above is the second sentence of chapter II.10, the scholium is in both manuscripts near to the intended *relatum*, and in Marc. gr. 313 exactly by the side of it. The ascription to Menelaus' *Sphaerica* is treated as a title, and therefore it is in majuscule in both manuscripts.

Sch. **1**
ἐκ τῶν Μενελάου σφαιρικῶν
τρίπλευρον σχῆμα λεγέσθω τὸ περιεχόμενον ἐν σφαιρικῇ ἐπιφανείᾳ ὑπὸ τριῶν περιφερειῶν ὧν ἑκάστη μεγίστου κύκλου ἐλάττων ἐστὶ ἡμικυκλίου· γωνίαι δὲ ἴσαι λεγέσθωσαν αἱ ὑπὸ περιφερειῶν περιεχόμεναι ἐν σφαιρικῇ ἐπιφανείᾳ ὅταν ἴσαι ὦσιν αἱ κλίσεις τῶν κύκλων ὧν εἰσιν αἱ περιέχουσαι τὰς γωνίας περιφέρειαι.

2 σφαιρικῇ] σφ^{αι} codd. **4** σφαιρικῇ] σφ^{αι} codd.

Transl. From Menelaus' Spherics
Let a trilateral figure be called the one contained by three arcs in a spherical surface,[25] each of which is less than a semicircle of a great circle; let angles contained by arcs in a spherical surface be called equal whenever the inclinations of the circles are equal to which the arcs containing the angles belong.

The situation with the definitions in the Arabic tradition is quite complex and its essential features are set out in the following table. Probably because of an accident of transmission, Gerard's Latin translation does not contain definitions.[26]

al–Māhānī[27]	Abū Naṣr	al–Harawī & al–Ṭūsī[28]
Triangle on a spherical surface	Trilateral figure	Spherical figures. Triangle and quadrilateral
Angle of a spherical triangle	Angle of a trilateral	Angle of a spherical triangle
Equal angles	Equal angles	Right, acute, obtuse angles
Angle greater than another		Angle less than another
Right angle		Equal angles

Several points are worth a short discussion.

[25] Here and in the next line, the abbreviation in the scholium is also compatible with the reading ἐν σφαίρας ἐπιφανείᾳ "in a surface of a sphere".
[26] See the table in Krause 1936, n. 7 on 119–20; details on the specific recensions are *ibid*., 27 (al–Māhānī), 36–7 (al–Harawī and al–Ṭūsī), 54–5 (al–Ṭūsī).
[27] As said at the beginning, al–Māhānī's revision can only be recovered by means of the Hebrew and Latin translations.
[28] Al–Ṭūsī completes def. 4 with that of angle greater than another and has two further definitions of "arc of inclination".

- The first definition virtually coincides with the one in Abū Naṣr's revision; it includes the condition that the sides of the τρίπλευρον must be less than half of a great circle. This condition will also be crucial in the proof of the Sector Theorem; Ptolemy repeatedly recalls it in the *Almagest*. The other Arabic sources either change τρίπλευρον to "triangle on the surface of a sphere" (both) or enlarge the definition to one of a generic spherical figure, whose first two species are the triangle and the quadrilateral (al–Harawī).
- One might wonder why the scholiast quotes the first definition, since Ptolemy has already used the word τρίπλευρον in *Alm*. II.3 and since what is at issue here is only to back up Ptolemy's implicit definition of "equal angles" with a definition taken from standard literature. It is also true, on the other hand, that Ptolemy will repeatedly use the term τρίπλευρον in the textual segment *Alm*. II.10–2.
- All Arabic sources have a definition of "angles of a trilateral" (or of a spherical triangle) inserted between the two transcribed in the scholium: such angles are the angles contained by the arcs forming the trilateral. Of course, the scholiast might well have omitted this definition, but I would favour the possibility that he is really transcribing a continuous stretch of text of the *Sphaerica*. One indication in this sense is that the definition attested in the Arabic *Sphaerica* is a vacuous truism, unless a definition of angle between two arcs on a surface of sphere is provided. This is done in the next definition quoted in the scholium, that quite appropriately replaces an "essential" definition (namely, the "what is" of an angle between arcs on a spherical surface) with an operative definition, indicating when two such angles are equal.
- The second definition in the scholium virtually coincides with the ones attested in the Arabic sources as the third definition of the *Sphaerica*: it defines equality of angles between arcs on the surface of a sphere in terms of equality of the "inclination" of the planar objects that "carry" the arcs (see next remark).
- The "inclination of the circles" in the Greek definition did not win the favour of the Arabic revisors: they changed it to "inclination of the semicircles" (Abū Naṣr and al–Harawī) or to "inclination of the planes" (al–Māhānī). All these plane objects contain the arcs that contain the equal angles.
- Only in Abū Naṣr's revision we read an addition, intended to clarify what the "inclination" between two planes is.[29] As the Greek scholium appears to confirm *ex silentio*, this was taken for granted by Menelaus to be simply represented by the arc cut off by the planes from circles, perpendicular to the common section of the planes, with center on such a common section and any radius.[30] If the planes are defined by

[29] This is the arc cut off by the semicircles that contain the arcs from the great circle passing through the poles of these semicircles. This addition entailed completing "inclination of the semicircles" to "arcs of the inclination of the semicircles" in the previous sentence.

[30] The addition in Abū Naṣr's version must be connected with the very convoluted proofs of *Sph*. I.1 we read both in Abū Naṣr's version and in Gerard's translation (= al–Māhānī). These proofs surely are the result of radical, and to some extent independent, rewritings. Such rewritings involve constructions of solid geometry, whereas to "cut and paste" an angle on the surface of a sphere it is enough to "cut and paste" two suitable

arcs on the surface of a sphere, such circles most naturally specialize to circles on the surface of the sphere whose center coincides with the intersection of the arcs and whose "radius" is less than the chord subtending half a great circle of the sphere: this much we may also infer from Ptolemy's definition.

- Menelaus makes the angle a species of the genus κλίσις "inclination" taken as a primitive notion, exactly as the *Elements* do in the case of the definition of a plane angle at *El.* I.def.8.[31] In book XI, however, the Greek text of the *Elements* (but not the Arabo–Latin tradition) introduces three definitions (XI.def.5–7) related to what we would call dihedral angles. The definitions are never used in the sequel and present obvious problems, among them inverting the genus–species relation with the κλίσις, this choice conflicting squarely with I.def.8 in the case of XI.def.5. All of this shows that these definitions are spurious.[32] Let us read *El.* XI.def.6–7: ἐπιπέδου πρὸς ἐπίπεδον κλίσις ἐστὶν ἡ περιεχομένη ὀξεῖα γωνία ὑπὸ τῶν πρὸς ὀρθὰς τῇ κοινῇ τομῇ ἀγομένων πρὸς τῷ αὐτῷ σημείῳ ἐν ἑκατέρῳ τῶν ἐπιπέδων. ἐπίπεδον πρὸς ἐπίπεδον ὁμοίως κεκλίσθαι λέγεται καὶ ἕτερον πρὸς ἕτερον ὅταν αἱ εἰρημέναι τῶν κλίσεων γωνίαι ἴσαι ἀλλήλαις ὦσιν "the inclination of a plane to a plane is the acute angle contained by the <straight lines> drawn, at the same point in each of the planes, at right <angles> with their common section; a plane to a plane is said to be similarly inclined as another to another whenever the said angles of the inclinations are equal to one another".
- The definition we read as Theodosius, *Sph.* I.def.6 can safely be considered spurious as well; it quite obviously results from a montage, with some slight adaptation, of the two Euclidean definitions: ἐπίπεδον πρὸς ἐπίπεδον ὁμοίως κεκλίσθαι λέγεται καὶ ἕτερον πρὸς ἕτερον ὅταν αἱ τῇ κοινῇ τομῇ τῶν ἐπιπέδων πρὸς ὀρθὰς ἀγόμεναι εὐθεῖαι ἐν ἑκατέρῳ τῶν ἐπιπέδων πρὸς τοῖς αὐτοῖς σημείοις ἴσας γωνίας περιέχωσιν "a plane to a plane is said to be similarly inclined as another to another whenever the straight lines drawn, at the same points in each of the planes, at right <angles> with the common section of the planes contain equal angles".[33]

As for the issue of authenticity, the previous discussion seems to me to corroborate the hypothesis that the definitions in the scholium are original with Menelaus' treatise, the several versions we read in the Arabic tradition being the result of a series of very specific, sometimes slight, and maybe independent modifications. In particular, one might seriously entertain the hypothesis that the definition of "angles of a trilateral" is

arcs, and this can be done under the sole assumption that any circle can be traced on a sphere with given pole and "radius" less than the chord subtending half a great circle of the sphere. On the issue see Gori 2002, 167–9. On the "radius" involved in the previous construction, and on the construction itself, a tacit postulate in Theodosius' and Menelaus' *Sphaerica*, see Sidoli 2004.

[31] On the several genera ancient exegesis made angles a species of, see Acerbi 2010, 161–2.

[32] See Vitrac 2001, 77–9, for a discussion. Definition 6 is not well–founded since one must prove that the angle in the *definiens* is univocally defined by the construction identifying it. This is obvious if one uses orthogonal circles instead of the construction of *El.* XI.def.6.

[33] Heiberg 1927, 2.13–16. A quotation of this definition is also added in the proof of *Sph.* II.21, *ibid.*, 98.2–5.

spurious, its presence in all Arabic versions suggesting that it was already interpolated in a Greek source.

3.2 The "Parallel" Case of the Sector Theorem

The most celebrated result of Greek spherical trigonometry is the Sector Theorem, also known as "Menelaus' Theorem" because of its being attested in the *Sphaerica* (proposition III.1 in Abū Naṣr's redaction). It is a powerful mathematical tool, devised to determine arcs of a great circle on the surface of a sphere. It is the keystone of some of the most important technical results of *Alm.*, where it is applied seventeen times.[34] It comes as no surprise, then, that the Sector Theorem is also proved in *Alm*. I.13 and, with many more cases on offer, in Theon, *in Alm. I.13*, *iA*, 535.10–570.12.[35]

The Sector Theorem is proved by Ptolemy as the last of a series of seven propositions.

1) First rectilinear lemma, "by composition" (*POO* I.1, 68.23–69.20); see **Fig. 1**. From the end–points B, Γ of two mutually intersecting straight lines AB, AΓ, two lines BE, ΓΔ are drawn across, meeting at Z and intersecting straight lines AΓ, AB at E, Δ, respectively (this will henceforth be called "rectilinear supine configuration"). It is required to show that $\Gamma A:AE = (\Gamma\Delta:\Delta Z)\circ(ZB:BE)$ ("rectilinear relation" henceforth).[36] The proof writes the "obvious" compounded ratio with a term common to the two compunding ratios: $\Gamma\Delta:HE = (\Gamma\Delta:\Delta Z)\circ(\Delta Z:HE)$, draws a suitable parallel HE to one of the assigned straight lines and readily argues by similar triangles and substitutions in compounded ratios.
2) Second rectilinear lemma, "by separation" (*ibid.*, 69.21–70.16); see **Fig. 2**. In the same configuration as lemma 1, one also has that $\Gamma E:EA = (\Gamma Z:Z\Delta)\circ(\Delta B:BA)$. The auxiliary parallel line HA is now drawn external to the assigned configuration.
3) First cyclic lemma (*ibid.*, 70.17–71.13); see **Fig. 3**. In a circle ABΓ of centre Δ, mark two consecutive arcs AB, BΓ, any of which is less than a semicircle, join ΔB and AEΓ intersecting at E, drop from A, Γ perpendiculars AZ, ΓH to radius ΔB. Then $ch(2AB):ch(2B\Gamma)::AE:E\Gamma$, where $ch(2AB)$ is the chord of twice arc AB. Since

[34] This happens in *Alm* I.14, 16, II.2, 3 (*ter*), 7 (*bis*), 10, 11, 12 (*bis*), VIII.5 (*ter*), 6 (*bis*).

[35] See Neugebauer 1975, 26–30, for a clear exposition of the mathematics involved, Björnbo 1902, 88–92, Rome 1933, 49–50, and Sidoli 2006 for discussions of the issue of authenticity. Note that, at *in Alm. VIII.5*, Theon offers again a proof of a particular of the Theorem: see pp. 365–6 of the Basel edition.

[36] The sign ∘ stands for "composition" of ratios (see Acerbi 2016 for a survey of all Greek and Byzantine sources). The two compounding ratios, in fact, are not "multiplied": what is multiplied, *iuxta El*. VI.def.5, are the πηλικότητες "<numerical> values" of the two ratios, namely, the fractions corresponding to them: "A ratio is said to be compounded of ratios when the <numerical> values of the ratios multiplied by one another make some <numerical value of a ratio>" (*EOO* II, 72.13–15). My translation of *El*. VI.def.5 includes a final integration based on Theon *in Alm. I.13*, in *iA*, 533.1–2, who is our sole independent source completing the final τινα of *El*. VI.def.5 with πηλικότητα λόγου. One must note that the sign "=" is also misleading: a ratio is said to be "compounded" of two or more ratios, it is never said to be "equal to" or "the same as" something like their "composition".

$ch(2AB) = 2AZ$ and $ch(2B\Gamma) = 2\Gamma H$, the proportion is an immediate consequence of the fact that triangles AZE and ΓHE are similar.

4) Second cyclic lemma (*ibid.*, 71.14–72.10); see **Fig. 4**. The theorem is formulated in the "language of the givens". Adopting the configuration of the first cyclic lemma, from centre Δ draw a straight line ΔZ perpendicular to AEΓ. It is required to show that, once arc AΓ and ratio $ch(2AB){:}ch(2B\Gamma)$ are given, each of arcs AB, BΓ is also given. The proof applies a series of theorems from Euclid's *Data*.
5) Third cyclic lemma (*ibid.*, 72.11–73.10); see **Fig. 5**. In a circle ABΓ of centre Δ, mark two consecutive arcs AB, BΓ, any of which is less than a semicircle, join ΔA and ΓB intersecting at E once produced, drop from B, Γ perpendiculars BZ, ΓH to radius ΔA, possibly produced. Then $ch(2\Gamma A){:}ch(2AB){::}\Gamma E{:}BE$. Now, since $ch(2\Gamma A) = 2\Gamma H$ and $ch(2AB) = 2BZ$, the proportion is an immediate consequence of the fact that triangles BZE and ΓHE are similar. Note that, when B and Γ are so placed that BΓ is parallel to radius ΔA, obviously $ch(2\Gamma A) = ch(2AB)$ and hence the ratio involved in the left–hand side of the above proportion is that of equality, but no such proportion holds since triangle ΓHE cannot be constructed.
6) Fourth cyclic lemma (*ibid.*, 73.11–74.8); see **Fig. 6**. It is formulated in the "language of the givens". In the configuration of the third cyclic lemma, from centre Δ join BΔ and draw ΔZ perpendicular to EBΓ. Then, if arc ΓB and ratio $ch(2\Gamma A){:}ch(2AB)$ are given, arc AB is also given. The proof applies a series of theorems from Euclid's *Data*. If B and Γ are so placed that BΓ is parallel to radius ΔA, the theorem still holds since in this case arc AB is given by the very straightforward argument we shall read in sch. **2**.
7) The Sector Theorem (*ibid.*, 74.9–76.9); see **Fig. 7**. From the endpoints B, Γ of two mutually intersecting arcs AB, AΓ of great circles on the surface of a sphere, two arcs BE, ΓΔ are drawn across, meeting at Z and intersecting arcs AΓ, AB at E, Δ, respectively (this will henceforth be called "spherical supine configuration"); all these arcs must be less than a semicircle. Then the following relations (any of them will henceforth be called "Menelaus relation") hold:

$$ch(2\Gamma E){:}ch(2EA) = [ch(2\Gamma Z){:}ch(2Z\Delta)]\circ[ch(2\Delta B){:}ch(2BA)] \text{ (“by separation”)}$$

$$ch(2\Gamma A){:}ch(2AE) = [ch(2\Gamma\Delta){:}ch(2\Delta Z)]\circ[ch(2ZB){:}ch(2BE)] \text{ (“by composition”).}$$

The proof introduces a suitable rectilinear supine configuration, derives a specific rectilinear relation associated with it and "lifts" it to the requited Menelaus relation associated with the assigned spherical supine configuration. Let us see this at work for the theorem "by separation" in the spherical supine configuration of **Fig. 7**. From the centre H of the sphere, radii HB, HZ, HE are joined; HB is produced to meet AΔ produced at Θ; ΓΔ, ΓA are joined and they meet HZ, HE at K, Λ, respectively; one shows that points Θ, K, Λ are on one and the same straight line. Applying the preceding lemmas to the rectilinear supine configuration in which from the endpoints Θ, Γ of two mutually intersecting straight lines ΑΘ, ΑΓ, two lines ΘΛ, ΓΔ are drawn across, meeting at Κ and intersecting straight lines ΑΓ, ΑΘ at Λ, Δ, respectively, one readily obtains the result, for the first or the second rectilinear lemma provide the rectilinear relation appropriate to the case at hand, the first or the third cyclic lemma "lift" each ratio of segments in the rectilinear relation to a ratio of chords in the spherical supine configuration.

The second and the fourth cyclic lemma are not applied in the proof of the Sector Theorem: they are intended to validate the calculations needed to determine the numerical value[37] of an arc involved in an assigned Menelaus relation once the values of four chords and the sum or difference of the arcs subtended by the other two chords (provided they feature in the same ratio) are given. Such a calculation is never performed in *Alm*. (Sidoli 2004a) but we find it three times in Pappus' commentary thereon.[38]

It is easy to see that several Menelaus relations, both "by separation" and "by composition", are associated with one and the same spherical supine configuration; each of them requires a specific construction and proof, in the lines of that outlined above but in some cases presenting subtle mathematical differencies as to the required construction. This explains the length of Theon's exposition, who treats in fact only a small number of cases.[39] A complete classification of the different theorems and cases was worked out by Thābit ibn Qurra (Lorch 2001); most of the valid cases can be deduced from one another by simple manipulations of compounded ratios, without any geometric argument.

Both the first and the third cyclic lemma are applied in the proofs of the theorem "by separation" and in that of the theorem "by composition";[40] Ptolemy proves in detail the former theorem, leaving the latter to the reader.[41] There is, however, a case of the Sector

[37] On the issue of "validation", see Acerbi 2011, 141–6, Acerbi 2012, 199–211, and Acerbi, Vitrac 2014, Étude complémentaire I.

[38] At *in Alm. V.13*, in *iA*, 84.3–85.22, *in Alm. V.14*, in *iA*, 102.16–103.11, *in Alm. VI.5*, in *iA*, 186.1–187.5. The fourth cyclic lemma is always at issue. Pappus identifies it by the expression διὰ τοῦ ιγʹ θεωρήματος πρώτου βιβλίου τῆς συντάξεως "by the 13rd theorem of the first book of the *Composition*" (*iA*, 84.3–4, 102.16, 186.1).

[39] One must keep separated the several relations already associated with one and the same rectilinear supine configuration from the cases (among which the "parallel" case) arising in the process of "lifting" each rectilinear relation to a Menelaus relation. Theon is quite effective in keeping these issues apart: a choice of the combinatorics issuing from the former issue is presented at *in Alm. I.13*, *iA*, 538.4–545.11, the latter issue being partly tackled at *in Alm. I.13*, *iA*, 557.27–566.13. Rome is quite clear on this in his notes: see *iA*, 535–7 n. 1, 539–40 n. 1, 560–1 n. 2, 564 n. 1.

[40] It is easy to see that, in every possible Menelaus relation associated with an assigned spherical supine configuration, there is at least one ratio whose terms contain partly overlapping arcs [just one ratio—in our example *ch*(2ΔΒ):*ch*(2ΒΑ)—in a relation "by separation" (actually, a ratio associated with an outer arc) and all ratios in a relation "by composition"]. The third cyclic lemma has the function to "lift" each ratio of partly overlapping segments in the rectilinear relation associated with the rectilinear supine configuration corresponding to the assigned spherical supine configuration, to a ratio of partly overlapping chords in the Menelaus relation associated with the spherical supine configuration.

[41] Contrary to what Ptolemy appears to imply (namely, that the theorem "by composition" requires a proof independent from that of the theorem "by separation"), both Theon (at *in Alm. I.13*, *iA*, 568.1–570.12) and

Theorem that cannot be covered by his proof, as we have seen under item 5 above: the "parallel" case. This case arises when, in the third cyclic lemma,[42] ΒΓ is parallel to radius ΔΑ. In this case, the constructions of the lemma cannot be completed. Nor can the proof of the theorem "by separation" just outlined under item 7: in this case (see **Fig. 7**), ΑΔ is parallel to ΗΒ and the rectilinear configuration does not "close" on point Θ. The proof of any theorem "by separation" presents the "parallel" case, but this can only happen with respect to one of the outer arcs of the spherical supine configuration. Since any theorem "by composition" can be deduced from a suitable theorem "by separation", no additional difficulties arise if the former theorem is to be proved or applied; for this reason, I shall implicitly refer in what follows to theorems "by separation".

Our Greek sources deal with the "parallel" case in the following ways.

- Ptolemy does not mention the "parallel" case, which in fact he never needs in the seventeen applications of the Sector Theorem one finds in the *Almagest*: see Rome's remarks at *iA*, 554–6 n. 1, and Rome 1933, 45–50.
- Theon does mention the "parallel" case of the third cyclic lemma but only to assert that it is non–constructible: ἀσύστατον ἔσται τὸ θεώρημα "the theorem will be non–constructible".[43] He also points out that Ptolemy οὐ προσχρῆται ταῖς οὕτως ἀσύστατον ποιούσαις τὸ πρόβλημα "does not use those <straight lines> that make the problem in this way non–constructible" (*in Alm. I.13*, *iA*, 554.11 and 554.16; the oscillating denomination "theorem"/"problem" has no relevance).

Still, by directly reasoning on the final configuration of the Sector Theorem (that is, without applying the cyclic lemmas), the "parallel" configuration, albeit as a limiting case, can be shown to give rise to the same relations between ratios of chords as those witten down under item 7 above: the peculiarity of the "parallel" case, as we shall see, is that one of the compounding ratios in the associated Menelaus relation is that of identity. Therefore, the Menelaus relation reduces in this case to a proportion.

Sph. III.1 derive the theorem "by composition" from that "by separation"; they use the obvious fact that the same chord subtends the arc double of a given arc and the arc double of its complement to a semicircle. In **Fig. 3** above, if we call Κ the other end–point of diameter ΒΔ, this amounts to the obvious equalities *ch*(2ΑΒ) = 2ΑΖ = *ch*(2ΑΚ): cf. Theon *in Alm. I.13*, *iA*, 567.1–10.

[42] As we shall see, the first mention of the "parallel" case will in fact occur in a scholium to the *fourth* cyclic lemma.

[43] It is not immediately clear what "non–constructible" means in this case. Maybe Theon really thought that the "parallel" case was impossible. In the same paragraph, he lists in fact two other non–constructible cases: when arc ΑΓ is greater than a semicircle, or equal to it (*iA*, 554.11–15). The plural ταῖς […] ποιούσαις in the clause at *iA*, 554.16 refers to these three cases. But maybe Theon simply remarked that it does not give rise to a compounded ratio, as we shall presently see. At any rate, Theon appears to perceive the "parallel" case as unproven (see the discussion in Rome 1933, 45 n. 1, who translates ἀσύστατον by "n'a pas lieu").

The contributions of the scholia amount to the following:

- Sch. **2** shows that the result of the fourth cyclic lemma is also valid when ΑΔ and ΒΓ are parallel; this fact is crucial in the proof of the "parallel" configuration of the Sector Theorem.
- Sch. **3** characterizes the "parallel" configuration as a limiting case of the configuration actually assumed by Ptolemy. The scholiast also outlines a correct proof of the "parallel" case.

Let us read the scholia; a discussion focusing on technical and linguistic detail will follow each of them.[44] A more general discussion will follow both. In particular, I shall show that the outline of proof found in sch. **3** is very much in the lines of the sketchy but sound proof of the "parallel" case found, with the variants to be discussed below, in the Arabo–Latin tradition of Menelaus' *Sphaerica*.

Sch. **2**

πολλάκις ἐν ταῖς κατὰ σύνθεσιν πτώσεσιν παράλληλος γίνεται ἡ ΒΓ τῇ ΔΑ· διὸ τότε αὐτόθεν δίδοται ἡ ΒΑ περιφέρεια, διὰ τὸ δοθείσης τῆς ὑπὸ ΖΔΒ δίδοσθαι καὶ τὴν λείπουσαν εἰς τὴν μίαν ὀρθήν, τουτέστι τὴν ὑπὸ ΒΔΑ· δίδοται ἄρα καὶ ἥ τε ΒΑ καὶ ὅλη ἡ ΓΒΑ.

1 ΔΑ] ΒΑ codd. **2** ΒΑ] ΓΑ **C** | ΖΔΒ] ΖΑΒ **K**

Transl. In the cases by composition, ΒΓ often becomes parallel to ΔΑ; this is the reason why in that case arc ΒΑ is immediately given, because, once <angle> ΖΔΒ is given, the complement to one right <angle> is also given, that is, ΒΔΑ; therefore both <arc> ΒΑ and ΓΒΑ as a whole are also given.

Comm. *a*) **B**, f. 25r marg. sup., **C**, f. 51r marg. sup., **K**, f. 31v. *b*) To *Alm*. I.13, 73.11–14 καὶ ἐνταῦθα δὲ αὐτόθεν παρακολουθεῖ, διότι, κἂν ἡ ΓΒ περιφέρεια μόνη δοθῇ καὶ ὁ λόγος ὁ τῆς ὑπὸ τὴν διπλῆν τῆς ΓΑ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΑΒ δοθῇ, καὶ ἡ ΑΒ περιφέρεια δοθήσεται "here too, it immediately follows that, even if both the single arc ΓΒ is given and the ratio of the <straight line> under the double of <arc> ΓΑ to that under the double of ΒΓ is given, arc ΑΒ will also be given" ff. *c*) A scholium to the

[44] The following *sigla* will be employed: **B** = Vat. gr. 1594; **C** = Marc. gr. 313; **K** = Vat. gr. 184. Since **K** is a copy of **B**, its readings should in principle be eliminated; I keep them since they give interesting information about the errors originating in the act of copying. The commentary provides the following information. *a*) Exact location of the scholium in the manuscripts. *b*) Transcription and translation of the passage of *Alm.* to which the scholium refers (called "the *relatum*"); the indication "*POO* I.1" is understood. In case it is possible to exactly identify the terms to which the scholium refers, or if the scholium is purposely (for instance, by means of a marginal sign) located beside a line of the text in **B**, the terms or the line are underlined. *c*) Discussion of textual issues and of the mathematical context, with identification of likely sources or of similar passages in other authors. *d*) Graphic and codicological features. *e*) Lexical and syntactical remarks.

fourth cyclic lemma (see **Fig. 6**), showing that the result is also valid when ΑΔ and ΒΓ are parallel. That angle ΖΔΒ is given is stated by Ptolemy, 73.16–74.2; one then applies *Data* 4, the fact that the arcs on a circumference and the angles at the centre subtending them are in one–to–one correspondence (use *Data* 89, *El*. III.20, *Data* 2), and *Data* 3. In this case one also immediately gets that, since ΒΖ = ΓΗ in the configuration of the third cyclic lemma (**Fig. 5**), the ratio *ch*(2ΓΑ):*ch*(2ΑΒ) mentioned in the *relatum* is that of equality, a fact that will prove crucial in the proof of the "parallel" configuration of the Sector Theorem outlined in sch. **3**. Why a similar scholium was not attached to Ptolemy's proof of the *third* cyclic lemma will remain a mystery. *d*) In **B**, the scholium is above the column in which the fourth cyclic lemma ends. In **C**, it is in the upper margin of the page containing the same lemma. In either case, no *signe de renvoi* is added. *e*) The scholiast's πολλάκις "often" at line 1 is something of a cheat: as we have noted above, the "parallel" configuration of the Sector Theorem is never required in *Alm*. The αὐτόθεν "immediately" at line 1 is both imitative of the *relatum*, where the same adverb occurs, and a typical metadiscursive modifier, of which Ptolemy is specially fond; one finds 69 occurrences in *Alm*., 5 in Pappus, *in Alm*. V–VI, 26 in Theon, *in Alm*. I–IV. The operator ἡ λείπουσα εἰς "the complement to" at line 2 is in this scholium applied to an angle; otherwise the expression ἡ λείπουσα εἰς τὸ ἡμικύκλιον means "the <chord> complement to a semicircle".

Sch. **3**

ὅταν μὲν ἡ ἀπὸ τοῦ Η ἐπὶ τὸ Α ἐπιζευγνυμένη ποιῇ τὰς ὑπὸ ΔΑΗ ΑΗΒ γωνίας δύο ὀρθῶν ἐλάττονας, τότε ἡ ΑΔ συμπεσεῖται τῇ ΗΒ κατὰ τὸ Θ ὡς νῦν· ὅταν δὲ δύο ὀρθῶν μείζονας, τότε ἡ ΔΑ τῇ ΒΗ ἐπὶ θατέρῳ μέρει συμπεσεῖται, προσαναπληρωθέντων τῶν ΒΔΑ ΒΖΕ ἡμικυκλίων καὶ τῆς ΒΗ διαμέτρου, καὶ ἡ δεῖξις προβαίνει· ὅταν δὲ παράλληλος ᾖ ἡ ΑΔ τῇ ΒΗ, τότε καὶ τῇ ΚΛ παράλληλος γίνεται ἐξ ἀνάγκης, καὶ ὁ τῆς ΓΛ πρὸς ΛΑ λόγος συναφθήσεται ἐκ τοῦ τῆς ΓΚ πρὸς ΚΔ· ὁ γὰρ τῆς ὑπὸ τὴν διπλῆν τῆς ΑΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΔ τότε ἰσότητός ἐστι λόγος, τουτέστι τοῦ αὐτοῦ πρὸς τὸ αὐτό· ὥστε καὶ οὕτως ἡ δεῖξις προβήσεται.

1 ὑπὸ] ἀπὸ **C** | ὀρθῶν] ὀρ$^{\theta}$ **BC** : ὀρθὸν **K** **3** μείζονας] comp. **B** : γωνίας comp. **C** : μιᾶς **K** | ΔΑ] ΑΔ **K** **3** θατέρῳ μέρει] **C** : θάτερα μὲν οὐ **BK** | προσαναπληρωθέντων] (προσ)ανα– **BC** : πρὸς ἀνα– **K** **5** ᾖ] ἦ̣ **K** **6** συναφθήσεται] ἀναφθ– codd. : dubitanter correxi | ΓΚ] Γ (καὶ) **K** | ΚΔ] scripsi : ΚΓ codd. | ὁ γὰρ τῆς] scripsi : ὃ γί/$^{\tau}$ **BC** : ὃ γίνεται **K** **8** οὕτως] (ου)$^{\tau}$ **B** : οὗ **K**

Transl. When the <straight line> joining Η and Α makes angles ΔΑΗ, ΑΗΒ less than two right angles, then ΑΔ will meet ΗΒ at Θ, as now; when <it makes angles ΔΑΗ, ΑΗΒ> greater than two right angles, then ΔΑ will meet ΗΒ on the other side, once semicircles ΒΔΑ, ΒΖΕ and diameter ΒΗ have been completed, and the proof can proceed. When ΑΔ is parallel to ΒΗ, then it necessarily becomes parallel to ΚΛ too, and the ratio of ΓΛ to ΛΑ will be compounded of that of ΓΚ to ΚΔ, for the <ratio> of the <straight line> under the double of <arc> AB to the <straight line> under the double of <arc> BΔ is in that case the ratio of equality, that is, of the same to the same—as a consequence, also in this way the proof will proceed.

Comm. *a*) **B**, f. 25r marg. ext., **C**, f. 51v marg. ext. et inf., **K**, f. 32r. *b*) To *Alm*. I.13, 74.20–76.2 εἰλήφθω γὰρ τὸ κέντρον τῆς σφαίρας […] καὶ τοῦ τῆς ὑπὸ τὴν διπλῆν τῆς ΔΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΑ "in fact, let the centre of the sphere be taken […] and of the <ratio> of the <straight line> under the double of <arc> ΔB to that under the double of BA". *c*) A scholium to the construction and proof of the theorem "by separation" of the Sector Theorem (see **Fig. 7**), describing the "parallel" configuration as a limiting case of the configuration actually assumed by Ptolemy. The straight line joining H and A does not feature in Ptolemy's construction; its function is simply to permit formulating a criterion of intersection *vs.* parallelism of straight lines AΔ and HB. The scholiast also summarizes in few but careful words the gist of the proof in that case (see just below for a more expanded version). *d*) In **B**, sch. **3** is located in the outer margin, beside the construction of the Sector Theorem; in **C**, its beginning is placed beside the last five lines of the proof; the remaining portion of the scholium continues beside the proof sketch of the theorem "by composition" (76.3–9). No *signe de renvoi* is added. *e*) Maybe the form ἀναφθήσεται the manuscripts have at line 5 need not to be corrected to συναφθήσεται: the point is that there is only one compounding ratio, namely, ΓΚ:ΚΔ, that "makes up" ratio ΓΛ:ΛΑ. *e*) At line 3, the form of προσαναπληροῦν with double preverb is slightly more canonical, in case of parts of circles, than the form of ἀναπληροῦν: after the isolated, seminal occurrences of the former at *El*. III.25 (what is completed is a circle) and of the latter at *El*. XII.2 (what is completed is a parallelogram), a mathematical Atticist such as Pappus only resorts to the former when completing circles (11 occurrences in *Coll*.).

The argument of the scholiast can be formalized as follows (an asterisk * marks the statements made by the scholiast).

(1) Take the rectilinear relation associated, by the first rectilinear lemma, with the rectilinear supine configuration assumed by Ptolemy: ΓΛ:ΛΑ = (ΓΚ:ΚΔ)∘(ΔΘ:ΘΑ).

(2)* Now, as seen in sch. **2** and as the scholiast points out, in the "parallel" case of the third cyclic lemma the ratio between chords *ch*(2ΔΒ):*ch*(2ΒΑ) mentioned in the *relatum* (underlined above), and that will be made to correspond to ratio ΔΘ:ΘΑ in the rectilinear relation of point (1), is that of equality.

(3)* On the other hand, as the scholiast points out, when AΔ becomes parallel to BH and hence to KΛ, by *El*. VI.2 the ratios ΓΛ:ΛΑ and ΓΚ:ΚΔ become identical.

(4) Now, we may use the third cyclic lemma applied to the following two ratios: *ch*(2ΓΕ):*ch*(2ΕΑ)::ΓΛ:ΛΑ and *ch*(2ΓΖ):*ch*(2ΖΔ)::ΓΚ:ΚΔ.

(5) By *El*. V.11, one immediately has *ch*(2ΓΕ):*ch*(2ΕΑ)::*ch*(2ΓΖ):*ch*(2ΖΔ).

(6) Since $ch(2\Delta B):ch(2BA)$ is the ratio of equality, this means that the Menelaus relation $ch(2\Gamma E):ch(2EA) = [ch(2\Gamma Z):ch(2Z\Delta)]\circ[ch(2\Delta B):ch(2BA)]$ associated with the assigned spherical supine configuration also holds in the "parallel" case, in the limiting non–compounded form $ch(2\Gamma E):ch(2EA)::ch(2\Gamma Z):ch(2Z\Delta)$.

Sch. **2** and **3** constitute the first direct evidence that a proof of the "parallel" configuration of the Sector Theorem was elaborated in Greek. The outline of proof provided by the scholiast is clear and omits no important step, since steps (1) and (4)–(6) above either involve trivial manipulations or are obvious given the context. One has to fill in the details and write down a formal argument, as I have just done, or as is reflected in the actual proof of this statement, attested as a case after that "by separation", in the Arabo–Latin tradition of the *Sphaerica*, to which I now turn.

The proofs of the Sector Theorem we read in Gerard's translation and in the al–Māhānī & al–Harawī recensions have been carefully compared by N. Sidoli, who takes them as "the versions of the theorem least removed from Menelaus" (2006, 51). For Abū Naṣr's revision one may obviously refer to Krause (1936, 195–6 and ٦٣–٤). In these versions, the proofs of the "parallel" case reads as follows (I adapt the lettering to an obvious modification of **Fig. 7**).

- *al–Māhānī & al–Harawī*. They state steps (2), (3), and (5), and only these. Step (2) is modified: it is not asserted that $ch(2\Delta B):ch(2BA)$ is the ratio of equality, but that $ch(2\Delta B) = ch(2BA)$.
- *Gerard*. He first provides a sketchy outline of the construction of the configuration of the "parallel" case. He then states the condition in step (3) by assuming that AΔ is parallel to KΛ, and therefore must also prove that it is also parallel to BH.[45] He then states: step (2) in the modified formulation just seen and identifying the two chords $ch(2\Delta B)$ and $ch(2BA)$ as the two perpendiculars dropped from points Δ and A, respectively, to straight line BH; steps (3) and (4) combined in one; the conclusion of step (6), backed up by a postposed explanation that we might take as a short form of step (5).
- *Abū Naṣr*. He first provides a lengthy outline of the construction of the configuration of the "parallel" case. At the beginning of the construction, he states steps (2) in the modified formulation just seen and identifying the two chords $ch(2\Delta B)$ and $ch(2BA)$ as the two perpendiculars dropped from points Δ and A, respectively, to straight line BH. He then embarks in a lengthy and pointless proof by *reductio* that KΛ is parallel to AΔ.[46] He then states steps (3), (5), and (6).

[45] See Lorch 2001, 328, on this feature of Gerard's version.
[46] Abū Naṣr simply forgets here that parallelism in space is transitive.

This outline shows that the Arabo–Latin tradition elaborates on one and the same core argument, adding steps whenever this was perceived to be too concise. If we take the al–Māhānī & al–Harawī version to be the least removed from Menelaus' original argument, then this virtually coincides with the sketchy outline we have read in sch. **3**.

I close this article with an assessment, in the form of three scenarios in each of which I shall argue in its favour, of the information on the Sector Theorem afforded by our sources. In principle, any of the actors mentioned in the scenarios might have had his own redaction of Menelaus' *Sphaerica*, and in any of these the (parallel case of the) Theorem might have been present or absent; this would solve all problems raised by our documentary record—still, *entia non sunt multiplicanda praeter necessitatem*. My discussion will not take up the issue of the origins of the Theorem (on this, see most recently Sidoli 2006), even if the first scenario makes the issue more urgent than the others.

1. Menelaus' *Sphaerica* did not contain the Sector Theorem. If one would insist on keeping the link between Menelaus and the Theorem, a very appropriate place for it could be the exposition on (rising and) setting times mentioned by Pappus in *Coll*. VI.110. No modern scholar seems to believe that the extreme scenario is possible. Still, a circumstantial argument can be adduced in its favour.[47] Neither Ptolemy nor Theon (nor, by implication, Pappus, whose commentary Theon surely took as a reference) ever associate the name of Menelaus with the Theorem that deserves the longest and most sustained mathematical argument in *Alm*. or in the commentaries thereon. If Ptolemy's silence comes as no surprise at all[48], to explain the silence of the commentators (part of whose job was exactly to make tacit references of this kind explicit) such typical distortions of hypercritical exegesis must be mobilized as supposing that Pappus' and Theon's acquaintance with the *Sphaerica* was limited to the propositions they quoted—or maybe, to the first book of the treatise. The fact that the Sector Theorem is applied in a number of subsequent propositions (*Sph*. III.2, 3, 13, 16, 22, 24 in Abū Naṣr's revision) has no relevance, since, for instance, a result as fundamental as the invariance of the cross–ratio on the surface of a sphere is applied without proof in *Sph*. III.5. On the contrary, this fact might provide a very simple explanation of the presence of the Sector Theorem in the *Sphaerica* we read: it was included to fill a perceived deductive gap. The fact that the Theorem is attested in the entire Arabic tradition suggests that this supplement to Menelaus' *Sphaerica* was already present in the Greek line of tradition. It may well be that the scholiast (who most likely writes, as we have seen, in the early 6th century) already had a "completed" edition of the *Sphaerica* in his hands.

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[47] Quite a strong case for severing the link between Menelaus and the Theorem, still holding that it was present in the *Sphaerica*, with many circumstantial and substantial arguments, is made in Sidoli 2004a. See Sidoli 2006, section V, for the presence of the Theorem in Menelaus' exposition on rising and setting times.

[48] Ptolemy never mentions Euclid even if, for instance, he refers to the enunciations of *El*. XIII.9 and 10 in *Alm*. I.10, *POO* I.1, 33.12–15 and 33.18–20. Of course, one might also be to some extent entitled to entertain the hypothesis that Ptolemy did not know the *Sphaerica*.

2. Menelaus' *Sphaerica* did contain the Sector Theorem but not the "parallel" case. It is the conclusion drawn by Rome (1933, 50):[49] "il semblerait bien, d'après ce qui précède, que Ptolémée et Théon ne trouvaient pas dans Ménélas la preuve complète du théorème qui est mis sous son nom". The crucial clue, of course, is Theon's claim that the "parallel" case is "non–constructible", but also Ptolemy's silence about it is significant: Rome takes pains to show, a task that is not obvious at all, that in none of the seventeen applications of the Theorem in *Alm*., the "parallel" case is needed, in at least one case the necessity of applying it being neutralized by a trifle. It is perverse, so Rome concludes, to think that Ptolemy spent a treasury of ingenuity during one thousand pages, just to avoid writing down the 5–line proof of the "parallel" case. This hypothesis has the advantage that we do not have to suppose that Theon (and, almost surely, Pappus) was ignorant of the later part of the *Sphaerica*. Again, we may think that the "parallel" case was added, in a sketchy form, at some point in the Greek line of transmission, possibly after Theon, or possibly independently of him; the form of the addition triggered the various completions and additions attested in the Arabo–Latin tradition; our scholiast just made a compendium of the argument he had found in his source (note that, contrary to what happens in sch. **1**, he does not mention the *Sphaerica*!). As for the apparent disadvantage of this hypothesis, namely, that we should think of a Menelaus (and of a Ptolemy after him) who did not realize that the proof he was about to give is incomplete, one might argue that in fact it is not, since the "parallel" case does not give rise to a compounded ratio (as we have seen, also Theon's remark can be read in this way). Therefore, it is debatable whether it can be regarded as a case of the Sector Theorem, or simply as a result similar to it and holding for the same spherical supine configuration as the Sector Theorem. As a matter of fact, the "parallel" case cannot occur in some of the propositions of the *Sphaerica* in which the Sector Theorem is applied: these are *Sph*. III.16, 22, 24 (and III.13 only applies the theorem "by composition"). Moreover, the "parallel" case is never used in *Alm*., and one might argue that in fact it could not.[50] Maybe it is simply at oversight on the part of Menelaus and Ptolemy, that Theon later transformed into an impossibility.
3. Menelaus' *Sphaerica* contained a complete proof of the Sector Theorem. This was ignored by Ptolemy and Pappus and unknown to Theon, but found by some scholar in the late ancient period. The main reason favouring Menelaus' authorship obviously resides in the fact that the Arabo–Latin tradition has the Sector Theorem in its complete form. If this were the case, however, we need to explain how Pappus and Theon either ignored or did not know this. As for Ptolemy, there is no indication that he read Menelaus' *Sphaerica*. At the very least he did not choose to use the material on spherical trigonometry that begins with the Sector Theorem, since this would have greatly

[49] This conclusion is virtually endorsed in Lorch 2001, 335.
[50] This was argued by N. Sidoli in a private communication. I hope he will fully develop his point.

simplified his spherical astronomy (see Nadal, Taha, Pinel 2004, 404). So he was probably following a previous, well–established tradition (Sidoli 2006, section V). Pappus clearly read some of Menelaus' treatise, but there is no indication that he went on to the final section on spherical trigonometry, which is anyway unnecessary for expounding Ptolemy's methods. As for Theon, there is nothing to indicate that he was familiar with the material on spherical trigonometry too, so all we need to assume is that Theon himself did not have Menelaus' book available to him, and quoted *Sph*. I.13 and 14 in *in Alm*. VI by lifting them from some other source. Note that this source cannot be Pappus' commentary, whose book VI we read and who does not mention the two propositions.

Someone might think that one of theses scenarios fits the documentary record on the Sector Theorem better than the others. I content myself with admitting that such a record fiercely resists being satisfactorily fitted.[51]

[51] I thank Nathan Sidoli for his critical remarks and Ramon Masiá for helping me with the diagrams. This research was supported in part by the project FFI2015-65118-C2-2-P "El autor bizantino II: Transmisión de los textos y bibliotecas" of the Spanish government, Ministerio de Economía y Competitividad.

Appendix. *Sph.* I.13 and I.14 in Theon's commentary

The text is established on the basis of Laur. Plut. 28.18, ff. 250v–251v (*siglum* **L**), Marc. gr. 303. f. 127r (**H**), Vat. gr. 183, ff. 189v–192r (**X**); the first critical apparatus presents the variant readings of these manuscripts, that Rome had shown to belong to different branches of tradition (**L** on the one side, **HX** on the other: *iA*, XXI–XXIV and LXXXVI–XCII) as far as Theon *in Alm*. I–IV is concerned. However, a few variant readings suggest either that **X** was a direct (and very bad) copy of **L** or that they are apographs of the same model. The second critical apparatus contains the variants of the main manuscripts of the two Byzantine recensions: Vat. gr. 198, f. 448r (**J**), and Marc. gr. 310, ff. 233v–234r (**E**), respectively. The Venice manuscript was penned by Isaac Argyros, to whom this recension must be ascribed—in fact, as it was usual with Argyros, his text is a correction *in scribendo* of the recension contained in the Vat. gr. 198.

Contrary to what we might have expected given the fact that Theon *in Alm*. I–IV is very correctly copied in **L** (*iA*, XXIII: "on le [*scil*. the copyist] prend rarement en faute"), the text in this manuscript is larded with mistakes, often very trivial (for instance, ἴσος "equal" is constructed several times with the genitive) and often to be found also in **HX**; such trivial mistakes are not contained in the summaries located in the margins of **L**, and here simply transcribed in three footnotes to the Greek text. This cannot be explained by the mere fact that a different copyist is at work in Theon *in Alm*. VI. One cannot draw conclusions from such a short text as the one edited here, but a good working hypothesis is that all extant manuscript witnesses of Theon *in Alm*. VI derive from an exemplar copied by a surprisingly unskilled copyist on a model in majuscule filled with abbrevations, conventional signs and truncated words.

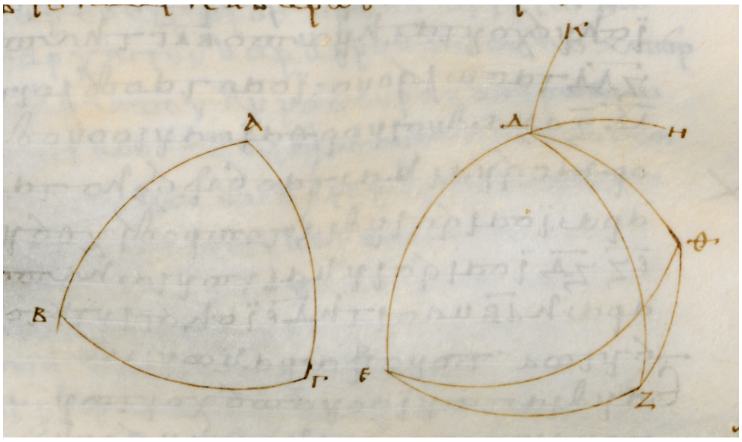
The Greek text is edited and (sparingly) punctuated according to the rules expounded in Acerbi, Vitrac 2014, 98. The diagrams of **L** are reproduced as **Pl**. **I** and **Pl**. **II** (ff. 250v and 251r, respectively). The sign | marks the beginning of a page of **L**. The first and the second critical apparatus are placed at the end of the Greek text and of the translation, respectively.

Greek Text

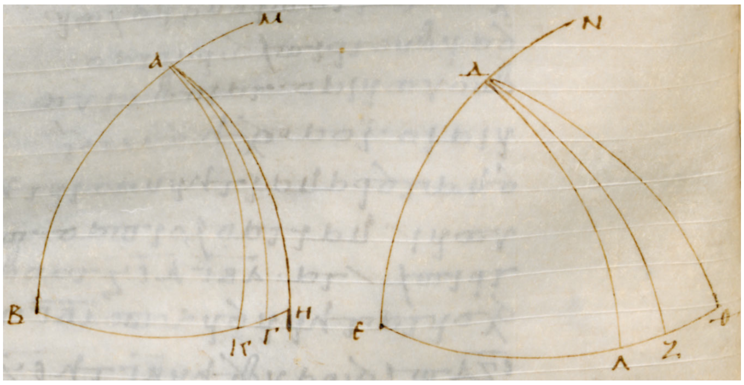
τούτων μὲν οὖν ἕνεκα ὡς ἐπὶ τὸ πλεῖστον παχυμερέστερον καταλαμβάνονται αἱ εἰρημέναι προσνεύσεις· δυνατὸν δέ ἐστιν ἀκριβέστερον αὐτὰς ἐφοδεύειν προλαμβανομένων τῶν τοιούτων δύο θεωρημάτων δειχθέντων ἐν τοῖς Μενελάου σφαιρικοῖς.

[52]ἐὰν δύο τρίπλευρα μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχῃ περὶ δὲ ἄλλας γωνίας τὰς πλευρὰς ἴσας ἑκατέραν ἑκατέρᾳ τὰς δὲ λοιπὰς γωνίας ἅμα δυσὶν ὀρθαῖς μὴ ἴσας, καὶ τὰς λοιπὰς πλευρὰς ἴσας ἀλλήλαις ἕξει.

[52] Marg. ext. **L** πῶς δείκνυται ἡ λεγομένη πρότασις ἐὰν δύο τρίπλευρα μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχῃ περὶ δὲ ἄλλας γωνίας τὰς πλευρὰς ἴσας ἑκατέραν ἑκατέρᾳ τὰς δὲ λοιπὰς γωνίας ἅμα δυσὶν ὀρθαῖς μὴ ἴσας, καὶ τὰς λοιπὰς πλευρὰς ἴσας ἀλλήλαις ἕξει.



Pl. I. The diagram of *Sph*. I.13 in Theon, *in Alm. VI.11*. Firenze, Biblioteca Medicea Laurenziana, Ms. Plut. 28.18, f. 250v. Su concessione del MiBACT. È vietata ogni ulteriore riproduzione con qualsiasi mezzo.



Pl. II. The diagram of *Sph*. I.14 in Theon, *in Alm. VI.11*. Firenze, Biblioteca Medicea Laurenziana, Ms. Plut. 28.18, f. 251r. Su concessione del MiBACT. È vietata ogni ulteriore riproduzione con qualsiasi mezzo.

Translation

By virtue of this, the said inclinations can be obtained quite loosely in most cases; still, it is possible to compute them more exactly if one assumes the following two theorems proved in Menelaus' Spherics as a preliminary.

If two trilaterals have one angle equal to one angle, the sides about the other angles respectively equal, and the remaining angles together not equal to two right <angles>, they will also have the remaining sides equal to one another.

ἔστω δύο τρίπλευρα τὰ ΑΒΓ ΔΕΖ ἴσας ἔχοντα τὰς γωνίας τὰς πρὸς τοῖς Β Ε τὰς δὲ περὶ τὰς Γ Ζ γωνίας πλευρὰς ἴσας, | τὴν μὲν ΒΓ τῇ ΕΖ τὴν δὲ ΑΓ τῇ ΔΖ, καὶ ἔτι τὰς πρὸς τοῖς Α Δ γωνίας ἅμα μὴ ἴσας δυσὶν ὀρθαῖς. λέγω ὅτι ἴση ἐστὶν ἡ ΑΒ τῇ ΔΕ.

ἐκβεβλήσθω γὰρ ἡ ΕΔ ἐπὶ τὸ Η. ἐπεὶ αἱ ὑπὸ ΒΑΓ ΕΔΖ ταῖς ὑπὸ ΕΔΖ ΖΔΗ ἄνισοί εἰσι, κοινῆς ἀφαιρουμένης τῆς ὑπὸ ΕΔΖ λοιπαὶ αἱ ὑπὸ ΒΑΓ ΖΔΗ ἄνισοί εἰσιν. ἔστω δὴ πρότερον μείζων ἡ ὑπὸ ΖΔΗ τῆς ὑπὸ ΒΑΓ, καὶ συνεστάτω τῇ Α γωνίᾳ ἴση ἡ ὑπὸ ΖΔΘ, καὶ κείσθω τῇ ΑΒ ἴση ἡ ΔΘ, καὶ διὰ τῶν Ζ Θ μέγιστος κύκλος γεγράφθω ὁ ΖΘ, καὶ ἔτι διὰ τῶν Θ Ε ὁ ΘΕ.

ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΔΘ ἡ δὲ ΑΓ τῇ ΔΖ, δύο δὴ αἱ ΒΑ ΑΓ δυσὶ ταῖς ΘΔ ΔΖ ἴσαι εἰσί· καὶ γωνία ἡ ὑπὸ ΒΑΓ γωνίᾳ τῇ ὑπὸ ΘΔΖ ἴση· βάσις ἄρα ἡ ΒΓ βάσει τῇ ΘΖ ἐστὶν ἴση· ἀλλὰ ἡ ΒΓ τῇ ΕΖ ὑπόκειται ἴση· καὶ ἡ ΕΖ ἄρα τῇ ΖΘ ἴση ἐστίν· ὥστε καὶ γωνία ἡ ὑπὸ ΖΕΘ γωνίᾳ τῇ ὑπὸ ΖΘΕ ἴση ἐστίν. καὶ ἐπεὶ ἴσον ἀπεδείχθη τὸ ΑΒΓ τρίπλευρον τῷ ΔΖΘ τριπλεύρῳ, ἴση ἐστὶ καὶ ἡ ὑπὸ ΖΘΔ γωνία τῇ ὑπὸ ΓΒΑ· ἀλλὰ ἡ ὑπὸ ΓΒΑ τῇ ὑπὸ ΖΕΔ ἐστὶν ἴση· ὥστε καὶ ἡ ὑπὸ ΖΘΔ τῇ ὑπὸ ΖΕΔ ἐστὶν ἴση, ὧν ἡ ὑπὸ ΕΘΖ τῇ ὑπὸ ΘΕΖ ἐδείχθη ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΔΕΘ λοιπῇ τῇ ὑπὸ ΔΘΕ ἐστὶν ἴση· ὥστε καὶ πλευρὰ ἡ ΕΔ πλευρᾷ τῇ ΔΘ ἴση ἐστίν· ἀλλ’ ἡ ΔΘ τῇ ΑΒ ἴση· καὶ ἡ ΑΒ ἄρα τῇ ΕΔ ἴση:—

[53]ἔστω δὴ ἐλάσσων ἡ ὑπὸ ΖΔΗ γωνία τῆς ΒΑΓ, καὶ συνεστάτω τῇ ὑπὸ ΒΑΓ ἴση ἡ ὑπὸ ΖΔΚ, καὶ ἔστω πάλιν ἴση ἡ ΑΒ τῇ ΔΚ. ὁμοίως πάλιν διὰ τῶν αὐτῶν ἐπιλογισμῶν δειχθήσεται τὸ προκείμενον.

[54]ἐὰν δύο τρίπλευρα τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ τὴν βάσιν τῇ βάσει ἴσην ἔχῃ τὴν πρὸς ταῖς ἴσαις γωνίαις, καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει.

ἔστω δύο τρίπλευρα τὰ ΑΒΓ ΔΕΖ τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχοντα, τὴν μὲν ὑπὸ ΑΒΓ τῇ ὑπὸ ΔΕΖ τὴν δὲ ὑπὸ ΒΓΑ τῇ ὑπὸ ΕΖΔ, πλευρὰν δὲ τὴν ΒΓ τῇ ΕΖ ἴσην. λέγω ὅτι καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει.

ἤτοι γὰρ ὀρθαί εἰσιν αἱ ὑπὸ ΑΒΓ ΔΕΖ ἢ ἐλάττονες ἢ μείζους.

ἔστωσαν πρότερον ὀρθαί· οἱ ἄρα τῶν ΑΒ ΔΕ κύκλων πόλοι ἐπὶ τῶν ΒΓ ΕΖ εἰσί· καί εἰσιν αἱ ΓΒ ΕΖ ἤτοι τεταρτημόρια ἢ μείζονες ἢ ἐλάσσονες.

ἔστωσαν πρότερον τεταρτημόρια· τεταρτημόρια ἄρα καὶ αἱ ΓΑ ΔΖ· δύο οὖν αἱ ΒΓ ΓΑ δυσὶ ταῖς ΕΖ ΖΔ ἴσαι εἰσί· καὶ γωνία ἡ ὑπὸ ΒΓΑ γωνίᾳ τῇ ὑπὸ ΔΖΕ ἴση· βάσις ἄρα ἡ ΑΒ βάσει τῇ ΔΕ ἴση ἐστίν.

ἔστωσαν δὴ ἐλάσσονες τεταρτημορίου αἱ ΒΓ ΕΖ, καὶ κείσθωσαν τεταρτημορίῳ ἴσαι αἱ ΒΗ ΕΘ, | καὶ διὰ τῶν Η Α Δ Θ μέγιστοι κύκλοι γεγράφθωσαν οἱ ΗΑ ΔΘ· τεταρτημόριον ἄρα ἑκατέρα.

[53] Marg. ext. **L** πῶς δειχθήσεται τὸ προκείμενον ἐὰν ὑποθώμεθα ἐλάσσονα εἶναι τὴν ὑπὸ ΖΔΗ γωνίαν τῆς ὑπὸ ΒΑΓ.

[54] Marg. ext. **L** πῶς δείκνυται τὸ ἐὰν δύο τρίπλευρα τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ τὴν βάσιν τῇ βάσει ἴσην ἔχῃ τὴν πρὸς ταῖς ἴσαις γωνίαις, καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει.

Let there be two trilaterals ΑΒΓ, ΔΕΖ, having the angles at Β, Ε equal, the sides about angles Γ, Ζ equal (ΒΓ to ΕΖ and ΑΓ to ΔΖ), and again the angles at Α, Δ together not equal to two right <angles>. I say that ΑΒ is equal to ΔΕ.

In fact, let ΕΔ be produced as far as Η. Since ΒΑΓ, ΕΔΖ are unequal to ΕΔΖ, ΖΔΗ, once ΕΔΖ in common is removed the remaining ΒΑΓ, ΖΔΗ are unequal. First, then, let ΖΔΗ be greater than ΒΑΓ, and let ΖΔΘ be constructed equal to angle Α, and let ΔΘ be set equal to ΑΒ, and let a great circle ΖΘ be traced through Ζ, Θ, and again ΘΕ through Θ, Ε.

Since ΑΒ is equal to ΔΘ and ΑΓ to ΔΖ, two <arcs> ΒΑ, ΑΓ are equal to two ΘΔ, ΔΖ; and angle ΒΑΓ is equal to angle ΘΔΖ; therefore base ΒΓ is equal to base ΘΖ; but ΒΓ has been supposed equal to ΕΖ; therefore ΕΖ is also equal to ΖΘ, so that angle ΖΘΕ is also equal to angle ΖΕΘ. And since trilateral ΑΒΓ has been proved equal to trilateral ΔΖΘ, angle ΖΘΔ is equal to ΓΒΑ; but ΓΒΑ is equal to ΖΕΔ, so that ΖΘΔ is also equal to ΖΕΔ, of which ΕΘΖ was proved equal to ΘΕΖ; therefore ΔΕΘ as a remainder is also equal to ΔΘΕ as a remainder, so that side ΕΔ is also equal to side ΔΘ; but ΔΘ is equal to ΑΒ; therefore ΑΒ is also equal to ΕΔ.

Then, let angle ΖΔΗ be less than ΒΑΓ, and let ΖΔΚ be constructed equal to ΒΑΓ, and let again ΑΒ be equal to ΔΚ. Again, what has been proposed will be similarly proved by means of the same arguments.

If two trilaterals have two angles respectively equal to two angles and have the base equal to the base (the one about the equal angles), they will also have the remaining sides equal to the remaining sides.

Let there be two trilaterals ΑΒΓ, ΔΕΖ having two angles equal to two angles (ΑΒΓ to ΔΕΖ and ΒΓΑ to ΕΖΔ) and side ΒΓ equal to ΕΖ. I say that they will also have the remaining sides equal to the remaining sides.

In fact, ΑΒΓ, ΔΕΖ either are, or are less, or are greater than a right <angle>.

First, let them be right <angles>; therefore the poles of circles ΑΒ, ΔΕ are on ΒΓ, ΕΖ; and ΓΒ, ΕΖ either are, or are less, or are greater than a quadrant.

First, let them be quadrants; therefore ΓΑ, ΔΖ also are; now, two <arcs> ΒΓ, ΓΑ are equal to two ΕΖ, ΖΔ; and angle ΒΓΑ is equal to angle ΔΖΕ; therefore base ΑΒ is equal to base ΔΕ.

Let ΒΓ, ΕΖ be less than a quadrant, and let ΒΗ, ΕΘ be set equal to a quadrant, and let great circles ΗΑ, ΔΘ be traced through Η, Α, Δ, Θ; therefore each of them is a quadrant.

καὶ ἐπεὶ ἴσαι εἰσὶν αἱ ΒΗ ΕΘ ἀλλήλαις, ὧν αἱ ΒΓ ΕΖ ἴσαι, λοιπαὶ ἄρα αἱ ΓΗ ΘΖ ἴσαι εἰσίν· εἰσὶ δὲ καὶ αἱ ΗΑ ΔΘ ἴσαι· καὶ γωνία ἡ ὑπὸ ΑΓΗ τῇ ὑπὸ ΘΖΔ ἴση διὰ τὸ καὶ τὰς ἐφεξῆς ἴσας εἶναι· δύο δὴ τρίπλευρά ἐστι τὰ ΑΓΗ ΔΖΘ μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχοντα, τὴν ὑπὸ ΑΓΗ τῇ ὑπὸ ΔΖΘ, περὶ δὲ τὰς ὑπὸ ΓΗΑ ΖΘΔ τὰς πλευρὰς ἴσας τὰς δὲ λοιπὰς τὰς ὑπὸ ΓΑΗ ΖΔΘ ἅμα δυσὶν ὀρθαῖς ἀνίσους διὰ τὸ ὅλας τὰς ὑπὸ ΒΑΗ ΕΔΘ δύο ὀρθὰς εἶναι· καὶ αἱ λοιπαὶ ἄρα πλευραὶ ταῖς λοιπαῖς πλευραῖς ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· ἴση ἄρα ἡ ΑΓ τῇ ΔΖ· ἔστι δὲ καὶ ἡ ΒΓ τῇ ΕΖ ἴση· δύο δὴ αἱ ΒΓ ΓΑ δυσὶ ταῖς ΕΖ ΖΔ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΒΓΑ τῇ ὑπὸ ΕΖΔ ἴση· βάσις ἄρα ἡ ΑΒ βάσει τῇ ΔΕ ἐστὶν ἴση.

ἔστωσαν δὴ πάλιν αἱ ΒΓ ΕΖ μείζονες τεταρτημορίου, καὶ ἀφῃρήσθωσαν τεταρτημόρια ΒΚ ΕΛ, καὶ διὰ τῶν Κ Α Λ Δ μέγιστοι κύκλοι γεγράφθωσαν οἱ ΚΑ ΛΔ· τεταρτημόριον ἄρα καὶ ἑκατέρα τῶν ΚΑ ΛΔ· εἰσὶν δὲ καὶ αἱ ΒΓ ΕΖ ἴσαι· καὶ λοιπαὶ ἄρα αἱ ΚΓ ΛΖ ἴσαι· δύο δὴ τρίπλευρά ἐστι τὰ ΑΚΓ ΔΛΖ μίαν γωνίαν μιᾷ ἴσην ἔχοντα, τὴν ὑπὸ ΒΓΑ τῇ ὑπὸ ΕΖΔ, περὶ δὲ τὰς ὑπὸ ΓΚΑ ΖΛΔ τὰς πλευρὰς ἴσας τὰς δὲ λοιπὰς γωνίας τὰς ὑπὸ ΚΑΓ ΛΔΖ ἅμα δυσὶν ὀρθαῖς ἀνίσους διὰ τὸ τὰς ὑπὸ ΒΑΚ ΕΔΛ δύο ὀρθὰς εἶναι καὶ τὰς ἐφεξῆς τὰς ὑπὸ ΚΑΜ ΛΔΝ· καὶ αἱ ΑΓ ΔΖ ἄρα ἴσαι εἰσὶ διὰ τὸ προδειχθέν· δύο οὖν αἱ ΒΓ ΓΑ δυσὶ ταῖς ΕΖ ΖΔ ἴσαι εἰσί· καὶ γωνία ἡ ὑπὸ ΒΓΑ τῇ ὑπὸ ΕΖΔ ἴση· βάσις ἄρα ἡ ΑΒ βάσει τῇ ΔΕ ἴση ἐστίν.

τοῦτο δὲ Μενέλαος ἀπέδειξεν ἐν τῷ α′ τῶν σφαιρικῶν :

2 αὐτὰς] ἃς **LHX** **4** μιᾷ γωνίᾳ] om. **X** | ἔχῃ] ἔχει **LX** : ἔχειν **H** **5** μὴ] s.l. **X** | καὶ τὰς — **6** ἀλλήλαις] om. **X** **6** ἀλλήλαις] ἀλλήλων **LH** **7** τρίπλευρα τὰ ΑΒΓ ΔΕΖ ἴσας] om. **X** | τοῖς] τῆς **X** **8** ΕΖ] ΕΓ **HX** **11** κοινῆς] κοινοῖς **X** | λοιπαὶ αἱ] αἱ λοιπαὶ **LHX** | δὴ] δὲ **L** **13** μέγιστος κύκλος] μ^{γ} κ^{υ} **LH** : μεγάλου κύκλου **X** **15** αἱ] ἡ **LHX** | δυσὶ] δύο **L** : β **H** | ΘΔ] ΔΘ **HX** **16** γωνία] γωνίαι **L** **17** τῇ ΕΖ] τῆς Ζ **LH** | ἡ^{2}] bis **H** **18** γωνίᾳ] om. **H** | ΖΘΕ] ΖΕΘ **LHX** | ἴσον] ἴσοι **X** **19** ἡ] om. **X** | γωνία] bis **X** **20** ἡ] s.l. **LX** | ὧν] ὡς καὶ **X** **21** ἐδείχθη] ἐδίχθη **L** | ΔΕΘ] ΔΕΖ **LHX** | τῇ] om. **X** | ΔΘΕ] ΘΔΕ **HX** **22** ΑΒ^{12}] ΑΜ **H** | τῇ^{3}] τῆς **LHX** | ἴση] ὥρα comp. **LHX** **23** τῆς ΒΑΓ] τῆς ΒΑ **H** : τῇ ΒΑ **X** | ἡ] om. **L** **24** τῇ] τῆς **LHX** **25** δειχθήσεται] δειχθεται **L** | προκείμενον] –μενων sed corr. **L** **26** ἔχῃ] ἔχει **HX** | ἑκατέρᾳ] om. **X** **27** ἔχῃ] ἔχει **X** | πλευρὰς] π^{λ} **L** **28** πλευραῖς] π^{λ} **L** **30** ΒΓΑ] ΒΑΓ ατη **L** : ΒΑΓ **HX** | ΕΖΔ] ΕΔΖ **HX** **31** καὶ τὰς] bis **X** **33** πόλοι] πολλοι **L** **34** ΓΒ] ΒΓ **X** | τεταρτημόρια] τετραγώνων **LHX** | ἐλάσσονες] ελασσον **L** **35** τεταρτημόρια^{1}] τετραγώνων **L** : om. **HX** | τεταρτημόρια^{2}] τετραγώνων **LHX** | δύο] διο **L** | ΓΑ] ΓΔ **LHX** **36** ΒΓΑ γωνίᾳ τῇ ὑπὸ] om. **X** **37** ΔΕ] δὲ ΔΕ **L** **38** δὴ] μὴ **LHX** | τεταρτημορίου] τετραγώνων **LHX** | τεταρτημορίῳ] τετράγωνον **LHX** **39** γεγράφθωσαν] γεγα– **L** | ΗΑ] ΚΑ **X** | τεταρτημόριον] τετραγων/ **LH** : τετράγωνοι **X** **41** ἀλλήλαις, ὧν] ἀλλήλων **LHX** **42** αἱ] om. **LHX** | ΗΑ] ΝΑ **H** **44** ΖΘΔ] ΖΘΑ **LHX** | πλευρὰς] π^{λ} **L** **45** δυσὶν] δύο **L** : β **H** | ἀνίσους] ἄνισα **X** **46** πλευραῖς] π^{λ} **L** | ἴσαι εἰσὶν] εἰσιν ἴσαι **X** **48** βάσις] βάσεις **L** **50** τεταρτημορίου] τετραγώνων **LHX** **51** τεταρτημόρια] τετραγώνων **LH** : τετράγωνα **X** | ΒΚ] ΑΒΚ **LHX** | μέγιστοι κύκλοι] μ^{γ} κ^{υ} **LH** : μέγα κύκλον **X** **52** τεταρτημόριον] τετραγ/ **L** : τετράγωνοι **HX** | δὲ] δὴ **LHX** **53** ἐστι] om. **LHX** **54** ΒΓΑ τῇ ὑπὸ ΕΖΔ] ΒΑΓ τῇ ὑπὸ ΕΔΖ **LHX** | ΖΛΔ τὰς — **55** ὑπὸ ΚΑΓ] om. **HX** | πλευρὰς] πλείους **L** **55** ΕΔΛ] ΕΔΖ **LHX** **56** ΛΔΝ] ΛΑΝ **LHX** | αἱ] om. **X** | ἴσαι εἰσὶ] εἰσὶν ἴσαι **X** **58** βάσει] βάσι **L** **59** Μενέλαος] Μενέλασος **LH** | α′] πρώτῳ **X**

And since ΒΗ, ΕΘ are equal to one another, of which ΒΓ, ΕΖ are equal, therefore ΓΗ, ΘΖ as remainders are equal; and ΗΑ, ΔΘ are also equal; and angle ΑΓΗ equal to ΘΖΔ because the adjacent <angles> are also equal; then, there are two trilaterals ΑΓΗ, ΔΖΘ having one angle equal to one angle (ΑΓΗ to ΔΖΘ), the sides about ΓΗΑ, ΖΘΔ equal, and the remaining <angles> ΓΑΗ, ΖΔΘ together unequal to two right <angles> because ΒΑΗ, ΕΔΘ as a whole are two right angles; therefore the remaining sides are also equal to the remaining sides, respectively; and angle ΒΓΑ is equal to ΕΖΔ; therefore base ΑΒ is equal to base ΔΕ.

Then, again, let ΒΓ, ΕΖ be greater than a quadrant and let quadrants ΒΚ ΕΛ be removed, and let great circles ΚΑ, ΛΔ be traced through Κ, Α, Λ, Δ; therefore each of ΚΑ, ΛΔ is also a quadrant; and ΒΓ, ΕΖ are equal; therefore ΚΓ, ΛΖ as remainders are also equal; then, there are two trilaterals ΑΚΓ, ΔΛΖ having one angle equal to one (ΒΓΑ to ΕΖΔ), the sides about ΓΚΑ, ΖΛΔ equal, and the remaining <angles> ΚΑΓ, ΛΔΖ together unequal to two right <angles> because ΒΑΚ, ΕΔΛ are two right angles and the adjacent <angles> ΚΑΜ, ΛΔΝ also are; therefore ΑΓ, ΔΖ are also equal because of what has been proved above; now, two <arcs> ΒΓ, ΓΑ are equal to two ΕΖ, ΖΔ; and angle ΒΓΑ is equal to ΕΖΔ; therefore base ΑΒ is equal to base ΔΕ.

Menelaus proved this in the 1st <book> of the Spherics.

2 αὐτὰς] ταύτας **E** **3** προλαμβανομένων] προσλαμβανομένων **E** **5** ἑκατέραν] ἑκατέρα **J** **6** πλευρὰς] π$^{\lambda}$ **E** **9** ἅμα] om. **E** | ἴσας] post ὀρθαῖς scr. sed eras. et hoc loco s.l. scr. **E** **10** ἐπεὶ] καὶ ἐπεὶ **JE** **11** κοινῆς — ἔστω δὴ] ἔστω **E** | λοιπαὶ αἱ] αἱ λοιπαὶ αἱ **J** **12** Α] ΒΑΓ **J** : πρὸς τῷ Α **E** **13** μέγιστος κύκλος] μεγίστου κύκλου περιφέρεια **E** | ἔτι] om. **J** **15** ἐπεὶ] καὶ ἐπεὶ **JE** | ἐστὶν] om. **J** | ΘΔ] ΔΘ **E** **16** εἰσί] εἰσίν ἑκατέρα ἑκατέρᾳ **J** | post ἴση marg. add. καὶ τὸ ΑΒΓ τρίπλευρον τῷ ΔΖΘ τρπλεύρῳ ἴσον ἐστίν **E** **17** ΖΘ ἴση ἐστίν] ΘΖ ἐστὶ ἴση **J** : ΖΘ ἐστὶ ἴση **E** **18** γωνίᾳ] om. **JE** | ἴση ἐστίν] ἐστὶν ἴση **E** **19** γωνία] om. **JE** **20** ΖΕΔ^{12}] ΔΕΖ **J** | ἐστὶν ἴση, ὧν] ἴση ἐστίν· ἐδείχθη δὲ καὶ **E** **21** ἐδείχθη ἴση· καὶ] ἴση ἐστί· **J** : ἴση· καὶ **E** | ΔΕΘ] ΔΕΖ **E** | ΔΘΕ] ΘΔΕ **E** | ἐστὶν ἴση· — **22** τῇ ΔΘ] om. **J** | πλευρὰ] π$^{\lambda}$ **E** **22** ἴση^{2}] ὑπόκειται ἴση **J** : ἴση ἐστί **E** | ἴση^{3}] ἐστὶ ἴση **J** : ἴση ἐστί **E** **23** τῆς ΒΑΓ] τῆς ὑπὸ ΒΑΓ **J** **24** ἴση ἡ ΑΒ] ἡ ΑΒ ἴση **E** | ὁμοίως] ὁμοίως δὴ **J** | ἐπιλογισμῶν] om. **J** **26** ante ἐὰν add. καὶ τοῦτο μέν ἐστι τὸ ἓν θεώρημα· ἕτερον δὲ ὅτι **E** **27** πλευρὰς] π$^{\lambda}$ **J** **28** πλευραῖς] π$^{\lambda}$ **J** **30** ΒΓΑ τῇ ὑπὸ ΕΖΔ] ΒΑΓ τῇ ὑπὸ ΕΔΖ **E** **31** πλευρὰς] π$^{\lambda}$ **J** | πλευραῖς] om. **J** **34** ΓΒ] ΒΓ **JE** | τεταρτημόρια] τετραγώνων **JE** **35** τεταρτημόρια^{12}] τετραγώνων **JE** **36** ΔΖΕ] ΕΖΔ **JE** | ἴση] ἴση ἐστί **E** **37** ΔΕ ἴση ἐστίν] ΕΔ ἐστὶ ἴση J **38** τεταρτημορίου] τετραγώνων **JE** | τεταρτημορίῳ ἴσαι] τετραγώνων **J** : τετραγώνων ἴσαι **E** **39** Δ Θ] Θ Δ **JE** | ΔΘ] ΘΔ **E** | τεταρτημόριον] τετραγώνου **JE** **40** ἑκατέρα] ἑκάτεροι **E** **41** αἱ ΒΗ ΕΘ ἀλλήλαις] ἀλλήλαις αἱ ΒΗ ΕΘ **E** | λοιπαὶ ἄρα — **42** εἰσίν] λοιπὴ ἄρα ἡ ΓΗ τῇ ΖΘ ἐστὶν ἴση **J** : καὶ λοιπαὶ ἄρα κτλ. **E** **42** ΘΖΔ] ΔΖΘ **JE** **44** τὰς πλευρὰς ἴσας] γωνίας τὰς π$^{\lambda}$ ἴσας τὰς ΓΑ ΗΑ ταῖς ΖΘ ΘΔ **J** : τὰς π$^{\lambda}$ ἴσας **E** **45** λοιπὰς] λοιπὰς γωνίας **E** **46** πλευραὶ ταῖς λοιπαῖς πλευραῖς] π$^{\lambda}$ ταῖς λοιπαῖς **J** **48** ΒΓΑ] ΒΓΑ γωνίᾳ **J** **49** ΔΕ] ΕΔ **J** | ἐστὶν ἴση] ἴση ἐστίν **E** **50** τεταρτημορίου] τετραγώνων **JE** **51** τεταρτημόρια] τετραγώνων **JE** | μέγιστοι κύκλοι γεγράφθωσαν] γεγράφθωσαν μέγιστοι κύκλοι **E** **52** τεταρτημόριον — ΛΔ] om. **E** | τεταρτημόριον] τετραγώνου **J** | ΚΑ ΛΔ] ΑΚ ΔΛ **J** | δὲ] δὴ **J** **54** ΒΓΑ — **55** γωνίας τὰς] (lac. 3 litt.) τῇ ὑπὸ (lac. 3 litt.) τὰς δὲ **E** | πλευρὰς] π$^{\lambda}$ **J** **56** ΛΔΝ] ΛΔΝ καὶ αὐτὰς ὁμοίως ὀρθὰς **J** **57** εἰσί] εἰσὶ ἑκατέρα ἑκατέρᾳ **J** **58** ἴση ἐστίν] ἐστὶ ἴση **J**

Diagrams

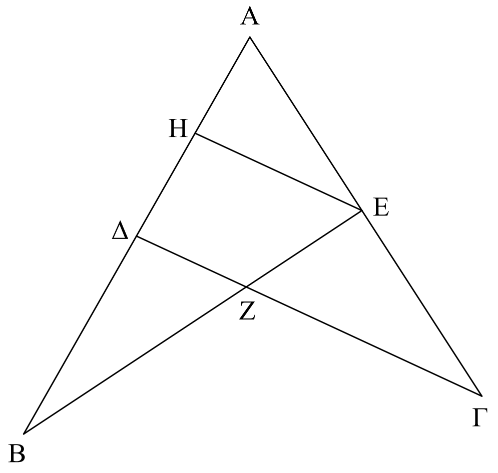


Fig. 1. First rectilinear lemma

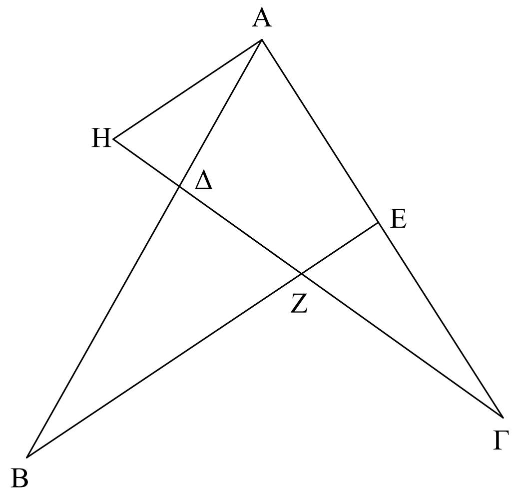


Fig. 2. Second rectilinear lemma

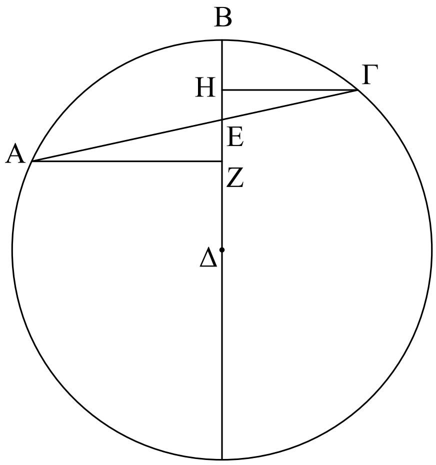


Fig. 3. First cyclic lemma

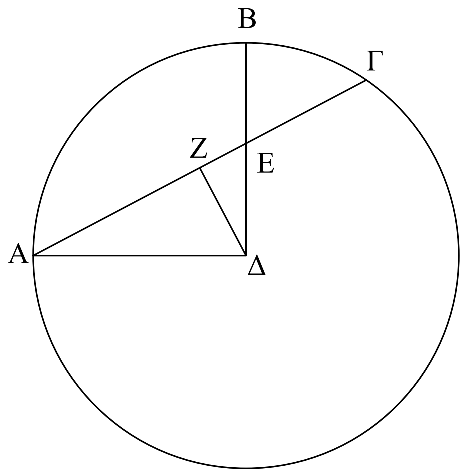


Fig. 4. Second cyclic lemma

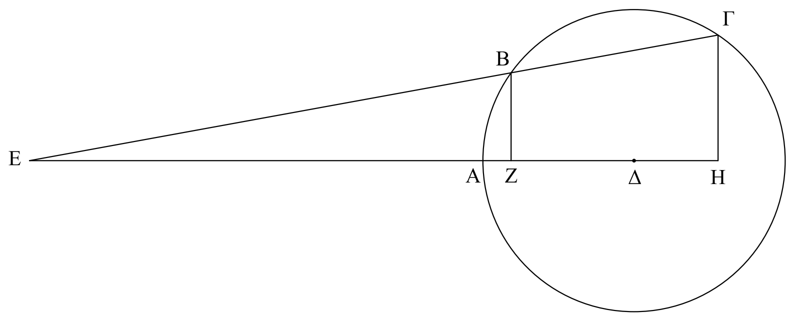


Fig. 5. Third cyclic lemma

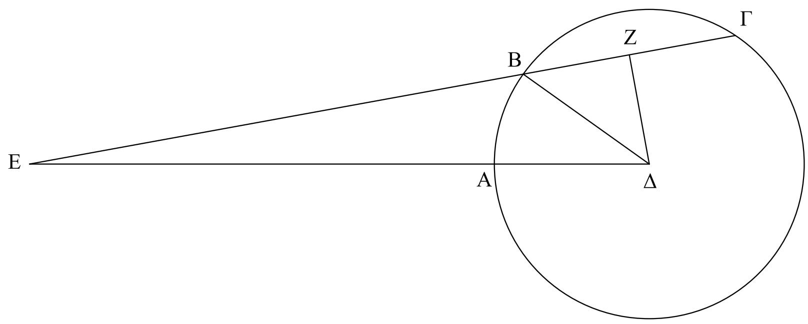


Fig. 6. Fourth cyclic lemma

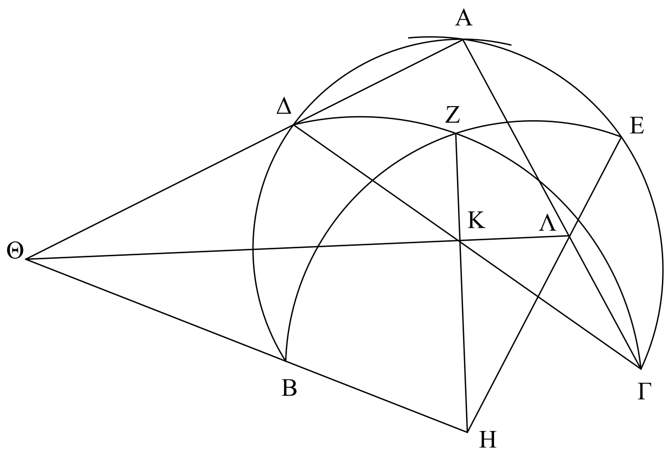


Fig. 7. The Sector Theorem

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