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Computation of Sines in Nityānanda's *Sarvasiddhāntarāja*

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Abstract

The *Sarvasiddhāntarāja* (1639) of Nityānanda is a monumental treatise that provides a comprehensive treatment of various aspects of astronomy. Since trigonometry was vital for professional astronomers, Nityānanda gives a systematic and detailed account of this topic in over sixty-five verses in one of the early chapters in this treatise. As is the case with much of the *Sarvasiddhāntarāja*, Nityānanda introduces certain novel features which are not found in prior treatments, some of which are his own insights, others of which were inspired by Arabic sources. We present a critically edited text of part of the trigonometry section on the basis of six manuscripts, as well as a translation and commentary of the technical content.

I Introduction

In 1639, Nityānanda, astronomer at the Mughal court of Shāh Jahān in Delhi, wrote his monumental *Sarvasiddhāntarāja* ('King of all *siddhāntas*'). This work was composed in the typical *siddhānta* format: two main sections—the *gaṇita* and *gola*—covering various topics relating to planetary computations (see Table 1). It has been observed that the work is notable for its incorporation of many Islamic parameters and concepts (Misra 2016; Pingree 1978, 1996, 2003). However, the full extent of this adoption is yet to come to light as more chapters of this text remain to be examined. Indeed, by the time of the composition of this treatise, Nityānanda was no stranger to the Islamic astronomical tradition, as around a decade earlier he had produced a translation of the massive set of astronomical tables of Ulugh Beg (ca. 1440) via the Persian intermediary, the *Zīj-i Shāh Jahānī*. Accordingly, his work is critical in tracing the transmission of Arabic astronomy and mathematics into Sanskrit sources.

Like most *siddhāntas*, Nityānanda devotes a large portion of his text (I 3, 19–85) to a systematic exposition of trigonometry. This appears in the chapter which concerns the determination of the true positions and velocities of the planets, as the orbital corrections involved in determining these require computing Sines and related trigonometric relations. In this portion, Nityānanda commences with a brief description of the method of reckoning the different values of Sines. He then moves on to discuss at length certain relations among the Sines, Cosines, and Versines of various arcs, including the rules for determining the Sines of the sums and differences of two arcs in five sections (*prakāra*) (I 3, 19–59). A notable feature of his exposition is the justification of the results by means of diagrammatic representation. Following this, Nityānanda introduces a sixth section (I 3, 60–85) in which a recursive algorithm, accompanying geometrical demonstrations, an algebraic derivation, and a worked example (*udāharaṇa*) are set out to determine the Sine of 1° and from that, the Sine of 0;01°. This section, albeit in verse, parallels in a few aspects the account given by his Islamic predecessor al-Kāshī,[1] although ingenious modifications have been made by Nityānanda to account for the fact that he uses Sines and not Chords.

Part	Names of the different chapters	
I *gaṇita*	1. *mīmāṃsā*	(Philosophical Rationales)
	2. *madhyama*	(Mean Positions)
	3. *spaṣṭa*	(True Positions)
	4. *tripraśna*	(Three Questions: Direction, Place, Time)
	5. *candragrahaṇa*	(Lunar Eclipses)
	6. *sūryagrahaṇa*	(Solar Eclipses)
	7. *śṛṅgonnati*	(Elevation of the Lunar Cusps)
	8. *bhagrahayuti*	(Conjunction of the Stars and Planets)
	9. *bhagrahāṇām unnatāṃśa*	(Measure of Altitudes of the Stars and Planets)
II *gola*	1. *bhuvanakośa*	(Cosmography)
	2. *golabandha*	(Armillary Sphere)
	3. *yantra*	(Astronomical Instruments)

Table 1: The contents of the *Sarvasiddhāntarāja*.

The current study examines the first five sections in detail, offering a critically edited text based on six available manuscripts, a literal English translation, and technical analysis of the contents of all the verses constituting these sections. An examination of the sixth section (I 3, 60-85) is in preparation and will appear in a forthcoming publication.

Our study of these sections of the *Sarvasiddhāntarāja* has revealed many fascinating features in this text. These include the use of the measure sixty as the

[1] See Aaboe (1954) and Rosenfeld and Hogendijk (2003).

Radius of the circle, the articulation of new trigonometric elements including the Coversine and the 'arc-hypotenuse' (*cāpakarṇa*), the juxtaposition of both algebraic as well as geometric techniques to demonstrate the validity of a result, and detailed instructions for ruler-and-compass type constructions to establish the equivalence of key line segments. Though some of these innovations present in the text have been inspired by Arabic and Persian sources, the resulting overall treatment of the topic made by Nityānanda clearly mirrors his extraordinary skills in weaving these with his own insightful findings within the framework of a traditional *siddhānta*.

II Critical Edition of the Verses

II.1 A Description of the Sources

The critical edition of the text presented in the following pages is based on six manuscripts of the *Sarvasiddhāntarāja*.[2] The manuscripts and the library or repository that currently holds them, along with the sigla we assigned them are presented in Table 2.[3]

Siglum	Description of the Manuscript
B_1	Sarasvati Bhavana Library, Benares, (1963) 35741
B_2	Sarasvati Bhavana Library, Benares, (1963) 37079
B_o	Bhandarkar Oriental Research Institute 206 of A, 1883–1884
N	National Archives Nepal, Kathmandu, Microfilm Reel No. B $\frac{354}{15}$
R	Rajasthan Oriental Research Institute (Alwar), 2619
W	Wellcome Institute for the History of Medicine, Y550

Table 2: Manuscripts of the *Sarvasiddhāntarāja*, their locations, and the sigla.

In what follows, we provide a brief description of the manuscripts listed in Table 2. For a comprehensive description of five of them, the reader is referred to Misra (2016, 45–100).

Manuscript B_1 This was copied in *Saṃvat* 1804 (= 1747 CE). It is comprised of 84 folia; the trigonometry passage is found between ff. 14r–20r. It is written in a tidy *Devanāgarī* hand, filling roughly 11-12 lines per page. The scribe has left space for the diagrams but has not filled them in. Part of f. 15v and 16r are left blank, presumably for the Sine table, as well as 18v-19r. This manuscript contained the most variant readings. In particular, this scribe had

[2] Many more copies exist. See Pingree (1970–1994, A3 173–4, A4 141, A5 184) which lists 16 altogether.

[3] The following concordance should be observed: the present B_1, B_2, B_o, N, R are Misra's Bn. I, Bn. II, Br. I, Np. I, Rr. I.

trouble correctly copying the *akṣara*'s that corresponded to lettered points on the diagram, often rendering them wrong. Of note is his repeated confusion in this context between the two symbols for *ṅa* and *ḍa*.

Manuscript B_2 This was copied in *Saṃvat* 1895 (= 1838 CE) and *Saṃvat* 1936 (= 1879 CE). It is comprised of 85 folia; the trigonometry passage is found between ff. 13v–18r. The scribe has a neat hand with rather vertical *Devanāgarī* and the manuscript is oriented in the book-layout. As is the case in B_1, spaces are left for the diagrams which have not been filled in.

Manuscript B_o This was copied in *Saṃvat* 1941 (= 1884 CE). It is comprised of 47 folia; the trigonometry passage is found between ff. 9r–11v. There are around 14 lines per page and the hand is a rough *Devanāgarī*. There are no diagrams nor spaces left by the scribe for them.

Manuscript N This contains no indication of the date it was copied. It is comprised of 96 folia; the trigonometry passage is found between ff. 15r–21v. There are around 9 lines per page in a compact *Devanāgarī* hand. This manuscript includes a Sine table with 90 entries (ff. 15r–15v) and a Sine of first differences table with 90 entries. The manuscript also includes roughly drawn diagrams some which are incomplete or badly smudged.

Manuscript R This was copied on Thursday 10 *śuklapakṣa* of Kārttika in *Saṃvat* 1903 (29 October 1846). It is comprised of 60 folia; the trigonometry passage is found between ff. 11r–14v. It is written in a clear broad hand, with around 14 lines per page. This manuscript is the most reliable one and includes many diagrams (see Figures 2, 6, 8, 10) most of which are well executed.

Manuscript W This contains no indication of the date it was copied. It is comprised of 67 folia; the trigonometry passage is found between ff. 18v–27r. It is written in a loose *Devanāgarī* hand, filling roughly 10 lines per page. The scribe has included spaces for the diagram. Many are filled in; a few have been left blank. Even those diagrams that have been drawn have been executed roughly and some are incomplete. This manuscript also includes a Sine table with 90 values, found on ff. 18v–19r.

II.2 Conventions Adopted

In preparing our critical edition, we have adopted the following conventions:

- Square brackets [ ] represent an addition to the translation for sense and meaning. Round brackets ( ) indicate an editorial gloss to clarify the text.
- Common orthographical variants have been emended silently and have not been recorded in the critical apparatus. These include misplaced *daṇḍas*, omitted *visarga, virāma*, or *avagraha*, *anusvāra* for a conjoined nasal, doubled consonants such as *ācāryya* or *arddha*, frequent confusion over *ṅa* and *ḍa*, and *ṭha* and *ṭa*.
- The use of letter labels compounded onto existing Sanskrit words have created some novel Sandhi resolutions. For instance, we have emended *anyaṅ ṅakāra* to *anyan ṅakāra*.
- We have not critically edited the diagrams, leaving that to a future dedicated study.
- All manuscripts had issues with the numeration of verses but as they are trivial we have not recorded them.

अथ ज्याकथनम्।

आचार्यवर्या गणकास्त एव जानन्ति ये ज्यानयनोपपत्तिम् ।
ततोऽधिगन्तुं पदवीं च तेषां महाजडो वाञ्छति मादृशोऽपि ॥ १९ ॥

षष्ट्यङ्गुलव्यासदलेन वृत्तं दिगङ्कितं भूमितले विधाय ।
प्रत्येकपादे नवतिं लवानां विभज्य तुल्यां विलिखेन्मनीषी ॥ २० ॥

पूर्वं परं वा यदिहास्ति चिह्नं प्रकल्प्य तल्लक्षकनामधेयम्।
अभीष्टचापस्य तु कोटियुग्मे रज्जुर्निबद्धा खलु सैव जीवा ॥ २१ ॥

ज्यां ज्यार्धमेव प्रवदन्ति तज्ज्ञाः व्यासार्धमेव त्रिभसिञ्जिनीञ्च ।
यतो ज्यकार्धाग्रगतो ग्रहेन्द्रः तिर्यक्स्थितो मध्यमसूत्रतः स्यात् ॥ २२ ॥

अथोच्यते ज्यानयनोपपत्तिः पञ्चप्रकारैर्गणितस्य तावत् ।
त्रित्र्यंशकोत्थानि परिस्फुटानि त्रिंशज्ज्यकार्धानि यतो भवन्ति ॥ २३ ॥

अथाद्यप्रकारे नवतित्रिंशदष्टादशभागादीनां जीवाज्ञानम् ।

नवत्यंशजीवा भवेद्व्यासखण्डं तदर्धं खरामांशजीवा निरुक्ता ।
तदर्धोनितं विस्तृतेरङ्घ्रिवर्गात् सपादात् पदं ज्याष्टचन्द्रांशकानाम् ॥ २४ ॥

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2 जानन्ति] जानन्त B_2, R
2 -नोपपत्तिम्] -नोपत्तिम् B_1
3 तेषां महाजडो] तेषामहो जडो N, W; तेषामहाजडो B_o
4 षष्ट्यङ्गुल-] षष्ट्याङ्गल- B_o
5 प्रत्येकपादे] प्राप्तंकपादे B_1
5 नवतिं ल-] नवतिर्ले-N; नवतीं ल-B_1
5 तुल्यां विलिखेन्म-] तुल्या विलिविन्म- B_2
6 पूर्वं परं वा] पूर्वपदं वा B_o; पूर्वं परन्वा N
6 -हास्ति] -हस्त B_1
6 तल्लक्षकनामधेयम्] तल्लक्ष्यक- B_1, N; -धेयाम् B_o
7 खलु] खल B_2
7 सैव] -सैक B_1
8 ज्यां] ज्या N
8 त्रिभसिञ्जिनीञ्च] -त्रिभसिञ्जिनीह N; त्रिभिसोजिनीं B_2, R
9 यतो] ततो N
9 ज्यकार्धाग्रगतो] ज्यकार्धाग्रतो B_2, R
10 -नयनोपपत्तिः] -नटानोपपत्तिः W; -नयनोपपपतिः B_2
10 गणितस्य] गसितस्य B_o
11 त्रित्र्यंशकोत्था-] त्रित्र्यं कोत्था- B_1
11 परिस्फुटानि] परिस्फुटानिः B_o
11 त्रिंशज्ज्यका-] त्रिंशज्यका- B_1, N
11 यतो भ-] तयोर्भ-B_1
12 - त्रिंशदष्टादशभागादीनां] - त्रिंशष्टादश- B_1; -भागानां B_1
12 -ज्ञानम्] -जानम् B_o
13 -व्यासखडं] -द्यासस्वडं B_2, R
13 निरुक्ता] प्रदिष्टा B_1
14 तदर्धो - कानाम्] विस्तृतें रङ्घ्रि- B_2, B_o, R; तदर्धोनयुक् व्यासपदोत्थवर्गात् सपादज्याद्दतेश्च त्रिनिघ्नः B_1

अत्रोपपत्तिः ।

पूर्वाङ्ककेन्द्रोपगसूत्रमध्ये कृत्वार्धवृत्तस्य च केन्द्रचिह्नम् ।
ततो नयेद्दक्षिणगामिसूत्रं कर्णानुकारं प्रथमं मनीषी ॥ २५ ॥

एकत्रिभज्ये भुजकोटिरूपे त्र्यस्राभिधे स्पष्टतरे च विन्द्यात् ।
कर्णः स्पृशेद्यत्र कृतार्धवृत्ते तद्याम्यचिह्नान्तरगं प्रमाणम् ॥ २६ ॥

षड्वह्निभागोन्मितचापकर्णः स्यात्तस्य खण्डं धृतिभागजीवा ।
सिद्ध्यन्त्यतो ज्या रसरामभागद्रिशैलभागाब्धिशरांशजाद्याः ॥ २७ ॥

त्रिज्यावर्गार्धतो मूलं सार्धराशिज्यका भवेत् ।
अत्र सार्धभजे जीवे दोःकोटी त्रिज्यका श्रुतिः ॥ २८ ॥

दोर्ज्यावर्गोनितात् त्रिज्यावर्गान्मूलञ्च कोटिजा ।
ज्या स्यात् कोटिज्यकावर्गहीनान्मूलं च दोर्ज्यका ॥ २९ ॥

अत्र त्र्यस्रे भुजो दोर्ज्या कोटिः कोटिज्यका तथा ।
कर्णस्त्रिज्या ततः साध्ये कोटिदोर्ज्ये परस्परम् ॥ ३० ॥

अथ द्वितीयप्रकारे चापज्याज्ञाने सति चापार्धज्याज्ञानं तदुपयोगित्वात्तावदुत्क्रमज्यानयनम् ।

कोटिज्यका व्यासदलाद्विशोध्या बाहूत्क्रमज्या विशिखाभिधा स्यात् ।
चेद्व्यासखण्डात् पतिता भुजज्या कोट्युत्क्रमज्या प्रभवेत् तदानीम् ॥ ३१ ॥

भुजोत्क्रमज्याकृतिसंयुतायाः दोर्ज्याकृतेर्मूलदलञ्च यत् स्यात् ।
तदर्धचापस्य भुजज्यका सा कोटिज्यकातोऽपि तथैव कार्या ॥ ३२ ॥

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16 -केन्द्रोपग-] -केन्द्रोपपग- B_1
17 नयेद्दक्षिण-] नये दक्षिण- W
18 -त्रिभज्ये भुजकोटि-] -त्रिभज्यजकोटि- B_1
18 त्र्यस्राभिधे] त्र्यस्त्रभि- B_1; त्र्यस्र अभिधे R, B_2
19 स्पृशेद्यत्र] स्पष्टतरेत्र B_1
19 कृतार्ध-] कृतावार्ध- B_o
19 -चिह्नान्तरगं] -चिह्नातरगं B_o
20 -तस्य खण्डं] -तत्र खण्डं W
20 -भागजीवा] -भागवीजा B_2
23 सार्धभजे] सार्धभवे N
23 दोःकोटी] दोः- ... कोटिजा (in next verse) om. R
25 च] om. B_1
26 त्र्यस्रे] त्र्यसे B_1; त्र्यस्रें B_o
27 -त्रिज्या] -त्रीज्या B_2
28 चापज्याज्ञाने] चाज्याज्ञाने B_1
28 -तावदुत्क्रमज्या-] -तावत् क्रमज्या- W; -तदुत्क्रमज्या- B_1
29 कोटिज्यका] कोटिज्याका B_1
29 विशिखाभिधा] विशिखाबिधा R
30 कोट्युत्क्रमज्या] कोकोट्युत्क्रमज्या B_1
30 तदानीम्] तदानी B_2
31 -संयुतायाः] -संघुतायाः B_o
31 दोर्ज्याकृतेर्मूलदलञ्च] दोकृतेमूल- B_1; -मूलदलं चं B_o
32 -चापस्य] -चापस्या W
32 सा] शा B_2
32 कार्या] साध्या B_1

अत्र त्र्यस्रे व्यस्तजीवास्ति कोटिः बाहुज्या दोश्चापकर्णोऽत्र कर्णः ।
यत्कर्णार्धं सैव चापार्धजीवा चापक्षेत्रे सर्वमेतद्विलोक्यम् ॥ ३३ ॥

अथवा

मूलञ्च यद्व्यासशरावघातात् तदर्धकं वा धनुरर्धजीवा ।
अव्यक्तबाणोऽत्र च तेन हीना त्रिज्या भवेत् सा निजकोटिजीवा ॥ ३४ ॥

तदीयवर्गस्त्रिगुणस्य वर्गात् च्युतो भुजज्याकृतिरेव सा स्यात् ।
शरस्य वर्गेण युता पुनः सा कोदण्डकर्णस्य कृतिर्विचिन्त्या ॥ ३५ ॥

एवं शरव्यासहतिः प्रपन्ना कोदण्डकर्णस्य तु वर्गतुल्या ।
कोदण्डकर्णस्य कृतेः पदं यत् तदर्धकं चापदलस्य जीवा ॥ ३६ ॥

अथ तृतीयप्रकारे चापस्य ज्याज्ञाने सति द्विघ्नचापदोर्ज्याज्ञानम् ।

दोर्ज्यावर्गश्चैकभज्याविभक्तो लब्धं व्यासात् पातयेच्छेषमित्या ।
लब्धं हत्वा तस्य मूलं यदाप्तं जीवा स्पष्टा द्विघ्नचापस्य सा स्यात् ॥ ३७ ॥

अत्र द्विनिघ्नीष्टधनुर्भुजज्या स्याद्विघ्नकोदण्डजचापकर्णः ।
चेच्चापकर्णस्य कृतिर्विभक्ता व्यासप्रमाणेन शरस्तदा स्यात् ॥ ३८ ॥

बाणोनितं व्यासदलञ्च यत् स्यात् सा द्विघ्नकोदण्डजकोटिजीवा ।
तद्वर्गहीनस्त्रिगुणस्य वर्गो दोर्ज्याकृतिर्द्विघ्नधनुर्भवा स्यात् ॥ ३९ ॥

अव्यक्तमार्गेण सुधीर्विदध्यात् एवं यथा बाणविवर्जितः स्यात् ।

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33 त्र्यस्रे] त्र्यस्त्रे B_1
34 सैव] सेव B_2
34 चापार्ध-] चापाद्र- B_1
37 सा निजकोटिजीवा] स्यान्निज- B_1; -कोटिवीजा R, B_2
39 वर्गेण] वर्गेणा B_1
39 कोदण्डकर्णस्य] कोदकर्णस्य B_o
40 तु] All manuscripts have च except B_1
40 वर्गतुल्या] वर्गतुल्याः B_2, R
42 -प्रकारे] -प्रकारेण B_1
42 द्विघ्न-] द्विघ्ना- R
43 दोर्ज्या-] दोज्या- B_1
43 लब्धं व्यासात्] लब्दं व्यासात् W; लब्धव्यासात् N
44 लब्धं] लब्दं W
44 -द्विघ्न-] -द्विघ्ना- B_2, R
44 सा] om. B_o
45 -भुजज्या] -भुज्या B_2, R
45 स्याद्विघ्नकोदण्डज-] स्याद्विघ्न- B_2, R; -चापकार्णः ... -कोटिजीवा (in next verse) om. N
46 कृतिर्विभक्ता] कृति द्विघ्नको विभक्ता B_1
46 शरस्तदा] शरस्वदा B_1
47 बाणोनितं] बाणोनित्यं B_2
47 स्यात्] स्यात् स्यात् B_1, B_2
47 सा] om. R
48 तद्वर्गहीनस्त्रि-] तद्वर्गहीनास्त्रि- B_1
48 दोर्ज्याकृतिर्द्विघ्न-] दोर्ज्याकृतेद्विघ्न- B_2; दोज्याकृतिर्द्विघ्न- B_1; दोर्ज्याकृतिर्द्विघ्न- B_o
49 -मार्गेण] -वर्गेण B_1
49 -विदध्यात्] -विदधात् B_1
49 स्यात्] स्या B_2

व्यासः पुनर्बाणगुणः समानः प्रजायते बाहुगुणस्य कृत्या ॥ ४० ॥

अथ चतुर्थप्रकारे धनुर्द्वयस्य भुजज्ययोर्ज्ञाने सति धनुर्योगज्याज्ञानम् ।

अन्योन्यकोटिमौर्व्या गुणिते ये चेष्टचापयोर्दोर्ज्ये।
त्रिज्योद्धृते तयोर्यः योगः सा चापयोगज्या ॥ ४१ ॥

अत्रोपपत्तिः ककारादिवर्णोपलक्षितरेखाभिरत्र क्षेत्रव्यवस्था दर्शनीया ।

कखागघं भूमितलेषु मण्डलं ङकेन्द्रकं कर्कटकेन साधयेत् ।
कचं चछं चापयुगं कङं चङं छङं क्रमाद्व्यासदलत्रयं लिखेत् ॥ ४२ ॥

चजं छझं चापयुगस्य दोर्ज्यके कङे चङे लम्बवदेव पातिते ।
कछस्य चापद्वययोगसंमितेः ज्यका कङे लम्बवदेव पातिता ॥ ४३ ॥

छटाज्झपातं चङयोगचिह्ने लिखेट्टकारं गणकप्रवीणः ।
सूत्रं झटाख्यं विलिखेच्चजेन तुल्यप्रमाणं खलु शिल्पसिद्धम् ॥ ४४ ॥

छटं धरा झाटझछे भुजौ द्वौ महात्रिकोणे झतसंज्ञलम्बः ।
त्र्यस्रत्रयं तत्र समीक्ष्यमाणं वर्णाङ्कितं तच्छ्रवणादि चिन्त्यम् ॥ ४५ ॥

त्र्यस्रं झटातं छङझस्वरूपं छझातमन्यन्ङचजोपमानम् ।
तृतीयमन्यत् झठतं तथैव ततोऽनुपातः परिकल्पनीयः ॥ ४६ ॥

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50 बाहुगुणस्य] बाहुर्गुणस्य W
50 कृत्या] कृत B_1; कृत्याः B_2
52 -कोटिमौर्व्या गुणिते] -कोटेमोर्व्यार्गुणिते B_2
52 -चापयोर्दोर्ज्ये] -चापदोर्योर्ज्ये B_1; -चापदोर्ज्यै B_2
53 तयोर्यः योगः] तयोर्योगः R
54 -भिरत्रक्षेत्र-] -भिरत्रक्षैत्र- B_2, R; -भिरक्षक्षेत्र- B_1
54 दर्शनीया] दर्शनीयाः N
55 -तलेषु] -तलेसु B_1, B_o
56 कङं चङं छङं] कड चड छड B_1; कङं छङं W
57 चजं छझं] चजं छंझं B_2, R; चजं छस्लं B_o
57 दोर्ज्यके] दोज्येके B_2; दोज्यके R
57 कङे] कङें B_2
58 कछस्य] कचछ B_1: छछस्य B_o
58 कङे] कने N
58 लम्बवदेव] लेबदेव B_1
59 झपातं चङयोग-] झपातत्रतच्चङयोग- W; झपातच्चङयोगा- B_2, R
59 टकारं गणकप्रवीणः] -गणस्मकप्रवीणः W; -गणकप्रवीवीणः B_o; टकारप्रवीणः B_1
60 -चजेन] -चयोजेन W; -चजेयेन B_o
61 छटं धरा झाटझछे] सिछटं धरा झाटंछे W; -झाटछे B_o; छटं धस- B_1
61 झतसंज्ञलम्बः] झतसंज्ञसंवं B_1
62 चिन्त्यम्] चिन्त्यंतयं B_1
63 त्र्यस्रं … ङचजो]

B_1
64 झठतं] All the manuscripts give झटतं except N, W
64 परिकल्पनीयः] परिकल्पनीय B_1, B_2

कर्णे छङे चेत् झङतुल्यकोटिः झटश्रुतौ कास्ति तटन्तदानीम् ।
चङे श्रुतौ चेज्झङतुल्यकोटिः कर्णे छझे कास्ति तदा छतं सा ॥ ४७ ॥

या कोटियुग्मस्य युतिस्तु सैव ज्या चापयोगस्य छटाभिधाज्या ।
चापैक्यखण्डस्य तु सिञ्जिनी या तत्रोपपत्तिर्गदिता पुरैव ॥ ४८ ॥

अथ पञ्चमप्रकारे धनुर्द्वयस्य भुजज्ययोर्ज्ञाने सति चापान्तरज्याज्ञानम् ।

अन्योन्यकोटिमौर्व्या गुणिते ये चेष्टचापयोर्दोर्ज्ये ।
त्रिज्योद्धृते तयोर्या वियुतिः सा चापविवरज्या ॥ ४९ ॥

अत्रोपपत्तिः ।

बृहद्धनुः कंछमितं छचं लघु छटं महाचापगुणं चझं लघोः ।
प्रकल्प्य तद्धन्मतिमांश्चजं लिखेत् धनुर्द्वयान्तर्गतचापसिञ्जिनीम् ॥ ५० ॥

छटं सदा जाझमितं विचिन्तयेत् झथं परं लम्बकमानयेत् कङे ।
जथाझसंज्ञं त्रिभुजं यथा तथा झचाङसंज्ञं परिचिन्तयेद्बुधः ॥ ५१ ॥

कर्णे चङे यदि झङप्रमितास्ति कोटिः झाजश्रुतौ भवति कोटिरियं तदा किम् ।
एवं भवेज्झथमितं किल सूत्रकं हि त्रैराशिकेन च वदामि झतप्रमाणम् ॥ ५२ ॥

त्र्यस्रं झचातं छटङोपमानं किं वा झथाङप्रमितं विचिन्त्यम् ।
छङश्रुतौ चेट्टङकोटिमानं चझश्रुतौ कास्ति झतं तदानीम् ॥ ५३ ॥

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65 छङे चेत्] चडे झङ B_1
65 झटश्रुतौ] झवश्रुतौ B_1
66 चेज्झङ-] चेझज- B_1
66 कर्णे] कणे B_1
66 छझे] झजे N
67 छटाभिधाज्या] छटाभिधासा B_1
68 -खण्डस्य तु सिञ्जिनी] -खण्डस्य तु सिजिनी W; -खण्डस्व च सिञ्जिना B_1; -खण्डस्य सिञ्जितानी N
68 पुरैव] पुरैवः B_2
70 अन्योन्यकोटिमौर्व्या] अन्यस्य कोटि- N; -मौर्ध्या B_1
71 -विवरज्या] -विरज्या R
73 कंछ-] All manuscripts except B_1 and N have कछ। Since we need a long syllable (गुरु) for metrical requirements we have chosen कंछ।
73 छचं लघु] छचं लघुंघु B_2; लछ लघु B_o; छलं चघु N
74 -सिञ्जिनीम्] -सिञ्जिनि B_2
75 जाझमितं] जाामितं B_2
75 विचिन्तयेत्] विचिन्तयेझ B_2
75 झथं] झटं N
75 लम्बक-] लक- B_1
76 परिचिन्तयेद्बुधः] परिचिन्तयेद्बध B_1
77 -प्रमितास्ति] -प्रमिता च B_1
77 झाजश्रुतौ] झचश्रुतो B_1
78 किल] किविल B_1
78 झतप्रमाणम्] झतमाणम् W
79 छट-] छत- B_1
79 किं] कि B_2
79 -प्रमितं] -प्रमित B_1; -प्रतितं R
80 छङश्रुतौ चेट्टङकोटिमानं] चाश्रुतौ- R; -चेच्छङकोटिमानं N; छङकोटिमानं W
80 कास्ति] कास्त B_1

झतोनितं चेज्झथसंज्ञसूत्रं तदा भवेच्चाजसमं सदैव ।
एषैव चापान्तरमानजीवा पूर्वैर्निरुक्ता गणकप्रवीणैः ॥ ५४ ॥

अथ प्रसङ्गात् किञ्चिच्छिल्पकथनम् ।

कर्कटकेन जटाभ्यां तिमिना तुल्यत्रिबाहुकं कृत्वा ।
बहिरिह कोणे धृत्वा गं गाझटदिक्प्रसारिते रेखे ॥ ५५ ॥

जचव्यासार्धमानेन वृत्तं यत्र स्पृशेत् क्वचित् ।
रेखां प्रसारितां तत्र डकारं विलिखेद्बुधः ॥ ५६ ॥

गडविस्तारखण्डेन मण्डलं साधयेत् परम् ।
तच्चापि संस्पृशेद्यत्र रेखामन्यां प्रसारिताम् ॥ ५७ ॥

तत्र विन्यस्य दं वर्णं टदव्यासार्धजं पुनः ।
वृत्तं कुर्यात् ततष्टादं टाझं जाचं जडं समम् ॥ ५८ ॥

विवरज्यानयनार्थं टछरेखाख्यां जचस्थाने ।
कृत्वा टाडं जादं कुर्यादपरं पुरोक्त्यैव ॥ ५९ ॥

अत्रापि चत्वारि सूत्राणि जदं जझं टछं टडं तुल्यानि ज्ञेयानि ।

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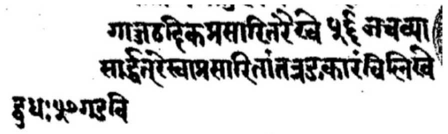
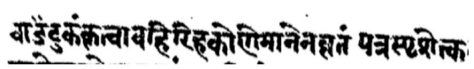
81 चेज्झथसंज्ञसूत्रं] चेज्झतसंज्ञसूत्रं W; चैकथसंसूत्रं B_1
82 चापान्तरमान-] चार्पतितरमान- B_1
82 -प्रवीणैः] -प्रवीणै B_2
84 तुल्य ... धृत्वा] तुल्यत्रिमानेन वृतं धृत्वा B_2; तुल्यं त्रि- B_1; तुल्यत्रिबाहुकं हि हरिकोणे N
85 गाझटदिक्प्रसारिते] गाझटदिकासरिते W; गाझटदिकप्रसारिते B_2, R गाजटदिक्प्रसारिते N गाझटदिक्प्रसारिते B_1
86 जचव्यासार्धमानेन वृत्तं यत्र स्पृशेत् क्वचित्]



B_2
87 रेखां] रेखा R
88 गडविस्तारखण्डेन]



B_2
88 परम्] पराम् B_1
89 तच्चापि] च्चापि R; तत्रापि B_1
90 दं] om. W; दंः B_2, B_o
90 -व्यासार्धजं] -व्यासार्द्धतं B_2
91 ततष्टादं] तत्तष्टादं B_2, R
91 टाझं] नझं B_1
92 टछरेखा-] टच्छरेखा- B_2, R
93 टाडं] टांदं B_1; All have टादं except B_o
94 जदं] जडं B_1; जदंगे W
94 जझं] झदं N
94 टडं] टङं N

III Text, Translation, and Technical Analysis

In the following sections, we present Nityānanda's text in *Devanāgarī*, with accompanying transliteration and English translation, verse by verse. The meter of each verse has been identified and noted on the right hand side after each verse, both in *Devanāgarī* and in transliteration.

At the end of each section, we offer a thorough analysis of the technical contents of the verses, often with recourse to modern symbolic styles of reasoning to help the reader follow the mathematics expressed in these verses. We have also included diagrams as well as original images from the manuscripts to show the ways in which the diagrams were executed by the scribes. In addition, where appropriate, we have inserted our own diagrams to further elucidate the contents of the text.

The structure of this excerpt of chapter three of the *Sarvasiddhāntarāja* is as follows:

verses 19–23		Preamble and definitions
verses 24–30	Section 1	The Sines of ninety, thirty, eighteen degrees
verses 31–36	Section 2	The Sine of half the arc
verses 37–40	Section 3	The Sine of double the arc
verses 41-48	Section 4	The Sine of the sum of two arcs
verses 49–54	Section 5	The Sine of the difference of two arcs
verses 55–59		Demonstration of equivalences by geometrical construction (*śilpa*)

Table 3: The structure and contents of the five sections on trigonometry.

Nityānanda introduces each section with a short statement in prose which describes the contents of the upcoming verses. These statements have a distinct structure: they announce the section number, indicate the relation that is given (using the locative absolute), and specify the relation to be derived.

III.1 Preamble and Definitions

Text and translation

अथ ज्याकथनम् ।

atha jyākathanam |

Now, the description of the Sines.

आचार्यवर्या गणकास्त एव
जानन्ति ये ज्यानयनोपपत्तिम् ।
ततोऽधिगन्तुं पदवीं च तेषां
महाजडो वाञ्छति मादृशोऽपि ॥ १९ ॥ ॥ रामा उपजातिका ॥

ācāryavaryā gaṇakās ta eva
jānanti ye jyānayanopapattim |
tato 'dhigantuṃ padavīṃ ca teṣāṃ
mahājaḍo vāñchati mādṛśo 'pi ||19|| || *rāmā upajātikā* ||

Oh revered teachers! Only those are considered mathematicians who know the rationale behind the computation of Sines. Therefore, [by venturing to explain that,] even a dullard like me wishes to accomplish their status.

षष्ट्यङ्गुलव्यासदलेन वृत्तं
दिगङ्कितं भूमितले विधाय ।
प्रत्येकपादे नवतिं लवानां
विभज्य तुल्यां विलिखेन्मनीषी ॥ २० ॥ ॥ भद्रा उपजातिका ॥

ṣaṣṭyaṅgulavyāsadalena vṛttaṃ
digaṅkitaṃ bhūmitale vidhāya |
pratyekapāde navatiṃ lavānāṃ
vibhajya tulyāṃ vilikhen manīṣī ||20|| || *bhadrā upajātikā* ||

Having drawn a circle on a [flat] surface of the earth with radius 60 *aṅgulas* [and] marked with directions, in each of the [four] quadrants, having divided them into equal [parts], may the intelligent one mark ninety degrees.

पूर्वं परं वा यदिहास्ति चिह्नं प्रकल्प्य तल्लक्षकनामधेयम् ।
अभीष्टचापस्य तु कोटियुग्मे रज्जुर्निबद्धा खलु सैव जीवा ॥ २१ ॥ ॥ माया उपजातिका ॥

pūrvaṃ paraṃ vā yad ihāsti cihnaṃ prakalpya tallakṣakanāmadheyam |
abhīṣṭacāpasya tu koṭiyugme rajjur nibaddhā khalu saiva jīvā || *21* |||| *māyā upajātikā* ||

Having placed labels that identify the points which have [already] been marked whether they are in the eastern [half] or in the western [half], the string tied at the two end-points of the desired arc is indeed [its] chord.

ज्यां ज्यार्धमेव प्रवदन्ति तज्ज्ञाः
व्यासार्धमेव त्रिभसिञ्जिनीञ्च ।
यतो ज्यकार्धाग्रगतो ग्रहेन्द्रः
तिर्यक्स्थितो मध्यमसूत्रतः स्यात् ॥ २२ ॥ ॥ शाला उपजातिका ॥

jyāṃ jyārdham eva pravadanti tajjñāḥ
vyāsārdham eva tribhasiñjinīñ ca |
yato jyakārdhāgragato grahendraḥ
tiryaksthito madhyamasūtrataḥ syāt || *22* || || *śālā upajātikā* ||

The knowledgeable in this [subject, conventionally] refer to *jyārdham* (literally: half-chord; i.e., the Sine) as "*jyā* (literally: chord)", and the Sine of three signs (i.e, 90°) as the Radius. This is because [in astronomical models, the position of] the planet, lying at the tip of the half-chord, situated perpendicular to the central line [intersecting the chord, is computed using this measure].

अथोच्यते ज्यानयनोपपत्तिः
पञ्चप्रकारैर्गणितस्य तावत् ।
त्रित्र्यंशकोत्थानि परिस्फुटानि
त्रिंशज्ज्यकार्धानि यतो भवन्ति ॥ २३ ॥ ॥ कीर्तिः उपजातिका ॥

athocyate jyānayanopapattiḥ
pañcaprakārair gaṇitasya tāvat |
tritryaṃśakotthāni parisphuṭāni
triṃśajjyakārdhāni yato bhavanti || 23 || || *kīrtiḥ upajātikā* ||

Now, the rationale behind the computation of Sines is indeed described through five sections. Since there are only 30 Sines, accurate values [of Sines of any angle] are obtained from [the Sine] of one third of three degrees (i.e., Sin 1°).

Technical analysis

In the opening five verses, Nityānanda describes the preliminaries necessary for his reckoning of Sines. First, he prescribes the construction of a circle with Radius equal to sixty *aṅgulas*.[4] He then instructs that the arc of the circumference for each quadrant be divided into 90° (*lava*), as indicated in Figure 1. In explicitly defining his Radius as sixty units, Nityānanda becomes one of the few Indian scholars to construct a Sine table with $R = 60$.[5] This value is almost certainly inspired by the Islamic tradition.[6]

Nityānanda then defines the chord of an arbitrary arc (verse 21) by the act of fixing a string (*rajju*) at two points on this circle (say, A and B in Figure 1). He then continues (verse 22) to clarify the use of some technical terminology, in order to avoid ambiguity. The Sanskrit word for chord is *jyā* and given that the Sine is defined as half of this chord, the Sine is thus called *jyārdham*, a compound of *jyā* and *ardham* meaning literally 'half of the chord'. However, Nityānanda notes many authors simply abbreviate the term *jyārdham* to *jyā* to refer to the Sine. Thus since *jyā* can refer to both 'Chord' and 'Sine', the reader must be cautious to distinguish, presumably from the context and/or with the aid of commentaries, which one is being referred to. While on the topic of terminology, Nityānanda also reminds the reader that one can substitute the term 'Radius' (*vyāsārdha*) with the term 'Sine

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[4] An *aṅgula* is a common unit of linear measure used in Indian works.

[5] He is not the first however. Around half a century earlier, Indian author Bhūdhara graduated the radius of his quadrant into 60 units in his *Turyayantraprakāśa* (*ca.* 1572). See SaKHYa (2014).

[6] See, for instance, Van Brummelen (2009, 136). More common Radii measures in the Sanskrit tradition included 3438, 3270, 150, and 120.

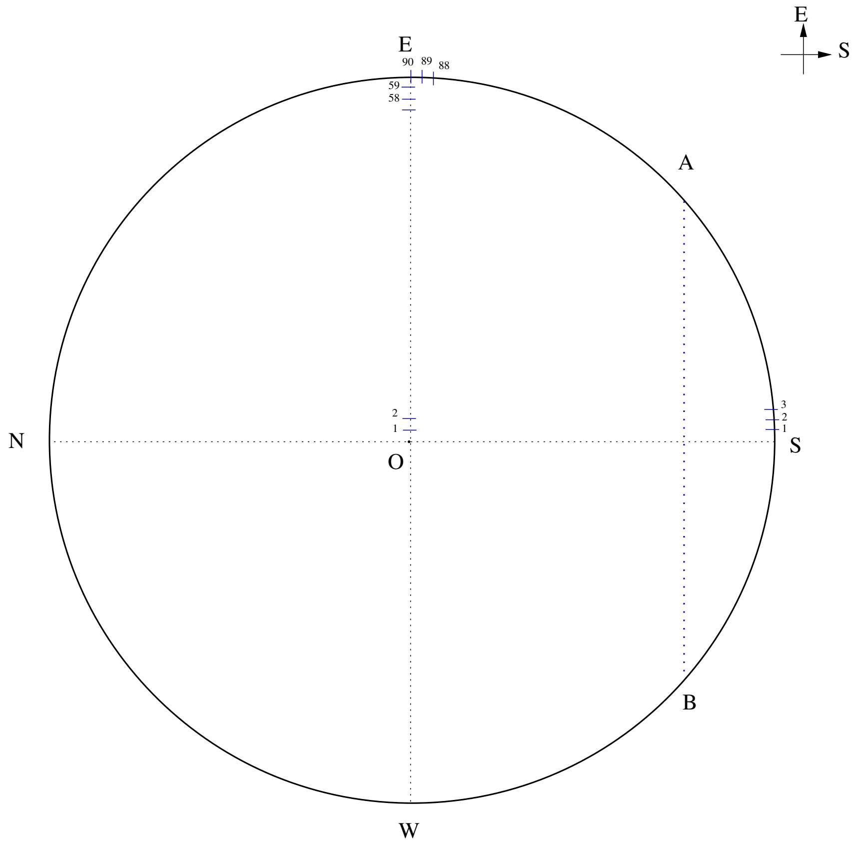

Figure 1: The circle with Radius 60 and quadrants marked on which the chord and the Sine can be visualised.

of three (zodiacal) signs' (*tri-bha-siñjinī*) as they are mathematically equivalent ($R = \mathrm{Sin}\,90$).[7]

It is also in verse 22 that Nityānanda gives a fascinating insight into the motivation for Indian scholars in originally adopting the 'half-chord' over the chord as employed by the Greeks. Given the extensive use of trigonometry in the computation of planetary positions, it is the half-chord—the measure of the segment from the apsidal line to the planet positioned on a circular orbit—that is of more immediate use to astronomers, rather than the entire chord itself.

Nityānanda concludes his introduction by noting that the techniques he shall be setting out in the following five sections provide a set of thirty Sine values corresponding to every 3° of arc.[8] However, he also prepares the reader for his treatment

[7] Given that these procedures were composed in a verse-format which has various metrical constraints, having synonyms in order to satisfy these constrainsts was a huge advantage to the composer.

[8] Sine tables with thirty values were used by several of Nityānanda's predecessors. For instance,

of the determination of the Sine of 1°, which follows the five sections in a 'sixth section.' With an accurate value of the Sine of 1°, one can find the Sine of any arc of integer degrees by the application of the sum or difference formula. This allows him to produce a Sine table with 90 entries, which is included in some manuscripts.

III.2 Section One: The Sines of Ninety, Thirty, and Eighteen Degrees

Text and translation

अथाद्यप्रकारे नवतित्रिंशदष्टादशभागादीनां जीवाज्ञानम् ।
athādyaprakāre navatitriṃśadaṣṭādaśabhāgādīnāṃ jīvājñānam |
Now, in the first section (*prakāra*), the knowledge of the Sines of ninety, thirty, and eighteen degrees [is described].

नवत्यंशजीवा भवेद्व्यासखण्डं
तदर्धं खरामांशजीवा निरुक्ता ।
तदर्धोनितं विस्तृतेरङ्घ्रिवर्गात्
सपादात् पदं ज्याष्टचन्द्रांशकानाम् ॥ २४ ॥ ॥ भुजङ्गप्रयातम् ॥
navatyaṃśajīvā bhaved vyāsakhaṇḍaṃ
tadardhaṃ kharāmāṃśajīvā niruktā |
tadardhonitaṃ vistṛter aṅghrivargāt
sapādāt padaṃ jyāṣṭacandrāṃśakānām || 24 || *|| bhujaṅgaprayātam ||*
The Sine of ninety degrees is the Radius (*vyāsakhaṇḍa*). The Sine of thirty (*kha-rāma*) degrees is stated to be half of that. The Sine of eighteen degrees is the square-root of the square of quarter of the Diameter increased by quarter [of that amount], decreased by half of that (i.e., half of quarter the diameter, that is, a quarter of the Radius).

अत्रोपपत्तिः ।
atropapattiḥ |
Now, the demonstration.

पूर्वाङ्ककेन्द्रोपगसूत्रमध्ये
कृत्वार्धवृत्तस्य च केन्द्रचिह्नम् ।
ततो नयेद्दक्षिणगामिसूत्रं
कर्णानुकारं प्रथमं मनीषी ॥ २५ ॥ ॥ शाला उपजातिका ॥
pūrvāṅkakendropagasūtramadhye
kṛtvārdhavṛttasya ca kendracihnam |
tato nayed dakṣiṇagāmisūtraṃ

Bhāskara II describes how to construct one in his *Jyotpatti* chapter of the *Siddhāntaśiromaṇi* (Śāstrī 1989).

karṇānukāraṃ prathamaṃ manīṣī || *25* || || *śālā upajātikā* ||

Making the central point of the half circle as the middle of the line going between the east point [of the main circle], may the intelligent one first draw a line from that [point] towards the south [point] which resembles the hypotenuse.

एकत्रिभज्ये भुजकोटिरूपे
त्र्यस्राभिधे स्पष्टतरे च विन्द्यात् ।
कर्णः स्पृशेद्यत्र कृतार्धवृत्ते
तद्याम्यचिह्नान्तरगं प्रमाणम् ॥ २६ ॥ || इन्द्रवज्रा ॥
षड्वह्निभागोन्मितचापकर्णः
स्यात् तस्य खण्डं धृतिभागजीवा ।
सिद्ध्यन्त्यतो ज्या रसरामभाग-
द्विशैलभागाब्धिशरांशजाद्याः ॥ २७ ॥ || बाला उपजातिका ॥

ekatribhajye bhujakoṭirūpe
tryasrābhidhe spaṣṭatare ca vindyāt |
karṇaḥ spṛśed yatra kṛtārdhavṛtte
tadyāmyacihnāntaragaṃ pramāṇam || *26* || || *indravajrā* ||
ṣaḍvahnibhāgonmitacāpakarṇaḥ
syāt tasya khaṇḍaṃ dhṛtibhāgajīvā |
siddhyanty ato jyā rasarāmabhāga-
dviśailabhāgābdhiśarāṃśajādyāḥ || *27* || || *bālā upajātikā* ||

In the clear [right-angled figure] called trilateral [whose hypotenuse has just been drawn], one may understand that the Sines of one and three signs form the upright (*bhuja*) and the lateral (*koṭi*) [respectively].[9] Wherever the hypotenuse (*karṇa*) touches the circle made into half [the Radius], the measure of the distance between this [point] and the south point gives the hypotenuse corresponding to the arc equal to thirty-six (*ṣaḍ-vahni*) degrees. Part (i.e., half) of it is the Sine of eighteen (*dhṛti*) degrees. From this, the Sines corresponding to thirty-six (*rasa-rāma*) degrees, seventy-two (*dvi-śaila*) degrees, and fifty-four (*abdhi-śara*) degrees and so on are obtained.

त्रिज्यावर्गार्धतो मूलं सार्धराशिज्यका भवेत् ।
अत्र सार्धभजे जीवे दोःकोटी त्रिज्यका श्रुतिः ॥ २८ ॥ || श्लोक ॥

trijyāvargārdhato mūlaṃ sārdharāśijyakā bhavet |
atra sārdhabhaje jīve doḥkoṭī trijyakā śrutiḥ || *28* || || *śloka* ||

The square-root of half the square of the Radius is the Sine of a sign and a half (i.e., 45°). In this case, the upright (*doḥ*) and the lateral (*koṭi*) are [both equal to] the Sine produced for a sign and a half (i.e., 45°). The hypotenuse is the Radius.

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[9] Given their clear orientation in the diagram in this text, we have opted to for the translations 'upright' and 'lateral' for *bhuja* and *koṭi* respectively. Commonly, *bhuja* can also be translated as 'side' or 'base.'

दोर्ज्यावर्गोनितात् त्रिज्यावर्गान्मूलञ्च कोटिजा ।
ज्या स्यात् कोटिज्यकावर्गहीनान्मूलं च दोर्ज्यका ॥ २९ ॥ ॥ श्लोक॥

dorjyāvargonitāt trijyāvargān mūlañ ca koṭijā |
jyā syāt koṭijyakāvargahīnān mūlaṃ ca dorjyakā || 29 || *|| śloka ||*

The square-root of square of the Radius decreased by the square of the Sine (*dorjyā*) is the Cosine (*koṭijā jyā*). And the square-root of [the square of the Radius] decreased by the square of the Cosine (*koṭijyā*) is the Sine.

अत्र त्र्यस्रे भुजो दोर्ज्या कोटिः कोटिज्यका तथा ।
कर्णस्त्रिज्या ततः साध्ये कोटिदोर्ज्ये परस्परम् ॥ ३० ॥ ॥ श्लोक॥

atra tryasre bhujo dorjyā koṭiḥ koṭijyakā tathā |
karṇas trijyā tataḥ sādhye koṭidorjye parasparam || 30 || *|| śloka ||*

Here, in [a right-angle] triangle, the Sine is the upright (*bhuja*), the Cosine is the lateral (*koṭi*) and the hypotenuse is the Radius. From it (i.e., the hypotenuse), the Sine and the Cosine can be obtained mutually from one another [using the *Śulba* ('Pythagorean')-theorem].

Technical analysis

In this section, Nityānanda specifies the values of the Sines of ninety, thirty-six, and eighteen degrees, and using these, the values of others. In verse 24, he gives the following Sine relations:

$$\operatorname{Sin} 90 = R, \tag{1}$$

$$\operatorname{Sin} 30 = \frac{R}{2}, \tag{2}$$

$$\operatorname{Sin} 18 = \sqrt{\left(\frac{D}{4}\right)^2 + \frac{1}{4}\left(\frac{D}{4}\right)^2} - \frac{1}{2}\cdot\frac{D}{4}. \tag{3}$$

These and associated Sine relations are justified in verses 25–27 by means of a demonstration (*upapatti*). This demonstration invokes a geometrical construction. A diagram depicting the construction has been presented in some of the manuscripts (Figures 2 and 3).

In Figure 4, U is the midpoint of OE. Having marked this, Nityānanda instructs one to join this point to the South point (S). Then he points out that in the resulting triangle UOS, the upright is equal to the Sine of 30° (OU) and the lateral (OS) is equal to the Radius (or the Sine of 90° or three 'signs'). The circle with the midpoint U as center and radius OU will intersect the hypotenuse (*karṇa*) at V. The resulting line segment (VS) is equal to the chord of 36°.[10] Half of this chord is stated to be

[10] The validity of this can be demonstrated by noting that the chord of 36° is equivalent to the

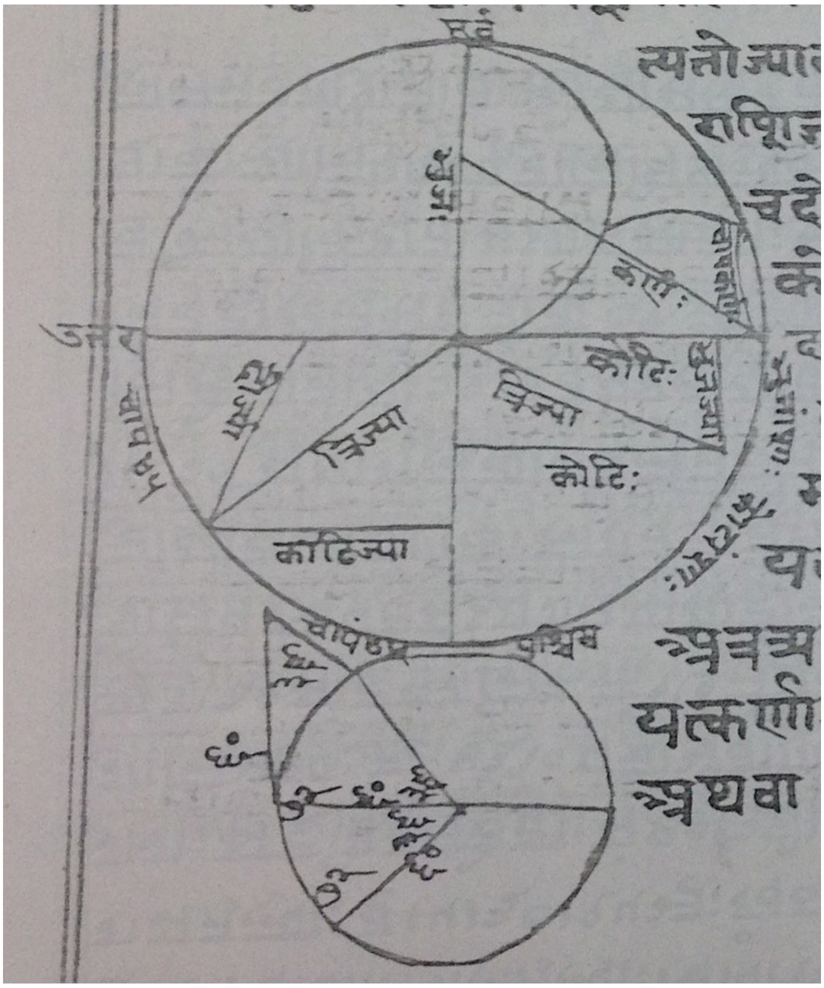

Figure 2: An excerpt from manuscript R (f. 11v) which depicts the geometrical construction related to obtaining the Sines of 18° and 45°, and, below it, the chord of 36°.

the Sine of 18°. This easily follows from the well-known relation that the Sine of an arc is half the chord of double that arc, or in modern notation, $\mathrm{Sin}\,\theta = \mathrm{Crd}\,2\theta/2$.

Now we will set out a simple approach by which we can arrive at the expression given in Equation (3). It is easily seen from Figure 4 that:

$$US = \sqrt{\left(\frac{D}{2}\right)^2 + \left(\frac{D}{4}\right)^2}$$

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side of a decagon inscribed in the circle. The steps that Nityānanda describes here are in fact also a common way to construct a decagon with ruler and compass. See, for instance, the demonstration in Richmond (1893).

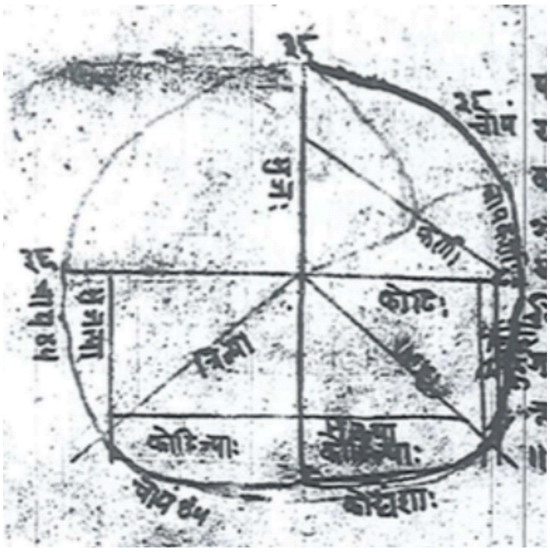

Figure 3: An excerpt from manuscript N (f. 16v) depicting the upper part of the diagram shown in Figure 2.

Hence,

$$\begin{aligned} VS &= US - UV \\ &= \sqrt{\left(\frac{D}{2}\right)^2 + \left(\frac{D}{4}\right)^2} - \frac{D}{4}. \end{aligned} \tag{4}$$

When half of VS is taken, there results:

$$\begin{aligned} \frac{VS}{2} &= \sqrt{\frac{1}{4}\left(\frac{D}{2}\right)^2 + \frac{1}{4}\left(\frac{D}{4}\right)^2} - \frac{1}{2}\cdot\frac{D}{4} \\ &= \sqrt{\left(\frac{D}{4}\right)^2 + \frac{1}{4}\left(\frac{D}{4}\right)^2} - \frac{1}{2}\cdot\frac{D}{4} \end{aligned}$$

as required.[11]

The expression for the Sine of 18°, given in Equation (3), may be rewritten in terms of R as:[12]

11 The same derivation is stated in Munīśvara's *Marīci* (before 1638). See Gupta (1976). We thank one of the anonymous referees for bringing this to our attention.

12 This is the more familiar form given by other astronomers, such as Bhāskara II (*Jyotpatti* verse 9). See Gupta (1976, 2).

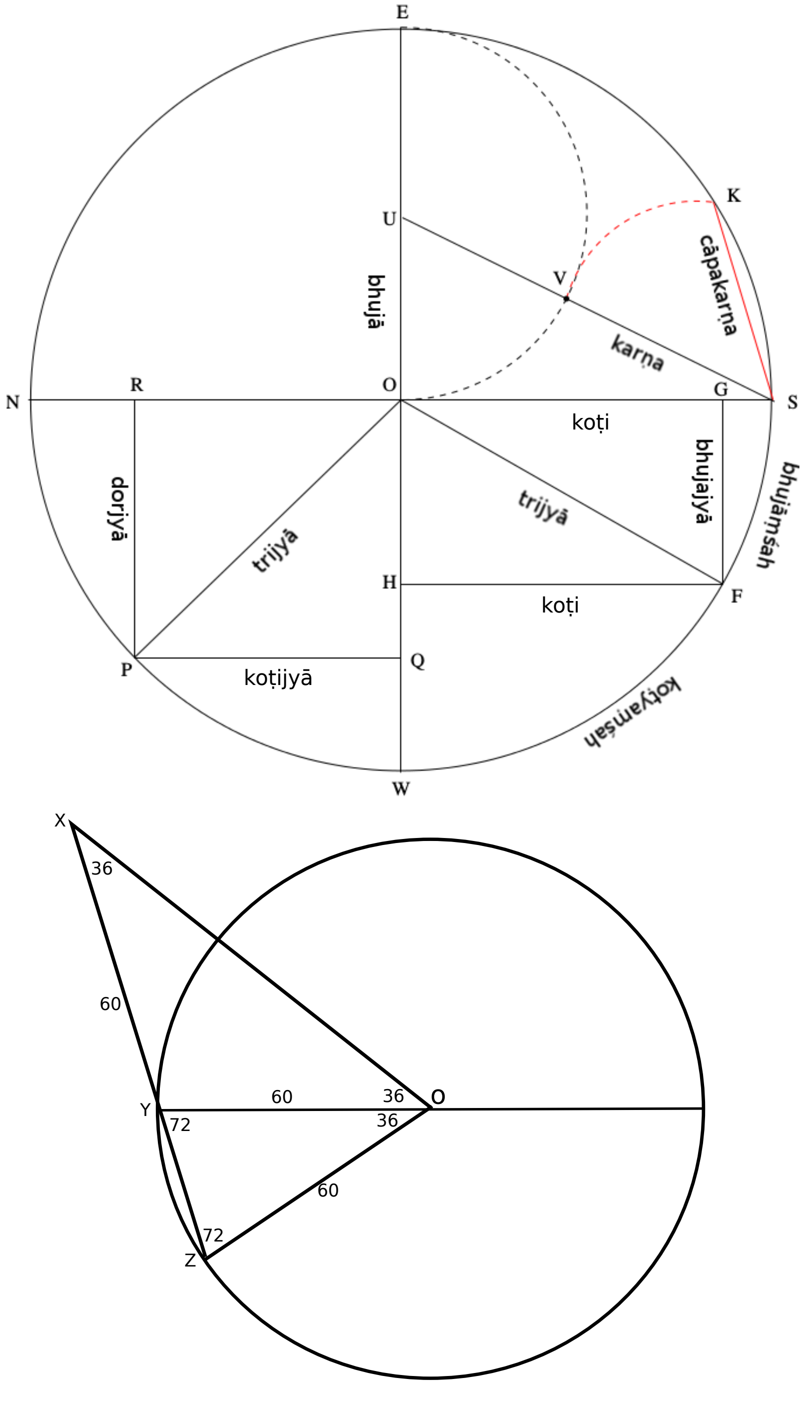

Figure 4: Top: The geometrical construction to determine the Sines of 36° and 18°, 45°, and the general relation between the Sine and the Cosine. Bottom: Another geometrical construction related to the determination of the Sines of 36° and 18° that appears below the construction in manuscript R (see Figure 2).

$$\begin{aligned}
\mathrm{Sin}\,18 &= \sqrt{\frac{4R^2}{4^2}+\frac{1}{4}\left(\frac{4R^2}{4^2}\right)}-\frac{R}{4}\\
&= \sqrt{\frac{5R^2}{4^2}}-\frac{R}{4}\\
&= \frac{\sqrt{5R^2}-R}{4}.
\end{aligned}$$

The expression for Sin 18° given in the text can be obtained from the construction shown in Figure 4 (bottom) which is essentially a reconstructed form of the lower image of 2. From the two similar isosceles triangles in the figure, we have:

$$\frac{XZ}{OZ}=\frac{OY}{YZ} \qquad \text{or} \qquad \frac{(x+60)}{60}=\frac{60}{x}. \tag{5}$$

From the above equation we get the following quadratic:

$$x^2+60x=60^2. \tag{6}$$

Solving the above quadratic:

$$x^2+60x+\left(\frac{60}{2}\right)^2=60^2+\left(\frac{60}{2}\right)^2 \tag{7}$$

so that,

$$x+\frac{60}{2}=\sqrt{60^2+\left(\frac{60}{2}\right)^2}. \tag{8}$$

Then,

$$x=\sqrt{60^2+\left(\frac{60}{2}\right)^2}-\frac{60}{2}=\sqrt{\left(\frac{D}{2}\right)^2+\left(\frac{D}{4}\right)^2}-\frac{D}{4}, \tag{9}$$

which is exactly the same as Equation (4) obtained above.

In verse 28, Nityānanda provides a rule for determining the Sine of 45°:

$$\mathrm{Sin}\,45=\sqrt{\frac{R^2}{2}}. \tag{10}$$

This relation is also demonstrated in the same diagram, but in the bottom left-hand quadrant in Figure 4. The arc of the quadrant is bisected into two equal halves of 45° each by the radial line *OP*. The resulting square produced by dropping perpendiculars onto the direction lines *NS* and *EW*, namely *OQPR*, is also a right-angled triangle made up of two identical triangles *ORP* and *OQP*. These triangles share a hypotenuse, OP, equal to the Radius, whose uprights and laterals are the same, and equal in measure to the Sine and Cosine of 45°. Application of the Pythagorean theorem to any of these triangles produces Equation (10) given above.

Verses 29–30 state the familiar rule relating the squares of the Sine and the Cosine:

$$\sqrt{R^2 - \mathrm{Sin}^2\,\theta} = \mathrm{Cos}\,\theta \qquad \text{and} \qquad \sqrt{R^2 - \mathrm{Cos}^2\,\theta} = \mathrm{Sin}\,\theta. \tag{11}$$

These relations are justified in the following verse by means of their status as the uprights and laterals of a right-angle triangle whose hypotenuse is the Radius. It appears that this relation too is depicted in the diagram, captured by an arbitrary configuration in the bottom right hand quadrant of Figure 4, with a Sine (GF) and a Cosine (HF) and hypotenuse (OF) as Radius.

III.3 Section Two: Determining the Sine of Half the Arc

Text and translation

अथ द्वितीयप्रकारे चापज्याज्ञाने सति चापार्धज्याज्ञानं तदुपयोगित्वात्तावदुत्क्रमज्यानयनम् ।

atha dvitīyaprakāre cāpajyājñāne sati cāpārdhajyājñānaṃ tadupayogitvāt tāvadutkrama-jyānayanam |

Now, in the second section, when the Sine of the arc is known, [the procedure to obtain] the knowledge of the Sine of half the arc [is described and also] the computation of the Versine [is described] since it will be useful for this (i.e., computing the Sine of half the arc).

कोटिज्यका व्यासदलाद्विशोध्या
बाहूत्क्रमज्या विशिखाभिधा स्यात् ।
चेद्व्यासखण्डात् पतिता भुजज्या
कोट्युत्क्रमज्या प्रभवेत् तदानीम् ॥ ३१ ॥ ॥ इन्द्रवज्रा ॥

koṭijyakā vyāsadalād viśodhyā
bāhūtkramajyā viśikhābhidhā syāt |
ced vyāsakhaṇḍāt patitā bhujajyā
koṭyutkramajyā prabhavet tadānīm || *31* || || *indravajrā* ||

The Cosine decreased from the Radius is the base (*bāhu*)-Versine, also refered to as 'arrow' (*viśikhā*). If the Sine is subtracted from the Radius (*vyāsakhaṇḍa*), then the lateral (*koṭi*)-Versine is produced.

भुजोत्क्रमज्याकृतिसंयुतायाः दोर्ज्याकृतेर्मूलदलञ्च यत् स्यात् ।
तदर्धचापस्य भुजज्यका सा कोटिज्यकातोऽपि तथैव कार्या ॥ ३२ ॥ ॥ हंसी उपजातिका ॥

bhujotkramajyākṛtisaṃyutāyāḥ dorjyākṛter mūladalañ ca yat syāt |
tadardhacāpasya bhujajyakā sā koṭijyakāto'pi tathaiva kāryā || *32* || || *haṃsī upajātikā* ||

Whatever [is obtained] from half the square-root of the sum of the squares of the Sine and Versine, that is the Sine corresponding to half the arc. Then, similarly [the Cosine of half the arc] can be obtained from the '*koṭijyā*s' (i.e., the Cosine and the Coversine).

अत्र त्र्यस्रे व्यस्तजीवास्ति कोटिः बाहुज्या दोश्चापकर्णोऽत्र कर्णः ।
यत्कर्णार्धं सैव चापार्धजीवा चापक्षेत्रे सर्वमेतद्विलोक्यम् ॥ ३३ ॥ ॥ शालिनी ॥

atra tryasre vyastajīvāsti koṭiḥ bāhujyā doś cāpakarṇo 'tra karṇaḥ |
yat karṇārdhaṃ saiva cāpārdhajīvā cāpakṣetre sarvam etad vilokyam || 33 || || śālinī ||

In this [right-angle] triangle, the lateral (*koṭi*) is the Versine (*vyastajīvā*), the upright (*doḥ*) is the Sine (*bāhujyā*), and the hypotenuse (*karṇa*) here is the arc-hypotenuse (*cāpakarṇa*). What is half the [arc]-hypotenuse, this is indeed the Sine of half the arc. All this is to be visualized in the figure (*kṣetra*) associated with the arc (*cāpa*).

अथवा

athavā

Alternatively,

मूलञ्च यद्व्यासशरावघातात् तदर्धकं वा धनुरर्धजीवा ।
अव्यक्तबाणोऽत्र च तेन हीना त्रिज्या भवेत् सा निजकोटिजीवा ॥ ३४ ॥ ॥ वाणी उपजातिका ॥

mūlañ ca yad vyāsaśarāvaghātāt tadardhakaṃ vā dhanurardhajīvā |
avyaktabāṇo 'tra ca tena hīnā trijyā bhavet sā nijakoṭijīvā || 34 || || vāṇī upajātikā ||

Or, otherwise, half the square-root of the product of the Diameter and the Versine is the Sine of half the arc. In this [right-angled triangle], the unseen[13] Versine when subtracted from the Radius gives its own Cosine.

तदीयवर्गस्त्रिगुणस्य वर्गाच्च्युतो भुजज्याकृतिरेव सा स्यात् ।
शरस्य वर्गेण युता पुनः सा कोदण्डकर्णस्य कृतिर्विचिन्त्या ॥ ३५ ॥ ॥ जाया उपजातिका ॥

tadīyavargas triguṇasya vargāc cyuto bhujajyākṛtir eva sā syāt |
śarasya vargeṇa yutā punaḥ sā kodaṇḍakarṇasya kṛtir vicintyā || 35 |||| jāyā upajātikā ||

The square of that (i.e., the Cosine of the half the arc) subtracted from the square of the Radius, indeed gives the square of the Sine [of the half the arc]. That [square of the Sine of half the arc] when added to the square of the Versine (*śara*) [of half the arc] is again to be understood as the square of the [new] arc-hypotenuse (*kodaṇḍakarṇa*).

एवं शरव्यासहतिः प्रपन्ना कोदण्डकर्णस्य तु वर्गतुल्या ।
कोदण्डकर्णस्य कृतेः पदं यत् तदर्धकं चापदलस्य जीवा ॥ ३६ ॥ ॥ बाला उपजातिका ॥

evaṃ śaravyāsahatiḥ prapannā kodaṇḍakarṇasya tu vargatulyā |
kodaṇḍakarṇasya kṛteḥ padaṃ yat tadardhakaṃ cāpadalasya jīvā || 36 |||| bālā upajātikā ||

In this way, the product of the Versine and the Diameter is the same as the square of

---

[13] Literally: 'non-manifest', in the sense of that which has not been indicated in the diagram. Hence we have used a dotted line to show this in Figure 5.

> the arc-hypotenuse. Whatever is square-root of the square of the arc-hypotenuse, half of that is the Sine of half the arc.

Technical analysis

The second section deals with rules for determining the Sine of half of the arc when the Sine of the arc is known. As he begins this section, Nityānanda introduces the Versine because it is useful in determining the half-arc relation. The Versine is defined (verse 31) as:

$$\mathrm{Vers}\,\theta = R - \mathrm{Cos}\,\theta.$$

In addition to the Versine, Nityānanda also defines the Coversine relation as

$$\mathrm{Coversin}\,\theta = R - \mathrm{Sin}\,\theta.$$

The former is more precisely referred to as the base-Versine (*bāhūtkramajyā*) and the latter as the lateral-Versine (*koṭyutkramajyā*).[14] In Figure 5 these are BS and FE respectively.

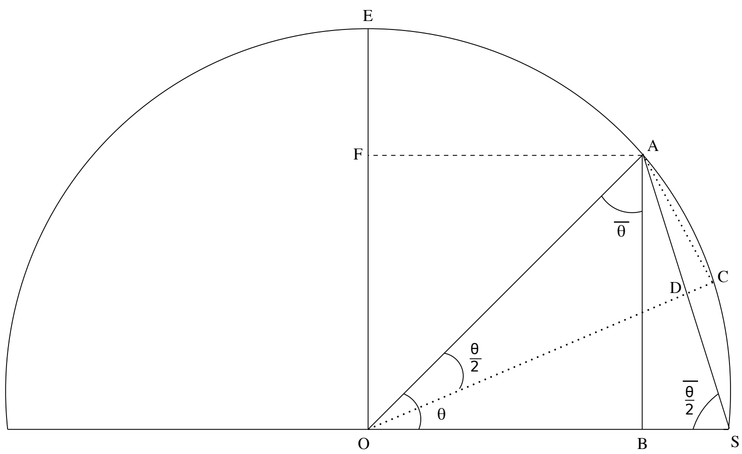

Figure 5: A diagram showing the Versine (BS), Coversine (FE), arc-hypotenuse (AS), and other key elements covered in this section.

Verse 32 gives rules for determining the Sine and Cosine of half the arc in terms of the Versine and Coversine, as follows:

$$\mathrm{Sin}\left(\frac{\theta}{2}\right) = \frac{\sqrt{\mathrm{Sin}^2\,\theta + \mathrm{Vers}^2\,\theta}}{2}, \tag{12}$$

[14] This latter relation is somewhat uncommon in the Sanskrit tradition.

and

$$\operatorname{Cos}\left(\frac{\theta}{2}\right) = \frac{\sqrt{\operatorname{Cos}^2\theta + \operatorname{Coversin}^2\theta}}{2}. \tag{13}$$

In the following verse, Nityānanda presents a diagrammatic justification of the result in Equation (12). For this, he considers the triangle ABS and then notes that in this triangle, the Versine (BS) is its lateral and the Sine (AB) is the upright. The hypotenuse (AS) of this triangle is called the 'arc-hypotenuse' (*cāpakarṇa*) which we denote by $\mathcal{K}$.[15] Then, the measure of $\mathcal{K}$ is simply the square-root of the sum of the squares of the lateral and upright (namely the Versine and the Sine), that is,

$$\mathcal{K} = \sqrt{\operatorname{Sin}^2\theta + \operatorname{Vers}^2\theta}. \tag{14}$$

Half of that, AD or DS, is the Sine of half the arc given in Equation (12). The rule for the Cosine of half the arc can be demonstrated in a similar way.

Nityānanda then states an alternative formula for the Sine of half the arc in the first half of verse 34:

$$\operatorname{Sin}\left(\frac{\theta}{2}\right) = \frac{\sqrt{D\cdot\operatorname{Vers}\theta}}{2}. \tag{15}$$

The equivalence of the RHS of Equations (15) and (12) can be seen as follows:

$$\begin{aligned} D\cdot\operatorname{Vers}\theta &= 2R\operatorname{Vers}\theta \\ &= 2R(R-\operatorname{Cos}\theta) \\ &= R^2+R^2-2R\operatorname{Cos}\theta \\ &= \operatorname{Sin}^2\theta + R^2 - 2R\operatorname{Cos}\theta + \operatorname{Cos}^2\theta \\ &= \operatorname{Sin}^2\theta + (R-\operatorname{Cos}\theta)^2 \\ &= \operatorname{Sin}^2\theta + \operatorname{Vers}^2\theta. \end{aligned} \tag{16}$$

Commencing from the latter half of verse 34, Nityānanda explains how the above procedure can be extended to find subsequent Sines of half-arcs. By drawing a perpendicular bisector OC to the arc AS, a new Versine CD corresponding to half the arc can be identified. Nityānanda refers to this by the term *avyakta-bāṇa*, unseen Versine, literally: non-manifest Versine. With this *avyaktabāṇa*, he then instructs (verse 35) one to find the lateral, and thence, the square of the Sine of half the arc. That is:

$$R - \operatorname{Vers}\left(\frac{\theta}{2}\right) = \operatorname{Cos}\left(\frac{\theta}{2}\right), \tag{17}$$

and

$$R^2 - \operatorname{Cos}^2\left(\frac{\theta}{2}\right) = \operatorname{Sin}^2\left(\frac{\theta}{2}\right). \tag{18}$$

[15] The term *cāpakarṇa* for this chord appears to be a novel term employed by Nityānanda. As far as we know, it does not appear earlier in the Sanskrit trigonometrical tradition.

Now, in terms of the quantities derived above and the *avyaktabāṇa*, the new arc-hypotenuse, corresponding to the arc AC, $\mathcal{K}_{\frac{\theta}{2}}$, is given by

$$AC = \sqrt{AD^2 + CD^2},$$
$$\mathcal{K}_{\frac{\theta}{2}} = \sqrt{\mathrm{Sin}^2\left(\frac{\theta}{2}\right) + \mathrm{Vers}^2\left(\frac{\theta}{2}\right)}. \tag{19}$$

This $\mathcal{K}_{\frac{\theta}{2}}$ is what is referred to by the term *kodaṇḍakarṇa* in verse 35. Half of this produces the Sine of quarter the arc $(\frac{\theta}{4})$. This procedure can be repeated to produce the Sines of subsequent half-arcs. Nityānanda concludes this section (verse 36) by summarizing the key relations that can be applied iteratively to obtain the Sines of subsequent half-arcs. In modern notation, this amounts to:

$$D \cdot \mathrm{Vers}\,\theta_i = \mathcal{K}^2_{\theta_i}, \tag{20}$$

and

$$\mathrm{Sin}\,\theta_{i+1} = \frac{\sqrt{\mathcal{K}^2_{\theta_i}}}{2}, \qquad \text{where } \theta_{i+1} = \frac{\theta_i}{2}. \tag{21}$$

III.4 Section Three: Determining the Sine of Double the Arc

Text and translation

अथ तृतीयप्रकारे चापस्य ज्याज्ञाने सति द्विघ्नचापदोर्ज्याज्ञानम् ।

atha tṛtīyaprakāre cāpasya jyājñāne sati dvighnacāpadorjyājñānam |

Now, in the third section, when the Sine of the arc is known, [the procedure to obtain] the knowledge of the Sine of double the arc [is described].

दोर्ज्यावर्गश्चैकभज्याविभक्तो लब्धं व्यासात् पातयेच्छेषमित्या ।
लब्धं हत्वा तस्य मूलं यदाप्तं जीवा स्पष्टा द्विघ्नचापस्य सा स्यात् ॥ ३७ ॥ ॥ शालिनी ॥

dorjyāvargaś caikabhajyāvibhakto labdhaṃ vyāsāt pātayec cheṣamityā |
labdhaṃ hatvā tasya mūlaṃ yad āptaṃ jīvā spaṣṭā dvighnacāpasya sā syāt || *37* |||| *śālinī* ||

The quantity obtained by dividing the square of the Sine by the Sine of one sign should be subtracted from the Diameter. Having multiplied the remainder by the quantity obtained earlier [by the process of division, and taking] its square-root, whatever is obtained, that is exactly the Sine of twice the given arc.

अत्र द्विनिघ्नीष्टधनुर्भुजज्या स्याद्विघ्नकोदण्डजचापकर्णः ।
चेच्चापकर्णस्य कृतिर्विभक्ता व्यासप्रमाणेन शरस्तदा स्यात् ॥ ३८ ॥ ॥ इन्द्रवज्रा ॥

atra dvinighnīṣṭadhanurbhujajyā syād dvighnakodaṇḍajacāpakarṇaḥ |
cec cāpakarṇasya kṛtir vibhaktā vyāsapramāṇena śaras tadā syāt || *38* || || *indravajrā* ||

Now, twice the Sine of the desired arc is the arc-hypotenuse (*cāpakarṇa*) of double the arc. If the square of the arc-hypotenuse is divided by the measure of the Diameter, then it is the Versine (*śara*) [of double the arc].

बाणोनितं व्यासदलञ्च यत् स्यात् सा द्विघ्नकोदण्डजकोटिजीवा ।
तद्वर्गहीनस्त्रिगुणस्य वर्गो दोर्ज्याकृतिर्द्विघ्नधनुर्भवा स्यात् ॥ ३९ ॥ ॥ इन्द्रवज्रा ॥

bāṇonitaṃ vyāsadalañ ca yat syāt sā dvighnakodaṇḍajakoṭijīvā ||
tadvargahīnas triguṇasya vargo dorjyākṛtir dvighnadhanurbhavā syāt || *39* |||| *indravajrā* ||

Whatever quantity is obtained by subtracting the Versine [of twice the arc] from the Radius, that is the Cosine of twice [the arc] born out of the arc-hypotenuse. The square of this removed from the square of the Radius is the square of the Sine of twice the arc.

अव्यक्तमार्गेण सुधीर्विदध्यात् एवं यथा बाणविवर्जितः स्यात् ।
व्यासः पुनर्बाणगुणः समानः प्रजायते बाहुगुणस्य कृत्या ॥ ४० ॥ ॥ बाला उपजातिका ॥

avyaktamārgeṇa sudhīr vidadhyāt evaṃ yathā bāṇavivarjitaḥ syāt |
vyāsaḥ punar bāṇaguṇaḥ samānaḥ prajāyate bāhuguṇasya kṛtyā || *40* |||| *bālā upajātikā* ||

May the intelligent one, by employing algebra (*avyakta*), demonstrate how, by subtracting the Versine from the Diameter, and again multiplying the Versine with that result, it becomes equal to the square of the Sine.

Technical analysis

The third section deals with rules for determining the Sine of double the arc when the Sine of the arc is known. In verse 37, Nityānanda describes the following procedure. First, one is instructed to find the product

$$\left(D-\frac{\operatorname{Sin}^2\theta}{\frac{R}{2}}\right)\times\frac{\operatorname{Sin}^2\theta}{\frac{R}{2}}.$$

Then it is stated that the square-root of this gives the expression for the Sine of double the arc. That is,

$$\operatorname{Sin}2\theta=\sqrt{\left(D-\frac{\operatorname{Sin}^2\theta}{\frac{R}{2}}\right)\times\frac{\operatorname{Sin}^2\theta}{\frac{R}{2}}}. \tag{22}$$

The following verses demonstrate this claim from relations already known. In verse 38, Nityānanda explicitly states that the arc-hypotenuse of double the arc is the same as twice the Sine of the arc:

$$\mathcal{K}_{2\theta}=2\operatorname{Sin}\theta. \tag{23}$$

He then states that

$$\operatorname{Vers}2\theta=\frac{(\mathcal{K}_{2\theta})^2}{D}, \tag{24}$$

which is essentially the same as Equation (20). Nityānanda then continues with relations for double the arc in verse 39, namely

$$R - \text{Vers}\, 2\theta = \text{Cos}\, 2\theta, \tag{25}$$

$$R^2 - \text{Cos}^2\, 2\theta = \text{Sin}^2\, 2\theta. \tag{26}$$

In verse 40, Nityānanda concludes the section by presenting the following trigonometric identity:

$$\text{Sin}^2\, 2\theta = (D - \text{Vers}\, 2\theta) \cdot \text{Vers}\, 2\theta, \tag{27}$$

and leaves this as an exercise for the intelligent reader to apply 'algebraic' techniques along with the necessary trigonometric principles to demonstrate the validity of Equation (27). We presume he intended something like the following:

$$\begin{aligned} \text{Sin}^2\, 2\theta &= R^2 - \text{Cos}^2\, 2\theta \\ &= (R + \text{Cos}\, 2\theta)(R - \text{Cos}\, 2\theta) \\ &= (D - \text{Vers}\, 2\theta)\, \text{Vers}\, 2\theta. \end{aligned} \tag{28}$$

Also from Equation (15), we have

$$\text{Vers}\, 2\theta = \frac{(2\,\text{Sin}\,\theta)^2}{D} = \frac{\text{Sin}^2\,\theta}{\frac{R}{2}}. \tag{29}$$

Using Equations (29) and (28) and taking the square-root, we get Equation (22), which is the expression for the Sine of double the arc given in verse 37.

III.5 Section Four: Determining the Sine of the Sum of Two Arcs

Text and translation

अथ चतुर्थप्रकारे धनुर्द्वयस्य भुजज्ययोर्ज्ञाने सति धनुर्योगज्याज्ञानम् ।

atha caturthaprakāre dhanurdvayasya bhujajyayorjñāne sati dhanuryogajyājñānam |

Now, in the fourth section, when the Sines of two [individual] arcs are known, [the procedure to obtain] the knowledge of the Sine of the sum of [those] two arcs [is described].

अन्योन्यकोटिमौर्व्या गुणिते ये चेष्टचापयोर्दोर्ज्ये ।
त्रिज्योद्धृते तयोर्यः योगः सा चापयोगज्या ॥ ४१ ॥ ॥ आर्या ॥

anyonyakoṭimaurvyā guṇite ye ceṣṭacāpayor dorjye |
trijyoddhṛte tayor yaḥ yogaḥ sā cāpayogajyā || *41* || || *āryā* ||

The Sines of the two desired arcs are mutually multiplied by the Cosines of the other two; when divided by the Radius, whatever is sum of the two, that is the Sine of the sum of the [two] arcs.

अत्रोपपत्तिः । ककारादिवर्णोपलक्षितरेखाभिः अत्र क्षेत्रव्यवस्था दर्शनीया ।

atropapattiḥ | kakārādivarṇopalakṣitarekhābhiḥ atra kṣetravyavasthā darśanīyā |

Here, the demonstration. The construction of the geometrical figure is to be demonstrated here by means of segments that are denoted by the letters beginning with *ka*.

कखागघं भूमितलेषु मण्डलं
ङकेन्द्रकं कर्कटकेन साधयेत् ।
कचं चछं चापयुगं कङं चङं
छङं क्रमाद्व्यासदलत्रयं लिखेत् ॥ ४२ ॥ ॥ वंशस्थम् ॥

kakhāgagham bhūmitaleṣu maṇḍalaṃ
ṅakendrakaṃ karkaṭakena sādhayet |
kacaṃ cachaṃ cāpayugaṃ kaṅaṃ caṅaṃ
chaṅaṃ kramād vyāsadalatrayaṃ likhet || 42 || *|| vaṃśastham ||*

May one draw a circle [labeled] *ka*, *kha*, *ga*, *gha* with a compass on a flat surface, with center [labeled] *ṅa*. [On that circle], may one mark two arcs [with pairs of letters] *ka-ca* and *ca-cha*. [Then], sequentially, may one draw the three radii *ka-ṅa*, *ca-ṅa* and *cha-ṅa*.

चजं छझं चापयुगस्य दोर्ज्यके कङे चङे लम्बवदेव पातिते ।
कछस्य चापद्वययोगसंमितेः ज्यका कङे लम्बवदेव पातिता ॥ ४३ ॥ ॥ वंशस्थम् ॥
छटाज्झपातं चङयोगचिह्ने
लिखेट्टकारं गणकप्रवीणः ।
सूत्रं झटाख्यं विलिखेच्चजेन
तुल्यप्रमाणं खलु शिल्पसिद्धम् ॥ ४४ ॥ ॥ माला उपजातिका ॥

cajaṃ chajhaṃ cāpayugasya dorjyake kaṅe caṅe lambavad eva pātite |
kachasya cāpadvayayogasaṃmiteḥ jyakā kaṅe lambavad eva pātitā || 43 |||| vaṃśastham||
chaṭāj jhapātaṃ caṅayogacihne
likheṭ ṭakāraṃ gaṇakapravīṇaḥ |
sūtraṃ jhaṭākhyaṃ vilikhec cajena
tulyapramāṇaṃ khalu śilpasiddham || 44 || *|| mālā upajātikā ||*

The two Sines corresponding to the two arcs, [denoted by] *ca-ja*, and *cha-jha* are dropped as perpendiculars onto [the radii] *ka-ṅa* and *ca-ṅa*. [Wherever] the Sine corresponding to the sum of the two arcs, *ka-cha* falls perpendicularly on *ka-ṅa*, may one write the letter *ṭa*. May the skillful mathematician draw the line denoted by *jha-ṭa* from [the tip *ṭa* of] *cha-ṭa* meeting the point of intersection *jha* with *ca-ṅa*.[16] [That this segment *jha-ṭa*] will be equal to the measure of *ca-ja* is ascertained by geometrical construction (*śilpa*).

[16] Here, the prose order is: ज्यका कङे (यत्र) लम्बवदेव पातिता (तत्र) टकारं लिखेत् । छटात् झटाख्यं सूत्रं झपातचङयोगचिह्ने गणकप्रवीणः विलिखेत् । (एतद्) चजेन तुल्यप्रमाणं (इति) शिल्पसिद्धम् ।

छटं धरा झाटझछे भुजौ द्वौ
महात्रिकोणे झतसंज्ञलम्बः ।
त्र्यस्रत्रयं तत्र समीक्ष्यमाणं
वर्णाङ्कितं तच्छ्रवणादि चिन्त्यम् ॥ ४५ ॥ ॥ माला उपजातिका ॥

chaṭaṃ dharā jhāṭajhache bhujau dvau
mahātrikoṇe jhatasaṃjñalambaḥ |
tryasratrayaṃ tatra samīkṣyamāṇaṃ
varṇāṅkitaṃ tacchravaṇādi cintyam || 45 || *|| mālā upajātikā ||*

In the big triangle, *cha-ṭa* forms the base, *jha-ṭa* and *jha-cha* are the two sides and [the segment denoted by] *jha-ta* forms the perpendicular. The three triangles that are clearly identified there, are marked with letters (*varṇa*). Their hypotenuses and so on (i.e., the uprights and laterals) are to be [now] carefully considered [for identifying similarity.]

त्र्यस्रं झटातं छङझस्वरूपं
छझातमन्यन्ङचजोपमानम् ।
तृतीयमन्यत् झठतं तथैव
ततोऽनुपातः परिकल्पनीयः ॥ ४६ ॥ ॥ बुद्धिः उपजातिका ॥

tryasraṃ jhaṭātaṃ chaṅajhasvarūpaṃ
chajhātam anyan ṅacajopamānam |
tṛtīyam anyat jhaṭhataṃ tathaiva
tato 'nupātaḥ parikalpanīyaḥ || 46 || *|| buddhiḥ upajātikā ||*

The triangle *jha-ṭa-ta* is similar to *cha-ṅa-jha*. The other [triangle] *cha-jha-ta* is similar to *ṅa-ca-ja*. The third one, *jha-ṭha-ta*, is also [similar to *ṅa-ca-ja*]. From these [triangles], the rule-of-three is to be employed.

कर्णे छङे चेत् झङतुल्यकोटिः
झटश्रुतौ कास्ति तटन्तदानीम् ।
चङे श्रुतौ चेज्जङतुल्यकोटिः
कर्णे छझे कास्ति तदा छतं सा ॥ ४७ ॥ ॥ माया उपजातिका ॥

karṇe chaṅe cet jhaṅatulyakoṭiḥ
jhaṭaśrutau kāsti taṭan tadānīm |
caṅe śrutau cej jaṅatulyakoṭiḥ
karṇe chajhe kāsti tadā chataṃ sā || 47 || *|| māyā upajātikā ||*

When the hypotenuse is *cha-ṅa*, the lateral (*koṭi*) is equal to *jha-ṅa*. If it were to be asked what [the lateral] is when the hypotenuse is *jha-ṭa*, then it is *ta-ṭa*. When the hypotenuse is *ca-ṅa*, then the lateral (*koṭi*) is *ja-ṅa*. If it were to be asked what [the lateral] is when the hypotenuse is *cha-jha*, then it is *cha-ta*.

या कोटियुग्मस्य युतिस्तु सैव ज्या चापयोगस्य छटाभिधा ज्या ।
चापैक्यखण्डस्य तु सिञ्जिनी या तत्रोपपत्तिर्गदिता पुरैव ॥ ४८ ॥ ॥ इन्द्रवज्रा ॥

yā koṭiyugmasya yutis tu saiva jyā cāpayogasya chaṭābhidhā jyā |

> *cāpaikyakhaṇḍasya tu siñjinī yā tatropapattir gaditā puraiva* || *48* || || *indravajrā* ||
> Whatever is the sum of the two laterals (*koṭi*), that indeed gives the Sine of the sum of the [two] arcs, called *cha-ṭa*. And the rationale (*upapatti*) for finding the Chord (*siñjinī*) corresponding to the sum of the two arcs has already been described.

Technical analysis

In section four, Nityānanda sets out the following rule for the Sine of the sum of two arcs in verse 41 in *ārya* meter:

$$\mathrm{Sin}(\theta+\phi)=\frac{\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\phi}{R}+\frac{\mathrm{Cos}\,\theta\,\mathrm{Sin}\,\phi}{R}. \tag{30}$$

In the subsequent six verses, he demonstrates diagrammatically how this relation can be arrived at by simply considering the similarity of triangles involved in the construction and the application of the rule-of-three. What is interesting to note here is Nityānanda's detailed description of how to construct a lettered diagram. As far as we are aware, this is the first Sanskrit astronomical treatise that gives such a prescription in the form of verses as to how to go about constructing geometrical figures with appropriate legends that facilitate the identification of various geometrical objects, directed at proving the result.

As for the prescription, a circle is to be drawn with a compass and four labels given, *ka*, *kha*, *ga*, *gha*.[17] These four consonants have to be placed at the four cardinal directions as per the convention. Although Nityānanda has not explicitly mentioned it here, it can be inferred from the description found in verse 20. The center is to be labeled *ṅa*. Then arcs and their related Sines are identified by lettered points.

The diagram as it appears in one of the manuscripts can be seen in Figure 6.[18] A better version to facilitate comprehension is given in Figure 7. In this figure, *ka-ca* and *ca-cha* represent the two arcs, the sum of whose Sine is to be determined. For convenience, we have denoted these two arcs to be subtending an angle θ and ϕ

[17] These are the first four consonants in the Sanskrit alphabet. We have opted to retain these as the lettered points in our technical analysis so that there is continuity between that and the translation. In general, letters with a diacritical dot underneath are the retroflex phonetic value. *ṅa* is the guttural nasal.

[18] In this diagram, the scribe has marked both *ca-ja* and *jha-ṭa* with the letter *bhu*, an abbreviation for the Sanskrit word *bhujā* ("Sine"). While the former is indeed a *bhuja*, the latter is not. This is clearly a scribal error, which is evident from the description in the text. It also appears that the base, *cha-ṭa*, of the triangle *cha-ṭa-jha* has been described as *pṛthvī* ("base") which is a common technical term in mathematics to describe the base of a triangle. There is one additional problem with the diagram. The scribe has imperfectly rendered the letter *ṭa*, making it look more like a *ṭha*. However the mathematical verses make it clear that the reading *ṭa* is the correct one.

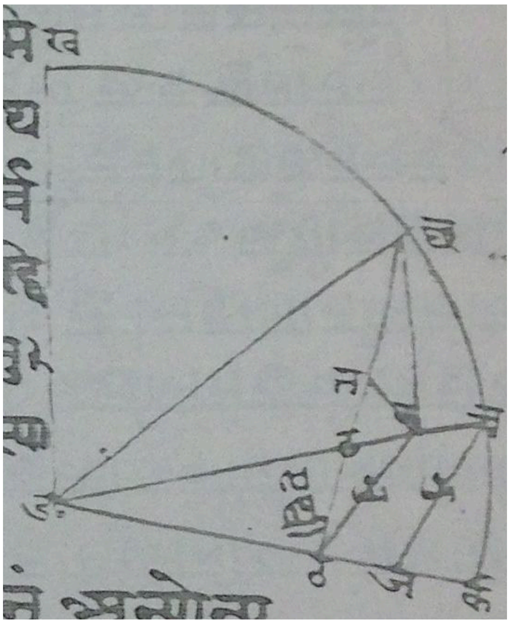

Figure 6: The rendering of the diagram in manuscript R (f. 12r) for the Sine of the sums of arcs.

respectively at the center of the circle, denoted by *ṅa*. It is evident from the triangles, *ca-ṅa-ja* and *cha-ṅa-jha* that *ca-ja* is $\mathrm{Sin}\,\theta$ and *cha-jha* is $\mathrm{Sin}\,\phi$. The Sine of the sum of the two arcs, $\mathrm{Sin}(\theta+\phi)$, is *cha-ṭa*. Since

$$\textit{cha-ṭa} = \textit{cha-ta} + \textit{ta-ṭa}, \tag{31}$$

in a sense then, the proof of Equation (30) rests on showing that the two line segments appearing in the right hand side of Equation (31) correspond to the two terms in the right hand side of Equation (30).

Nityānanda next draws our attention to the oblique triangle *jha-ṭa-cha* (verse 45) and then asks us to draw *jha-ta* such that it is perpendicular to *cha-ṭa*. Two right-angled triangles, *jha-ṭa-ta* and *jha-cha-ta* are formed, which will be made use of in identifying similar triangles. The one crucial equality that the proof of Equation (31) rests on is that

$$\textit{jha-ṭa} = \textit{ca-ja}, \tag{32}$$

which is stated in verse 44 but not proved at this point by Nityānanda. Although the text does not explicitly show in this section how the measure of *jha-ṭa* is equal to *ca-ja* ($= \mathrm{Sin}\,\theta$), it can be easily understood as follows. The triangles *cha-ṅa-jha*

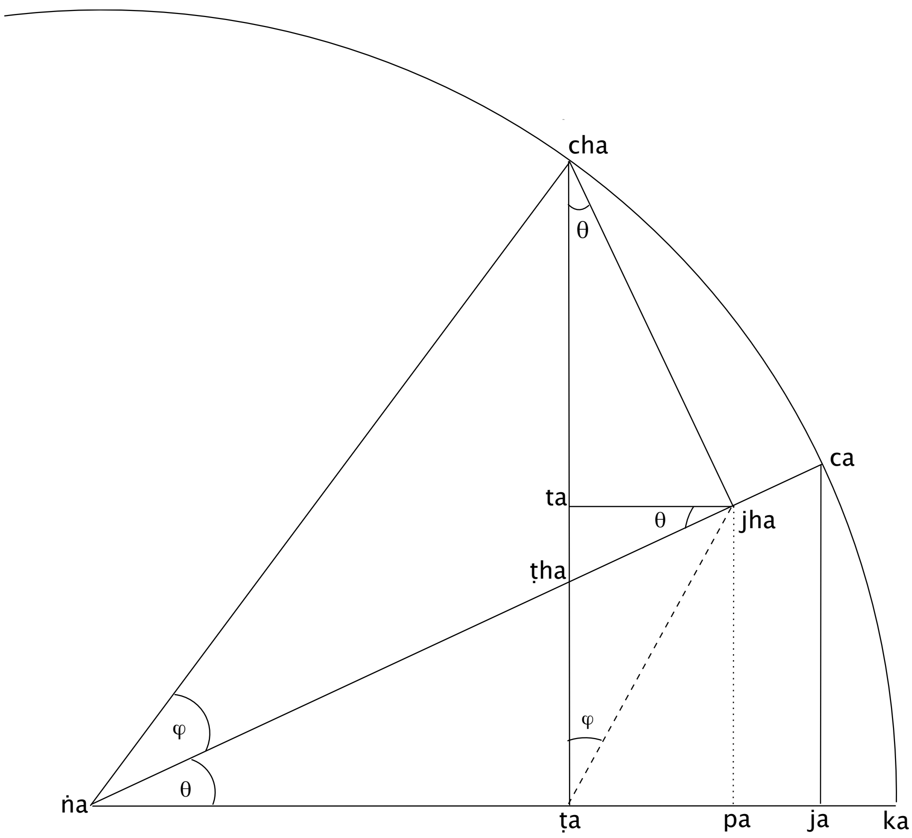

Figure 7: The geometrical construction for finding the Sine of the sum of the two arcs. Here it may be noted:

ca-ja = *jha-ṭa*	$\mathrm{Sin}\,\theta$	*ṅa-ja*	$\mathrm{Cos}\,\theta$
cha-jha	$\mathrm{Sin}\,\phi$	*jha-ṅa*	$\mathrm{Cos}\,\phi$
cha-ṅa = *ṅa-ca*	R	*cha-ṭa*	$\mathrm{Sin}(\theta+\phi)$
jha-pa = *ta-ṭa*	$\frac{\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\phi}{R}$	*jha-ta*	$\frac{\mathrm{Sin}\,\theta\,\mathrm{Sin}\,\phi}{R}$

and *jha-ṭa-pa* are similar.[19] Hence,

$$\frac{\textit{jha-pa}}{\textit{ca-ja}} = \frac{\textit{jha-ṅa}}{\textit{cha-ṅa}}, \tag{33}$$

or

$$\textit{jha-pa} = \frac{\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\phi}{R}. \tag{34}$$

Then, since *jha-pa* is equal to *ta-ṭa* it straightaway follows that the angle *ta-ṭa-jha* is equal to ϕ. Hence, *jha-ta* is $\frac{\mathrm{Sin}\,\theta\,\mathrm{Sin}\,\phi}{R}$. Now considering the triangle *jha-ta-ṭa* and

[19] We have taken the liberty of added the point *pa*, though it has not been given by Nityānanda, such that *jha-pa* is equal to *ta-ṭa* and its measure is more obviously derived from similar triangles.

applying the Pythagorean theorem:

$$\begin{aligned}
\textit{jha-ṭa} &= \sqrt{(\textit{jha-ta})^2 + (\textit{ta-ṭa})^2} \\
&= \sqrt{(\textit{jha-ta})^2 + (\textit{jha-pa})^2} \\
&= \sqrt{\left(\frac{\operatorname{Sin}\theta \operatorname{Sin}\phi}{R}\right)^2 + \left(\frac{\operatorname{Sin}\theta \operatorname{Cos}\phi}{R}\right)^2} \\
&= \sqrt{\frac{\operatorname{Sin}^2\theta(\operatorname{Cos}^2\phi + \operatorname{Sin}^2\phi)}{R^2}} \\
&= \operatorname{Sin}\theta. \qquad (35)
\end{aligned}$$

This equality is also demonstrated by means of a geometrical construction a little later (see section III.7).

In verses 46-47, Nityānanda identifies pairs of similar triangles, and expresses the appropriate rule-of-threes for each one to find the desired segments, *ta-ṭa* and *cha-ta*. Firstly, he states triangle *jha-ṭa-ta* is similar to triangle *cha-ṅa-jha*.[20] This leads to the following rule-of-three between the hypotenuses (*karṇa*) and the laterals (*koṭi*)

tryaśra	*karṇa*		*koṭi*
cha-ṅa-jha	*cha-ṅa*	::	*jha-ṅa*
jha-ṭa-ta	*jha-ṭa*	::	*ta-ṭa*

so that the desired lateral, *ta-ṭa* can be expressed as:

$$\begin{aligned}
\textit{ta-ṭa} &= \frac{\textit{jha-ṅa} \times \textit{jha-ṭa}}{\textit{cha-ṅa}} \\
&= \frac{\operatorname{Cos}\phi \times \operatorname{Sin}\theta}{R}. \qquad (36)
\end{aligned}$$

Then, Nityānanda states *cha-jha-ta* is similar to *ṅa-ca-ja*.[21] This leads to the following rule-of-three (*anupāta*) between the hypotenuses (*karṇa*) and the laterals (*koṭi*) of these triangles, namely

tryaśra	*karṇa*		*koṭi*
ṅa-ca-ja	*ṅa-ca*	::	*ṅa-ja*
cha-jha-ta	*cha-jha*	::	*cha-ta*

[20] As discussed above, this is because angle ϕ is common to both triangles. Without making any assumptions on the length of the segment *jha-ṭa* or the angle which the segment makes with *cha-ṭa*, considering the two triangles *ca-ṅa-ja* and *jha-ṅa-pa*, and applying the rule-of-three, it is easy to see that *jha-pa* is equal to $\frac{R\sin\theta R\cos\phi}{R}$. Similarly, it can be seen from the triangle *cha-jha-ta* that *jha-ta* is equal to $\frac{R\sin\theta R\sin\phi}{R}$. Since the two sides are products of $\sin\phi$ and $\cos\phi$ with the same multiplicative factor, the angle that *jha-ṭa* makes with *ṭa-ta* has to be ϕ.

[21] This is because angle θ is common to both triangles. He also indicates that one more triangle *jha-ṭha-ta* is similar to *cha-jha-ta*, which is not made use of in the derivation.

so that the desired lateral, *cha-ta* can be expressed as:

$$\begin{aligned} \textit{cha-ta} &= \frac{\textit{ṅa-ja} \cdot \textit{cha-jha}}{\textit{ṅa-ca}} \\ &= \frac{\mathrm{Cos}\,\theta \cdot \mathrm{Sin}\,\phi}{R}. \end{aligned} \tag{37}$$

The sum of Equations (36) and (37) yields the desired result (the Sine of the sum of two arcs):

$$\begin{aligned} \textit{cha-ṭa} &= \textit{cha-ta} + \textit{ta-ṭa} \\ \mathrm{Sin}(\theta+\phi) &= \frac{\mathrm{Sin}\,\phi\,\mathrm{Cos}\,\theta}{R} + \frac{\mathrm{Cos}\,\phi\,\mathrm{Sin}\,\theta}{R} \end{aligned} \tag{38}$$

as required.

III.6 Section Five: Determining the Sine of the Difference of Two Arcs

Text and translation

अथ पञ्चमप्रकारे धनुर्द्वयस्य भुजज्ययोर्ज्ञाने सति चापान्तरज्याज्ञानम् ।
atha pañcamaprakāre dhanurdvayasya bhujajyayor jñāne sati cāpāntarajyājñānam |
Now, in the fifth section, when the Sines of two [independent] arcs are known, [the procedure to obtain] the knowledge of the Sine of the difference of the [two] arcs [is described].

अन्योन्यकोटिमौर्व्या गुणिते ये चेष्टचापयोर्दोर्ज्ये ।
त्रिज्योद्धृते तयोर्या वियुतिः सा चापविवरज्या ॥ ४९ ॥ ॥ आर्या ॥
anyonyakoṭimaurvyā guṇite ye ceṣṭacāpayor dorjye |
trijyoddhṛte tayor yā viyutiḥ sā cāpavivarajyā || *49* || || *āryā* ||
The Sines of the two desired arcs are mutually multiplied by the Cosines of the other arcs; when divided by the Radius, whatever is the difference of the two, that is the Sine of the difference of the [two] arcs.

अत्रोपपत्तिः ।
atropapattiḥ |
Here, the demonstration.

बृहद्धनुः कंछमितं छचं लघु
छटं महाचापगुणं चझं लघोः ।
प्रकल्प्य तद्वन्मितिमांश्चजं लिखेत्
धनुर्द्वयान्तर्गतचापसिञ्जिनीम् ॥ ५० ॥ ॥ वंशस्थम् ॥
bṛhaddhanuḥ kaṃchamitaṃ chacaṃ laghu
chaṭaṃ mahācāpaguṇaṃ cajhaṃ laghoḥ |

prakalpya tadvan matimāṃś cajaṃ likhet
dhanurdvayāntargatacāpasiñjinīm || 50 || || *vaṃśastham* ||

The measure of the large arc is *ka-cha* and the short one is *cha-ca*. Having set out *cha-ṭa*, the Sine (*guṇa*) of the large arc, and *ca-jha*, the one corresponding to the smaller [arc], may the intelligent one in a similar manner draw *ca-ja* which is the Sine of the arc that lies inside (i.e., is the difference of) the two arcs.

छटं सदा जाझमितं विचिन्तयेत्
झथं परं लम्बकमानयेत् कङे ॥
जथाझसंज्ञं त्रिभुजं यथा तथा
झचाङसंज्ञं परिचिन्तयेद्बुधः ॥ ५१ ॥ ॥ वंशस्थम् ॥

chaṭaṃ sadā jājhamitaṃ vicintayet
jhathaṃ paraṃ lambakam ānayet kaṅe ||
jathājhasaṃjñaṃ tribhujaṃ yathā tathā
jhacāṅasaṃjñaṃ paricintayed budhaḥ || 51 || || *vaṃśastham* ||

May one conceive of *cha-ṭa* as always equal in measure with *ja-jha*. Then, may one drop the perpendicular *jha-tha* onto *ka-ṅa*. May the intelligent one conceive of the triangle denoted by *ja-tha-jha* to be similar to *jha-ca-ṅa*.

कर्णे चङे यदि झङप्रमितास्ति कोटिः
झाजश्रुतौ भवति कोटिरियं तदा किम् ।
एवं भवेत् झथमितं किल सूत्रकं हि
त्रैराशिकेन च वदामि झतप्रमाणम् ॥ ५२ ॥ ॥ वसन्ततिलका ॥

karṇe caṅe yadi jhaṅapramitāsti koṭiḥ
jhājaśrutau bhavati koṭir iyaṃ tadā kim |
evaṃ bhavet jhathamitaṃ kila sūtrakaṃ hi
trairāśikena ca vadāmi jhatapramāṇam || 52 || || *vasantatilakā* ||

When *ca-ṅa* is the hypotenuse, then the lateral (*koṭi*) is *jha-ṅa*. When the hypotenuse is *jha-ja*, what then is this lateral? The string whose measure is *jha-tha* will be indeed be [the lateral]. And [now], by means of a rule-of-three, I state the measure *jha-ta*.

त्र्यस्रं झचातं छटङोपमानं
किं वा झथाङप्रमितं विचिन्त्यम् ।
छङश्रुतौ चेट्टङकोटिमानं
चझश्रुतौ कास्ति झतं तदानीम् ॥ ५३ ॥ ॥ रामा उपजातिका ॥

tryasraṃ jhacātaṃ chaṭaṅopamānaṃ
kiṃ vā jhathāṅapramitaṃ vicintyam |
chaṅaśrutau ceṭ ṭaṅakoṭimānaṃ
cajhaśrutau kāsti jhataṃ tadānīm || 53 || || *rāmā upajātikā* ||

The triangle *jha-ca-ta* is similar to *cha-ṭa-ṅa*, and[22] what that would be with respect to *jha-tha-ṅa* is to be thought over. When the hypotenuse is *cha-ṅa*, the measure of the lateral (*koṭi*) is *ṭa-ṅa*. If it were to be asked what is the lateral when the hypotenuse is *ca-jha*, then it is *jha-ta*.

झतोनितं चेज्झथसंज्ञसूत्रं तदा भवेच्चाजसमं सदैव ।
एषैव चापान्तरमानजीवा पूर्वैर्निरुक्ता गणकप्रवीणैः ॥ ५४ ॥ ॥ माला उपजातिका ॥

jhatonitaṃ cej jhathasaṃjñasūtraṃ tadā bhavec cājasamaṃ sadaiva |
eṣaiva cāpāntaramānajīvā pūrvair niruktā gaṇakapravīṇaiḥ || 54 || || mālā upajātikā ||

When the segment denoted by *jha-tha* is decreased by *jha-ta*, then [the difference] is always equal to *ca-ja*. This indeed is stated to be the Sine of the measure of the difference of the [two] arcs by the mathematical experts (*gaṇakapravīṇa*) of the past.

Technical analysis

Nityānanda sets out the following rule for the Sine of the difference of two arcs:

$$\mathrm{Sin}(\theta-\phi)=\frac{\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\phi}{R}-\frac{\mathrm{Cos}\,\theta\,\mathrm{Sin}\,\phi}{R}, \tag{39}$$

again in a single verse, as was done in the case of the sum of the two arcs. It is noteworthy that verses 41 and 49 are almost identical but for the word *yoga* ("sum") in the former verse replaced by the word *viyuti*/*vivara* ("difference") in the latter.[23]

The diagram as it appears in one of the manuscripts can be seen in Figure 8. A better version to facilitate comprehension is given in Figure 9. In this figure, *ka-cha* and *ca-cha* represent the two arcs, the Sine of whose difference is to be determined. For convenience, we have denoted these two arcs to be subtending an angle θ and ϕ respectively at the center of the circle, denoted by *ṅa*. It is evident from the triangles, *cha-ṅa-ṭa* and *ca-ṅa-jha* that *cha-ṭa* is $\mathrm{Sin}\,\theta$ and *ca-jha* is $\mathrm{Sin}\,\phi$. The Sine of the difference of the two arcs, $\mathrm{Sin}(\theta-\phi)$, is *ca-ja*. Since,

$$\textit{ca-ja} = \textit{jha-tha} - \textit{jha-ta}, \tag{40}$$

in a sense then, the proof of Equation (39) rests on showing that the two segments appearing in right hand side of Equation (40) correspond to the two terms in the right hand side of Equation (39).

[22] *vā*, 'or' is translated as its less common meaning 'and.' Such uses made to satisfy metrical requirements are generally clarified in the commentaries.

[23] This of course should be of no surprise, since the only change that needs to be done in the formula is to replace the operation of addition by subtraction.

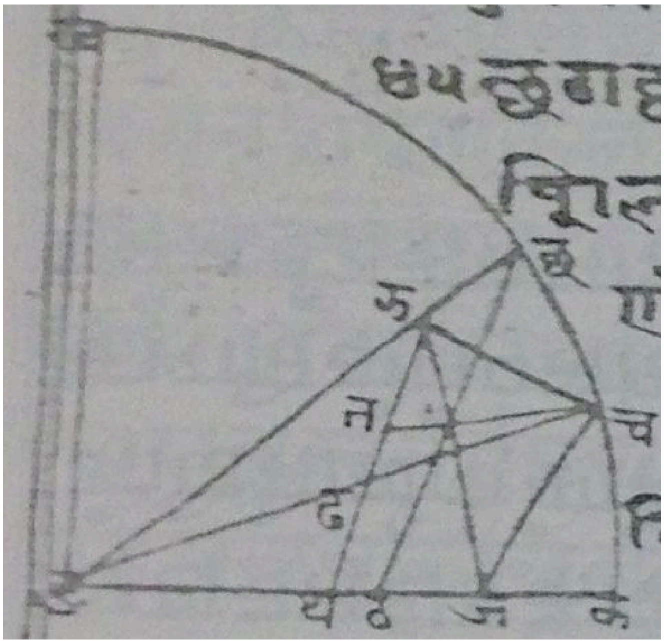

Figure 8: The rendering of the diagram in manuscript R (f. 12v) for the Sine of the difference of arcs.

As with the Sine of the sum formula, one crucial equivalence that this proof rests on is that

$$\textit{ja-jha} = \textit{cha-ṭa}, \tag{41}$$

which is stated in verse 51 but not proved at this point by Nityānanda. However, the equivalence can be understood as follows. Consider the triangle *jha-ca-ta*.[24] In this triangle, the hypotenuse *ca-jha* is equal to $\mathrm{Sin}\,\phi$ and the angle *ca-jha-ta* is equal to θ. Now the projection of the hypotenuse along the horizontal is given by

$$\textit{ca-ta} = \mathrm{Sin}\,\phi \sin\theta = \frac{\mathrm{Sin}\,\phi\,\mathrm{Sin}\,\theta}{R}. \tag{42}$$

Considering the triangle, *jha-ṅa-ṭa*, since *jha-ṅa*= $\mathrm{Cos}\,\phi$, its projection along the vertical is given by

$$\textit{jha-tha} = \mathrm{Cos}\,\phi \sin\theta = \frac{\mathrm{Cos}\,\phi\,\mathrm{Sin}\,\theta}{R}. \tag{43}$$

Since, *ca-ta* is equal to *ja-tha*, in the triangle *jha-tha-ja*, the two sides are known.

[24] It may be mentioned here that though Nityānanda meticulously takes care to prescribe how the various points have to be identified in the geometrical construction, with reference to the point *ta* he has not explicitly mentioned where it has to be marked. However it is implicit form the prescription given in verse 52.

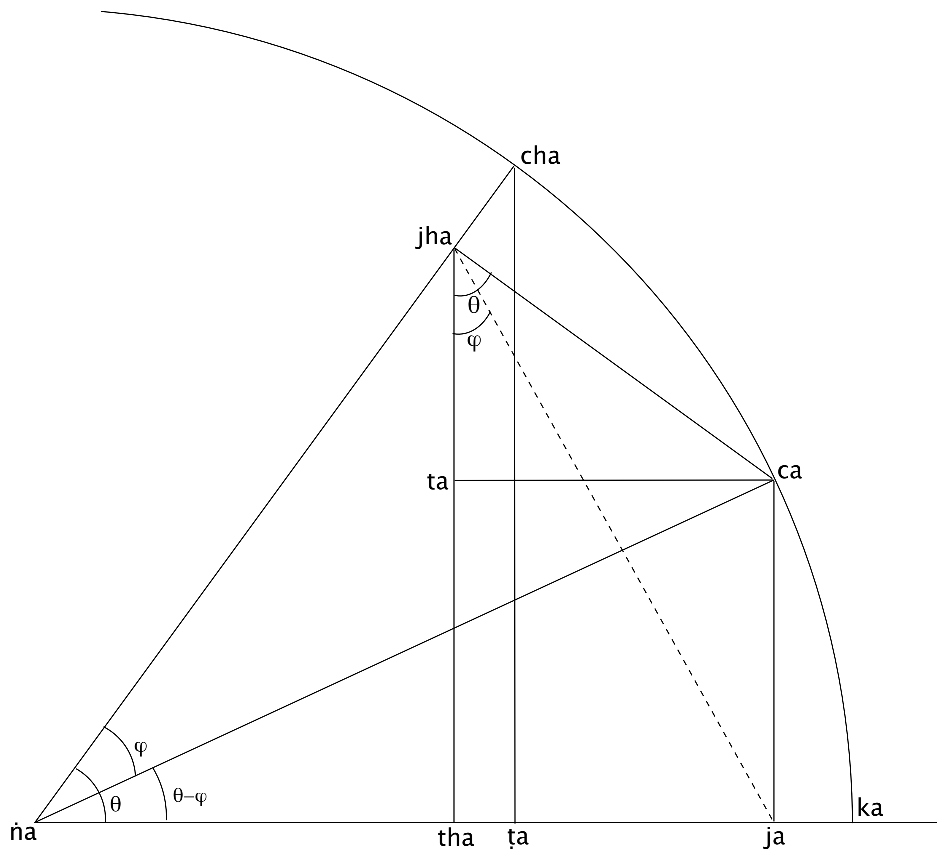

Figure 9: The geometrical construction for finding the Sine of the difference of the two arcs. Here it may be noted:

cha-ṭa = *ja-jha*	$\mathrm{Sin}\,\theta$	*ṭa-ṅa*	$\mathrm{Cos}\,\theta$
ca-jha	$\mathrm{Sin}\,\phi$	*jha-ṅa*	$\mathrm{Cos}\,\phi$
cha-ṅa = *ca-ṅa*	R	*ca-ja*	$\mathrm{Sin}(\theta-\phi)$
tha-ja = *ca-ta*	$\frac{\mathrm{Sin}\,\theta\,\mathrm{Sin}\,\phi}{R}$	*jha-tha*	$\frac{\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\phi}{R}$

Using Equations (42) and (43), and applying the Pythagorean theorem:

$$\begin{aligned}
\textit{jha-ja} &= \sqrt{(\textit{jha-tha})^2 + (\textit{tha-ja})^2} \\
&= \sqrt{(\textit{jha-tha})^2 + (\textit{ca-ta})^2} \\
&= \sqrt{\left(\frac{\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\phi}{R}\right)^2 + \left(\frac{\mathrm{Sin}\,\theta\,\mathrm{Sin}\,\phi}{R}\right)^2} \\
&= \sqrt{\frac{\mathrm{Sin}^2\,\theta(\mathrm{Cos}^2\,\phi + \mathrm{Sin}^2\,\phi)}{R^2}} \\
&= \mathrm{Sin}\,\theta. \qquad (44)
\end{aligned}$$

This equivalence shown here analytically, is carefully demonstrated by means of an interesting geometrical construction a little later in the text itself (see section III.7).

In verses 52–54, Nityānanda identifies pairs of similar triangles, and expresses the appropriate rule-of-threes for each one to find the desired segments, *jha-tha* and *jha-ta*. Firstly, he states triangle *ja-tha-jha* is similar to triangle *ca-jha-ṅa*.[25] This leads to the following rule-of-three between the hypotenuses (*karṇa*) and the laterals (*koṭi*)

tryaśra	*karṇa*		*koṭi*
ca-jha-ṅa	*ca-ṅa*	::	*jha-ṅa*
ja-tha-jha	*jha-ja*	::	*jha-tha*

so that the desired lateral, *jha-tha* can be expressed as:

$$\begin{aligned} \textit{jha-tha} &= \frac{\textit{jha-ṅa} \times \textit{jha-ja}}{\textit{ca-ṅa}} \\ &= \frac{\operatorname{Cos}\phi \times \operatorname{Sin}\theta}{R}. \end{aligned} \tag{45}$$

Then Nityānanda states *ca-jha-ta* is similar to *ṅa-cha-ṭa*. He also points out that the triangle *jha-tha-ṅa* is also similar to the triangles mentioned above. This leads to the following rule-of-three between the hypotenuses (*karṇa*) and the laterals (*koṭi*) of the first two of these triangles, namely:

tryaśra	*karṇa*		*koṭi*
ṅa-cha-ṭa	*ṅa-cha*	::	*ṅa-ṭa*
ca-jha-ta	*ca-jha*	::	*jha-ta*

so that the desired lateral, *jha-ta* can be expressed as:

$$\begin{aligned} \textit{jha-ta} &= \frac{\textit{ṅa-ṭa} \times \textit{ca-jha}}{\textit{ṅa-cha}} \\ &= \frac{\operatorname{Cos}\theta \cdot \operatorname{Sin}\phi}{R}. \end{aligned} \tag{46}$$

The difference of Equations (45) and (46) yields the desired result (the Sine of the difference of two arcs):

$$\begin{aligned} \textit{ca-ja} &= \textit{jha-tha} - \textit{jha-ta} \\ \operatorname{Sin}(\theta - \phi) &= \frac{\operatorname{Cos}\phi \operatorname{Sin}\theta}{R} - \frac{\operatorname{Sin}\phi \operatorname{Cos}\theta}{R} \end{aligned} \tag{47}$$

as required.

---

[25] As discussed above, this is because angle ϕ is common to both triangles. Without making any assumptions on the length of the segment *jha-ja* or the angle which the segment makes with *jha-tha*, considering the two triangles *cha-ṅa-ṭa* and *jha-ṅa-tha*, and applying the rule-of-three, it is easy to see that *jha-tha* is equal to $\frac{R\sin\theta R\cos\phi}{R}$. Similarly, it can be seen from the triangle *jha-ca-ta* that *ca-ta* (=*ja-tha*) is equal to $\frac{R\sin\theta R\sin\phi}{R}$. Since the two sides are products of $\sin\phi$ and $\cos\phi$ with the same multiplicative factor, the angle that *jha-ja* makes with *jha-tha* has to be ϕ.

III.7 Demonstration of Equivalences by Geometrical Construction

Text and translation

अथ प्रसङ्गात् किञ्चिच्छिल्पकथनम् ।

atha prasaṅgāt kiñcicchilpakathanam |

Now, in this context, a bit of description of geometrical constructions (*śilpa*) [are described].

कर्कटकेन जटाभ्यां तिमिना तुल्यत्रिबाहुकं कृत्वा ।
बहिरिह कोणे धृत्वा गं गाज्जटदिक्प्रसारिते रेखे ॥ ५५ ॥ ॥ गीति ॥

karkaṭakena jaṭābhyāṃ timinā tulyatribāhukaṃ kṛtvā |
bahir iha koṇe dhṛtvā gaṃ gāj jaṭadik prasārite rekhe || 55 || || *gīti* ||

By means of a compass, with points *ja* and *ṭa* [as centers], by means of a fish-figure,[26] having constructed an equilateral triangle (*tulyatribāhuka*), having placed [the letter] *ga* here at the exterior vertex, [may one draw] two lines extended from *ga* in the direction of *ja* and *ṭa*.

जचव्यासार्धमानेन वृत्तं यत्र स्पृशेत् क्वचित् ।
रेखां प्रसारितां तत्र डकारं विलिखेद्बुधः ॥ ५६ ॥ ॥ श्लोक॥
गडविस्तारखण्डेन मण्डलं साधयेत् परम् ।
तच्चापि संस्पृशेद्यत्र रेखामन्यां प्रसारिताम् ॥ ५७ ॥ ॥ श्लोक॥
तत्र विन्यस्य दं वर्णं टदव्यासार्धजं पुनः ।
वृत्तं कुर्यात् ततष्टादं टाझं जाचं जडं समम् ॥ ५८ ॥ ॥ श्लोक॥

jacavyāsārdhamānena vṛttaṃ yatra spṛśet kvacit |
rekhāṃ prasāritāṃ tatra ḍakāraṃ vilikhed budhaḥ || 56 || || *śloka* ||
gaḍavistārakhaṇḍena maṇḍalaṃ sādhayet param |
tac cāpi saṃspṛśed yatra rekhām anyāṃ prasāritām || 57 || || *śloka* ||
tatra vinyasya daṃ varṇaṃ ṭadavyāsārdhajaṃ punaḥ |
vṛttaṃ kuryāt tataṣ ṭādaṃ ṭājhaṃ jācaṃ jaḍaṃ samam || 58 || || *śloka* ||

Wherever the circle, [drawn with center *ja* and] with radius *ja-ca*, touches the extended line (i.e., the one through *ja*), may one write there (i.e., at that intersection of the circle and the straight line) the letter *ḍa*. Then, [with *ga* as center and] with radius (*vistārakhaṇḍa*) *ga-ḍa*, may the intelligent one (*budha*) draw a circle, and wherever it touches the other extended line, there having placed the letter *da*, once again may one draw a circle with *ṭa-da* as radius. Thence, [it may be noted that] *ṭa-da* *ṭa-jha*, *ja-ca* and *ja-ḍa* are equal [in measure].

[26] The phrase 'fish-figure' refers to the fish-like shape produced from a pair of arcs intersecting at two points, creating, in effect, an elongated almond-shaped body, and a fin-like tail emerging at either end.

विवरज्यानयनार्थं टछरेखाख्यां जचस्थाने ।
कृत्वा टाडं जादं कुर्यादपरं पुरोक्तयैव ॥ ५९ ॥ ॥ उद्गीति ॥

vivarajyānayanārthaṃ ṭacharekhākhyāṃ jacasthāne |
kṛtvā ṭāḍaṃ jādaṃ kuryād aparaṃ puroktyaiva || 59 || || udgīti ||

For the sake of computing the Sine of the difference [of two arcs], having substituted the segment *ja-ca* with *ṭa-cha*, one should draw [circles with radii] *ṭa-ḍa* and *ja-da*. The rest [is to be completed] as described earlier.

अत्रापि चत्वारि सूत्राणि जदं जझं टछं टडं तुल्यानि ज्ञेयानि ।

atrāpi catvāri sūtrāṇi jadaṃ jajhaṃ ṭachaṃ ṭaḍaṃ tulyāni jñeyāni |

Here also, the equivalence of the four segments *ja-da*, *ja-jha*, *ṭa-cha* and *ṭa-ḍa* is to be understood.

Technical analysis

In this section, Nityānanda presents two geometrical constructions in order to demonstrate Equations (32) and (41), which were taken to be true during the proofs for the Sine of sum and difference formulae discussed in sections four and five. In our technical analysis, we showed how these results can be arrived at analytically. Here, Nityānanda presents a novel geometrical construction by which one can convince oneself diagrammatically that in the former case the measure of *jha-ṭa* is equal to *ca-ja* and in the latter case that *cha-ṭa* is equal to *ja-jha* (see Figures 7 and 9).

In our attempt to understand this demonstration, the constructions that have been attempted by the scribe in RORI (see Figure 10) were found to be quite useful. Reproductions of them can be seen in Figures 11 and 12.

For the Sine of the sum case, Nityānanda first instructs the reader to draw an equilateral triangle on the base *ja-ṭa*. This figure is generated by drawing arcs with *ja* and *ṭa* as center and radius equal to *ja-ṭa*. The point at which these arcs intersect outside the diagram is taken to be *ga*. Now, by construction, *ja-ṭa-ga* forms an equilateral triangle with *ga* as the external vertex of the triangle. Then, having constructed this triangle, Nityānanda instructs that the two oblique sides of this equilateral triangle, *ga-ja* and *ga-ṭa*, be sufficiently extended from *ga* along the directions *ja* and *ṭa*.

Next, one should draw a circle with *ja* as center and *ja-ca* as radius. This circle intersects the extension of *ga-ja* at *ḍa*. With *ga* as center and *ga-ḍa* as radius, an arc has to be drawn such that it intersects the extension of *ga-ṭa* at point *da* (only part of this circle is represented in Figure 11 by the dash-dot-dash arc). Then with *ṭa* as center and *ṭa-da* as radius another circle should be drawn. It will now be noted that this circle passes through the point *jha*. Since, by construction, *ja-ca* = *ṭa-da*, and since the circle with radius *ṭa-da* passes through the point *jha*, *ja-ca* = *jha-ṭa*, which was what was needed to be shown in Equation (32). It may be reiterated here

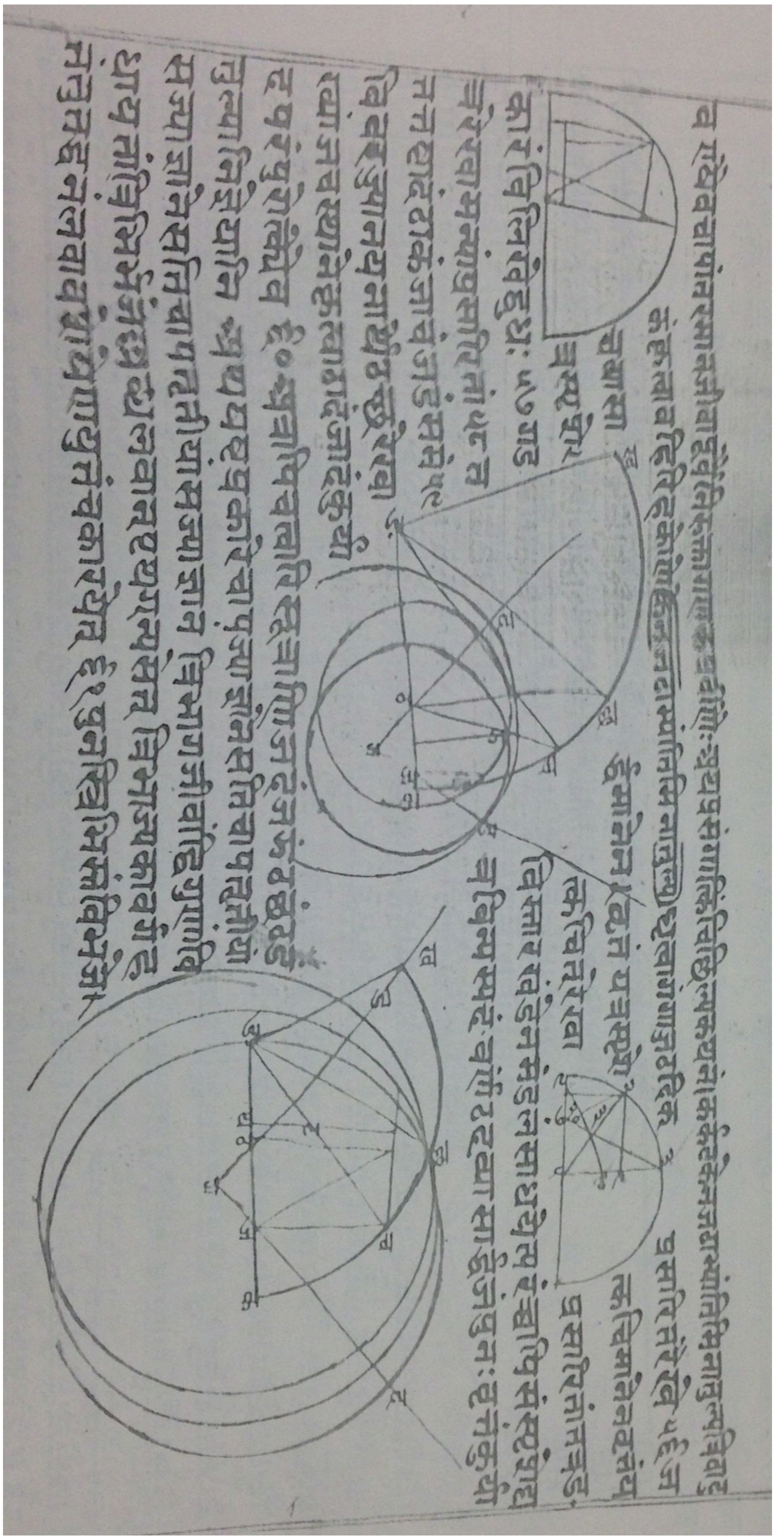

Figure 10: The rendering of the geometrical constructions in manuscript *R* (f. 13r) for showing the equivalence of sum and difference segments.

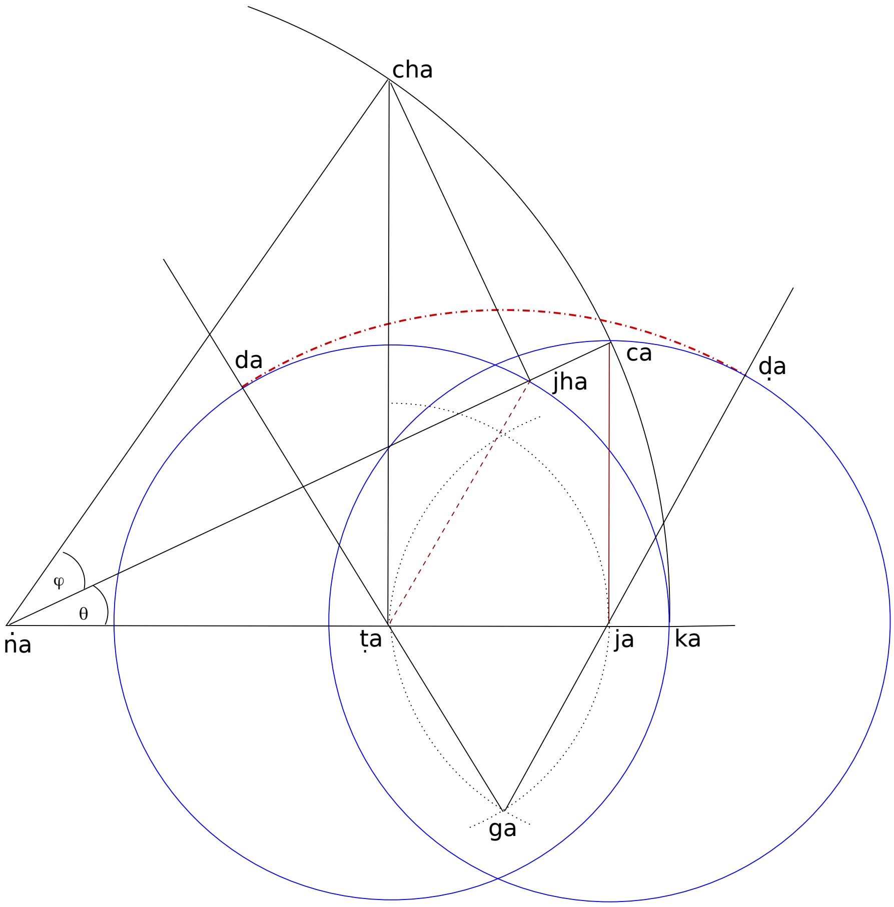

Figure 11: A reproduction of the geometrical construction to show that the measure of *ja-ca* is equal to that of *ṭa-jha*.

that this geometrical construction is meant only to demonstrate the equivalence of the two segments in question, which is not straightforward by similarity of triangles. However, Nityānanda as *śilpasiddham* in verse 44 refers to the fact that one can easily convince oneself of this equivalence by passing the compass through the point in question.

Having explained this construction in great detail for the Sines of sums, Nityānanda gives pointers as to how a similar geometrical construction can be used for the Sines of differences (verse 59). Here too we first construct the equilateral triangle *ga-ja-ṭa*, and then extend the lines *ga-ja* and *ga-ṭa*, as earlier. Now with *ṭa* as center and *ṭa-cha* as radius, we draw a circle which intersects the extension *ga-ṭa* at *ḍa*. Then with *ga* as center and *ga-ḍa* as radius, we draw an arc which intersects the extenion *ga-ja* at *da*. Next, with *ja* as center and *ja-da* as radius we draw a circle which intersects *cha-ṅa* at *jha*. Since, by construction, *ṭa-ḍa* = *ja-da*, and since the circle with radius *ja-da* passes through the point *jha*, *ja-jha* = *ṭa-cha*, which was what was needed to be shown in Equation (41).

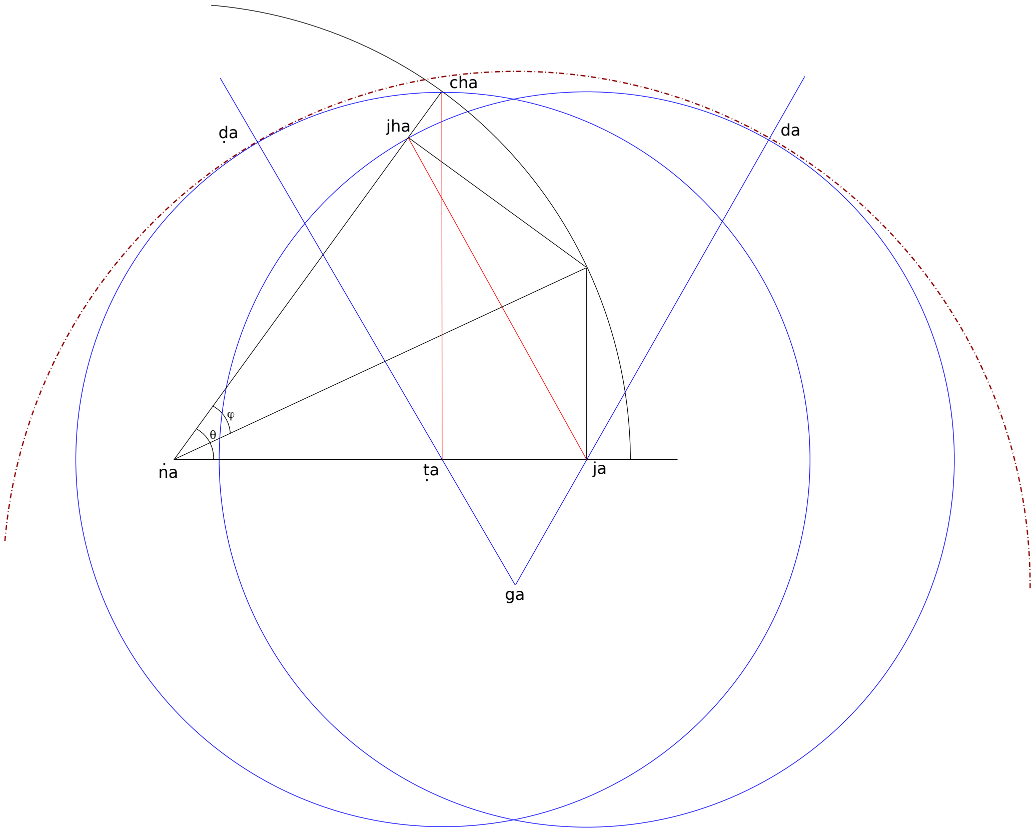

Figure 12: A reproduction of the geometrical construction to show that the measure of *ṭa*-*cha* is equal to that of *ja*-*jha*.

IV Appendix: Note on the Different Meters Employed by Nityānanda

In this appendix, we present a brief note on the different meters that have been employed by Nityānanda in the verses discussed in this paper. In classical Sanskrit prosody, there are two main metrical classes: the *mātrāvṛtta*s and the *varṇavṛtta*s. While *mātrāvṛtta*s are identified by the morae count, the *varṇavṛtta*s are identified by the syllable count. In both cases, a short syllable (*laghu*) is counted to be one *mātrā* and a long one (*guru*) two.

In what follows, we present a mnemonic (source unknown) that is useful to define, as well as identify, the different metrical patterns (*gaṇas*, that is a group of three syllables). We also denote it using the notation ∪ and −, representing short and long syllables respectively.[27]

य	मा	ता	रा	ज	भा	न	स	ल	गम्
ya	mā	tā	rā	ja	bhā	na	sa	la	gam
∪	−	−	−	∪	−	∪	∪	∪	−

This mnemonic is self-descriptive, in that the syllables of each *gaṇa* represent its own syllable length (*laghu* or *guru*) and the two immediately following it. Thus *ya–gaṇa* is ∪ − −, *mā–gaṇa* is − − −, and so on.

IV.1 Summary of the Meters Employed

In this paper, we have discussed the mathematical contents of 41 verses (Chapter 3, 19–59) of the *Sarvasiddhāntarāja*. These verses have been composed by Nityānanda using ten different meters. Table 4 below presents the names of the different meters that he used along with its type (*mātrāvṛtta* or *varṇavṛtta*), the number of verses composed in it, and so on.

IV.2 The *Mātrāvṛtta*s Used and their Definitions

The metrical compositions in Sanskrit consist of four quarters. If all the four quarters are identical they are known as *samavṛttas*. If only two are identical, they are called *ardhasamavṛtta*. If neither of the above, they are called *viṣamavṛtta*. We move on to provide the definitions of the three *mātrāvṛtta*s used by Nityānanda.

Āryā – This consists of 12, 18, 12, and 15 *mātrā*s in the four quarters respectively. It is defined in the following verse, composed in the same meter.[28]

यस्याः पादे प्रथमे द्वादशमात्रास्तथा तृतीयेऽपि ।

---

[27] In addition, the notation ⊻ indicates that the syllable can be either short or long.

[28] This is true of all the other definitions presented henceforth as well.

No.	Name of the meter	Type of the meter	Number of morae/syllables	Verse numbers
1	*Āryā*	*Mātrāvṛtta*	12, 18, 12, 15	41, 49
2	*Udgīti*	"	12, 15, 12, 18	59
3	*Gīti*	"	12, 18, 12, 18	55
4	*Śloka*	*Varṇavṛtta*	8	28, 29, 30, 56, 57, 58
5	*Indravajrā*	"	11	26, 31, 38, 39, 48
6	*Upajātikā*	"	11	19, 20, 21, 22, 23, 25, 27, 32, 34, 35, 36, 40, 44, 45, 46, 47, 53, 54
7	*Bhujaṅgaprayāta*	"	12	24
8	*Vaṃśastha*	"	12	42, 43, 50, 51
9	*Vasantatilakā*	"	14	52
10	*Śālinī*	"	11	33, 37

Table 4: Meters employed by Nityānanda in the trigonometry sections of the *Sarvasiddhāntarāja*.

अष्टादशद्वितीये चतुर्थके पञ्चदश सार्या ||
yasyāḥ pāde prathame dvādaśamātrās tathā tṛtīye 'pi |
aṣṭādaśadvitīye caturthake pañcadaśa sāryā || (*Śrutabodha* 1894, p. 2)

Gīti – This differs from *ārya* in one quarter only. Here, the number of *mātrās* in the fourth quarter is 18 instead of 15. Thus, *gīti* meter has 12, 18, 12, and 18 *mātrā*s in the four quarters respectively and its definition is:

आर्यापूर्वार्धसमं द्वितीयमपि भवति यत्र हंसगते |
छन्दोविदस्तदानीं गीतिं ताम् अमृतवाणि भाषन्ते ||
āryāpūrvārdhasamaṃ dvitīyam api bhavati yatra haṃsagate |
chandovidas tadānīṃ gītiṃ tām amṛtavāṇi bhāṣante || (*Śrutabodha* 1894, p. 3)

Udgīti – This is obtained by simply reversing the pattern of the first and second half of the *ārya*. Thus, *udgīti* has 12, 15, 12, and 18 *mātrā*s respectively in its four quarters. Its definition is:

आर्याशकलद्वितयं व्यत्ययरचितं भवेद् यस्याः |
सोद्गीतिः किल कथिता तद्वद्यत्यंशभेदसंयुक्ता ||
āryāśakaladvitayaṃ vyatyayaracitaṃ bhaved yasyāḥ |
sodgītiḥ kila kathitā tadvad yatyaṃśabhedasaṃyuktā || (*Vṛttaratnākara* 1894, p. 33)

IV.3 The *Varṇavṛtta*s Used and their Definitions

As indicated in the table above, Nityānanda has used seven different *varṇavṛtta*s. These are *śloka*, *indravajrā*, *upajātikā*, *bhujaṅgaprayāta*, *vaṃśastha*, *vasantatilakā*, and *śālinī*. Among them, *śloka* has 8 syllables per quarter, and *indravajrā*, *upajātikā*, and *śālinī* belong to the *triṣṭubh chandas* with 11 syllables. The *bhujaṅgaprayāta*, and *vaṃśastha* belong to the *jagatī chandas* with 12 syllables, and *vasantatilakā* belongs to the *śakvarī chandas* which has 14 syllables.

Śloka – This is the most generic type of meter, called *śloka*, in *anuṣṭubh chandas*. It is defined as:

> श्लोके षष्ठं गुरुज्ञेयं सर्वत्र लघुपञ्चमम् |
> द्विचतुष्पदयोर्ह्रस्वं सप्तमं दीर्घमन्ययोः ||
> *śloke ṣaṣṭhaṃ gurujñeyaṃ sarvatra laghupañcamaṃ* |
> *dvicatuṣpadayor hrasvaṃ saptamaṃ dīrghamanyayoḥ* || (*Śrutabodha* 1894, p. 5)

As per this definition, the general characteristic of *śloka* meter is:

- The fifth syllable in all the quarters has to be short and the sixth one has to be long.
- The seventh syllable in the second and the fourth quarters has to be short, whereas it has to be long in the other two quarters.
- There is no restriction on the nature of the other syllables.

Hence it may be represented as:

$$\underline{\cup}\ \underline{\cup}\ \underline{\cup}\ \underline{\cup}\ \cup\ -\ -\ \underline{\cup} \qquad \underline{\cup}\ \underline{\cup}\ \underline{\cup}\ \underline{\cup}\ \cup\ -\ \cup\ \underline{\cup}$$
$$\underline{\cup}\ \underline{\cup}\ \underline{\cup}\ \underline{\cup}\ \cup\ -\ -\ \underline{\cup} \qquad \underline{\cup}\ \underline{\cup}\ \underline{\cup}\ \underline{\cup}\ \cup\ -\ \cup\ \underline{\cup}$$

Indravajrā – This is in *triṣṭubh chandas* having 11 syllables per quarter. The 11 syllables are made up of two *ta–gaṇas*, one *ja–gaṇa*, and two *gurus*. It is defined as:

> स्यादिन्द्रवज्रा यदि तौ जगौ गः |
> *syād indravajrā yadi tau jagau gaḥ* | (*Vṛttaratnākara* 1894, p. 42)
> $- - \cup, - - \cup, \cup - \cup, - -$

Upajātikā – This is obtained by combining *indravajrā* and *upendravajrā*, both of which are in *triṣṭubh chandas*, having 11 syllables in each quarter. It is defined as:

अनन्तरोदीरितलक्ष्मभाजौ पादौ यदीयावुपजातयस्ताः |
इत्थं किलान्यास्वपि मिश्रितासु स्मरन्ति जातिष्विदमेव नाम ||

anantarodīritalakṣmabhājau pādau
yadīyāv upajātayas tāḥ |
itthaṃ kilānyāsv api miśritāsu
smaranti jātiṣvidam eva nāma || (*Vṛttaratnākara* 1894, pp. 42–43)

In this verse, the compound '*anantarodīritalakṣmabhājau*' means 'that which possesses the characteristic of the immediately preceding ones.' It is *indravajrā* (I) and *upendravajrā* (U) whose definition appears before this verse in *Vṛttaratnākara*. Since *upajātikā* can arise from any of the possible combinations of the two, there are 14 possibilities[29] that can be produced, as shown in the following table:

No.	Quarter				Name of the meter	Verse numbers
	1^{st}	2^{nd}	3^{rd}	4^{th}		
1	*U*	*I*	*I*	*I*	*Kīrti upajātikā*	23
2	*I*	*U*	*I*	*I*	*Vāṇī upajātikā*	34
3	*U*	*U*	*I*	*I*	*Mālā upajātikā*	44, 45, 54
4	*I*	*I*	*U*	*I*	*Sālā upajātikā*	22, 25
5	*U*	*I*	*U*	*I*	*Haṃsī upajātikā*	32
6	*I*	*U*	*U*	*I*	*Māyā upajātikā*	21, 47
7	*U*	*U*	*U*	*I*	*Jāyā upajātikā*	35
8	*I*	*I*	*I*	*U*	*Bālā upajātikā*	27, 36, 40
9	*U*	*I*	*I*	*U*	*Ādrā upajātikā*	-
10	*I*	*U*	*I*	*U*	*Bhadrā upajātikā*	20
11	*U*	*U*	*I*	*U*	*Premā upajātikā*	-
12	*I*	*I*	*U*	*U*	*Rāmā upajātikā*	19, 53
13	*U*	*I*	*U*	*U*	*Ṛddhi upajātikā*	-
14	*I*	*U*	*U*	*U*	*Buddhi upajātikā*	46

Śālinī – This meter too has 11 syllables in every quarter, each of which is made up of a *ma–gaṇa*, two *ta–gaṇas* followed by two *guru*s. It is defined as:

शालिन्युक्ता म्तौ तगौ गोऽब्धिलोकैः |

śālinyuktā mtau tagau go'bdhilokaiḥ | (*Vṛttaratnākara* 1894, p. 43)

− − −, − − ∪, − − ∪, − −

[29] The two special cases in which all the four quarters are either U or I have been left out from the table.

Bhujaṅgaprayāta – This meter has 12 syllables in every quarter, each of which is made up of four repeated occurrences of the *ya–gaṇa*. It is defined as:

भुजङ्गप्रयातं भवेद्यैश्चतुर्भिः ।
bhujaṅgaprayātaṃ bhaved yaiś caturbhiḥ | (*Vṛttaratnākara* 1894, pp. 44–45)
∪ – –, ∪ – –, ∪ – –, ∪ – –

Vaṃśastha – This meter has 12 syllables in every quarter, each of which is made up of a *ja–gaṇa*, a *ta–gaṇa*, a *ja–gaṇa*, and a *ra–gaṇa*. It is defined as:

जतौ तु वंशस्थमुदीरितं जरौ ।
jatau tu vaṃśastham udīritaṃ jarau | (*Vṛttaratnākara* 1894, p. 44)
∪ – ∪, – – ∪, ∪ – ∪, – ∪ –

Vasantatilakā – This meter has 14 syllables in every quarter, each of which is made up from a *ta–gaṇa*, a *bha–gaṇa*, two *ja–gaṇas*, and two *guru* syllables. It is the longest meter employed by Nityānanda in this section and it is used only once. It is defined as:

उक्ता वसन्ततिलका तभजा जगौ गः ।
uktā vasantatilakā tabhajā jagau gaḥ | (*Vṛttaratnākara* 1894, p. 47)
– – ∪, – ∪ ∪, ∪ – ∪, ∪ – ∪, – –

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References

Manuscripts and Sanskrit Source Works

Sarasvati Bhavana Library, Benares, (1963) 35741

Sarasvati Bhavana Library, Benares, (1963) 37079

Bhandarkar Oriental Research Institute 206 of A, 1883–1884

National Archives Nepal, Kathmandu, Microfilm Reel No. B $\frac{354}{15}$

Rajasthan Oriental Research Institute (Alwar), 2619

Wellcome Institute for the History of Medicine, Y550

Śrutabodha and *Vṛttaratnākara* with commentary *Subodha*. Nirṇayasāgara Press, Mumbai, 1894 (*Śaka* 1816).

The Siddhāntaśiromaṇi, a Treatise on Astronomy, edited by Śāstrī, Bāpū Deva, (1989). 2nd ed. rev. Gaṇapati Deva Śāstrī (*Kashi Sanskrit Series* 72). Chaukhambha Sanskrit Sansthan, Varanasi.

Modern Scholarship

Aaboe, A. 1954. "Al-Kāshī's iteration method for the determination of sin 1°," *Scripta Mathematica* 20, 24–29.

Gupta, R.C., 1976. "Sine of eighteen degrees in India up to the eighteenth century," *Indian Journal of History of Science* 11(1), 1–10.

Misra, A., 2016. *The Golādhyāya of Nityānanda's Sarvasiddhāntarāja: An examination of 'The Chapter on Spheres' in a seventeenth century text on Mathematical Astronomy*, PhD thesis, University of Canterbury.

Pingree, D., 1970–1994. *Census of the Exact Sciences in Sanskrit*, Series A, vols. 1–5. Philadelphia.

——— 1978. “Islamic astronomy in Sanskrit,” *Journal for the History of Arabic Science* 2, 315–330.

——— 1981. *Jyotiḥśāstra*. Wiesbaden.

——— 1996. “Indian reception of Muslim versions of Ptolemaic astronomy,” In Ragep, F.J., Ragep, S.P., Livesey, S., eds., *Tradition, Transmission, Transformation*, Leiden, pp. 285–346.

——— 2003. “The Sarvasiddhāntarāja of Nityānanda,” In Hogendijk, J.P., Sabra, A.I., eds., *The Enterprise of Science in Islam*, Cambridge, Mass., pp. 269–284.

Richmond, H.W., 1893. “A construction for a regular polygon of seventeen sides.” *The Quarterly Journal of Pure and Applied Mathematics* 26, 206–207.

Rosenfeld B.A., and J. P. Hogendijk. 2003. “A mathematical treatise written in the Samarqand observatory of Ulugh Beg”. *Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften* 15, 25-65.

SaKHYa, 2014. “The Turyayantraprakāśa of Bhūdhara: Chapters one to ten,” *SCIAMVS* 15, 3-55.

Van Brummelen, G., 2009. *The Mathematics of the Heavens and the Earth: The early history of trigonometry,* Princeton.

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