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The Sector Theorem Attributed to Menelaus

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The Sector Theorem is now generally known as the Menelaus Theorem. At first glance, it appears to be one of the few pieces of mathematics that we find both in Ptolemy and his sources. Unfortunately, the textual transmission of the theorem turns out to be quite involved and it has now become clear that we do not possess any version of it that can simply be taken as that which Menelaus wrote. Nevertheless, it is possible, by examining the textual dependencies of the theorem as it was transmitted by the Arabic mathematicians, to discern basic types of the theorem and to decide which one of these should be attributed to Menelaus.

This exercise allows us to discuss the relationship between Menelaus' and Ptolemy's treatment of the theorem, and to situate their differing approaches to this particular proposition within the broader context of their mathematical and scientific aims. The tendency of Greek mathematical authors to differentiate their texts by subject areas allows us to see that the theorem was intended for quite different purposes by Menelaus and Ptolemy.

The opinion put forward by Neugebauer [1975, 301] favors Menelaus as the author of the eponymous theorem.[1] Although some historians are not convinced by this position, no one since has cogently argued against it. An examination of Menelaus' version of the sector theorem and its function in his *Spherics*, however, shows that it is unlikely that he intended the sector theorem to be read as his original contri-

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[1]Before Neugebauer published *A History of Ancient Mathematical Astronomy*, it was common to hear doubts expressed as to Menelaus' authorship of the theorem. For example, Bulmer-Thomas [1974, 299] argues that it was known before Menelaus in the *Dictionary of Scientific Biography*. After 1975, however, it became more common for scholars to simply accept that Menelaus had written the theorem. Toomer [1984, 69, n. 84], for example, states this as a fact in his translation of the *Almagest*. Another example of the weight of Neugebauer's authority is shown in two studies carried out by Nadal and Brunet. Starting from the hypothesis that Hipparchus used a star globe to solve the problems encountered in writing his *Commentary on the Phenomena of Aratus*, they use statistical analysis to describe certain physical characteristics of the hypothetical globe [Nadal and Brunet 1984, 1989]. They do not, however, take into consideration the fact that there are numerous possibilities for Hipparchus' actual practice: various methods of calculation, multiple star globes, other modes of analog calculation and so forth.

bution. An examination of the astronomical evidence in the writings of Hipparchus and others gives support to the conclusion that the sector theorem was known to, and used by, Hipparchus.[2] Ptolemy's presentation of his spherical astronomy, for which the sector theorem is fundamental, gives evidence for an older tradition that relied exclusively on the sector theorem to solve just those problems that we know Hipparchus also solved using precise methods of computation.

We will see that the most likely story is that Menelaus found the sector theorem as a well known tool of predictive spherical astronomy and applied it to his own needs in the production of a new theory of advanced spherical trigonometry. These findings support other, independent reasons for believing that the trigonometric methods based on the sector theorem were available to Hipparchus [Sidoli 2004]. Once we understand Hipparchus' role in the history of the sector theorem, it becomes easier to understand Ptolemy's approach to spherical astronomy in books I, II and VIII of his *Almagest* [Toomer 1984, 64–130, 410–417].

I The sector theorem

The sector theorem was the fundamental theorem of ancient spherical trigonometry. It asserts a compound proportion that holds for combinations of the chords of six arcs of great circles forming a concave quadrilateral on the surface of a sphere. The most important form of the theorem was known to the ancient and medieval mathematical astronomers as *Disjunction*.[3] It is always the first, and often the only, form of the theorem that is fully demonstrated on the basis of the underlying chords.

Consider Figure 1. Where the arcs of the figure are less than semicircles and $Crd(\alpha)$ is the chord subtending arc α, *Disjunction* asserts that

$$\frac{Crd(2\,\overset{\frown}{GE})}{Crd(2\,\overset{\frown}{EA})} = \frac{Crd(2\,\overset{\frown}{GZ})}{Crd(2\,\overset{\frown}{ZD})} \times \frac{Crd(2\,\overset{\frown}{BD})}{Crd(2\,\overset{\frown}{BA})}.$$

By disjunction, we mean that the two components of one of the outer arcs of the sector figure, $\overset{\frown}{GE}$ and $\overset{\frown}{EA}$, are taken separately; that is disjointly. In *Disjunction*, a

[2]Björnbo [1902, 72 ff.] in his seminal study of Menelaus' *Spherics*, had already come to this opinion, in which he was followed by Heath [1921, vol. 2, 270]. Rome [1933, 42] agreed that this was a real possibility. Schmidt [1943, 66–68] also had no problem considering the sector theorem as one of the first tools of quantitative spherical astronomy.

[3] Neugebauer [1975, 28] called this combination of the arcs *Menelaus Theorem II* (M.T.II). My terminology follows Lorch [2001, 2–4]. *Sector theorem* itself is a common Arabic name for the theorem (الشكل القطاع). *Conjunctus* and *disjunctus* are the Latin technical terms for ratio types that Greek authors generally denoted with the terms σύνθεσις and διαίρεσις. This way of naming the two canonical forms of the theorem is found already in Theon's commentaries to the *Almagest* [Rome 1931–1943, 558, 562].