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The Abacus at Echternach in ca. 1000 A.D.

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The single sheet of manuscript from the Benedictine monastery of Echternach, catalogued as no. 770 in the Bibliothèque nationale de Luxembourg, is probably the nearest thing we have to the abacus board devised by the famous mathematician and educator of the late tenth century, Gerbert d'Aurillac (d. 1003). This article describes the board in the light of what we know of Gerbert's own instrument and also introduces another smaller example of exactly the same board. Both examples have hitherto been unnoticed by scholars.[1]

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1 The Gerbertian Abacus

From the late tenth century until at least the mid twelfth century the principal method of studying practical arithmetic in the schools of Western Europe was that of the abacus with marked counters. This was a a board or sheet of parchment ('abacus board') on which parallel lines were drawn to provide columns for powers of ten. Counters marked with each of the numerals were placed in the columns and successively replaced in the course of an arithmetical calculation. Rules were written to describe the sequences of procedures, especially for division and multiplication.

[1]I am very grateful to Luc Deitz, the curator of the manuscripts and rare books collection (Réserve précieuse) of the Bibliothèque nationale de Luxembourg for introducing me to the manuscript-sheet, and inviting me to give a talk at the library on its relation to the early history of the introduction of Arabic numerals into Europe, on 19 October 2001, as well as for showing me a preliminary study of his on the logical schemata contained on the sheet and correcting an early draft of this paper. Throughout this article I am indebted to the research of Menso Folkerts, whose article, 'Frühe Darstellungen des Gerbertschen Abakus' (in *Itinera mathematica: Studi in onore di Gino Arrighi per il suo 90a compleanno*, eds R. Franci, P. Pagli and L. Toti Rigatelli, Siena 1996, pp. 23–43) constitutes the starting point for the study of early depictions of the abacus, and who has generously shared his knowledge and research materials with me. I have also benefited from the advice of David Juste, Daibhi O Croinin, Alison Peden and Thomas Falmagne, and from the services of Reiner Nolden of the Stadtarchiv of Trier. The contents of the manuscript sheet as a whole will be the subject of a monograph written jointly by Luc Deitz, Thomas Flamagne and the present author.

The purpose of the exercise was to demonstrate how numbers interacted with each other rather than to facilitate calculation, which could more easily be done on the fingers, and the abacus was regarded as an instrument of geometry rather than of arithmetic. When the counters were marked with Arabic numerals, this kind of abacus mimicked with columns and marked counters the appearance of calculation with the same Arabic numerals but using pen and paper (or quill and parchment) — a process later called the 'algorism'. This mimicry extended to the use of a counter for zero, which is strictly not necessary when columns are drawn for each of the decimal places: one simply leaves a column empty.[2]

The origin of this form of abacus is still unknown.[3] Our earliest testimonies rather associate a *revival* of its use with Gerbert d'Aurillac, especially with his period as a teacher at Reims (972 to 983). It seems likely that Gerbert introduced the practice of marking the counters with Arabic numerals (which he would have come across when he studied in Catalonia, before coming to Reims), and established a form of the abacus board that became an exemplar for most subsequent teachers of the abacus. However, although several texts on calculating with the abacus (starting with one by Gerbert himself) survive,[4] they do not include diagrams of the instrument itself. The 'Gerbertian' form of the table can only be deduced from two descriptions of the table, and a few early manuscript depictions of the table occurring separately from the texts on calculation.[5] Hence the newly-discovered Echternach manuscript sheet

[2]For examples of how to calculate on this abacus see K. Vogel, 'Gerbert von Aurillac als Mathematiker', *Acta historica Leopoldina*, 16, 1985, pp. 9–23, G. Ifrah, *The Universal History of Numbers*, English translation by D. Bellos etc., London, 1998, pp. 579–85, and G. Beaujouan, 'Les Chiffres arabes selon Gerbert: L'abaque du Pseudo-Boèce', in *Autour de Gerbert d'Aurillac*, 1996, pp. 322–8; see also G.R. Evans, '*Difficillima et ardua*: theory and practice in treatises on the abacus 950–1150', *Journal of Medieval History*, 3, 1977, pp. 21–38.

[3]This problem is most fully discussed in W. Bergmann, *Innovationen im Quadrivium des 10. und 11. Jahrhunderts*, Wiesbaden, 1985. Bergmann, however, did not take account of the evidence of the early depictions of the abacus tables occurring separately from the texts on calculation.

[4]Several of these are edited in N. Bubnov, *Gerberti postea Silvestri II papae Opera mathematica* (972–1003), Berlin, 1899, pp. 1–22 and 197–245.

[5]These are: Bern, Burgerbibliothek, Cod. 250, fol. 1r, s. x^{ex} (**B**); Paris, Bibliothèque nationale de France, lat. 8663, fol. 49v, s. xi^{in} (**P**); ibid., lat. 7231, fol. 85v, s. xi^{in} (**Q**); Vatican, Biblioteca apostolica Vaticana, lat. 644, fols 77v–78r, s. xi^{in} (**V**); Rouen, Bibliothèque municipale, 489, fols 68v–69r, s. xi^{ex} (**R**); Oxford, St John's College 17, fols 48v–49r, s. xii, (**J**); and Chartres, Bibliothèque municipale 498, fols 165v–167r, ca. 1140 (**C**). **BPV** are illustrated and described in M. Folkerts, 'Frühe Darstellungen' (n. 1 above), **R** in idem, 'The names and forms of the numerals on the abacus in the Gerbert tradition', to be published in the proceedings of a conference on Gerbert d'Aurillac held at Bobbio, and **J** in Evans, '*Difficilis et ardua*' (n. 2 above). **C** survives in microfilm only.

may be a unique example of the artefact itself.

2 The Echternach abacus board

The Echternach abacus board (Plate 1) is a sheet of thick parchment 603 millimetres wide and 420 millimetres long. Since the sheet was used as a paste down in the binding of a large Echternach Bible that was written for the Abbot Regimbertus (1051–81) and has survived intact as the 'Riesenbibel' ('Bible géante'; now Bibliothèque nationale de Luxembourg, no. 264), the absolute terminus *ante quem* of the writing on the sheet is 1081.[6] However, since the material on the sheet must have been regarded as obsolete when it was reused as a paste down, it could have been produced much earlier, as the handwriting also suggests. The sheet was cut down to fit the dimensions of the bible and was once at least one-eighth wider, and somewhat longer.[7] There is no evidence that the sheet was ever a folio of a manuscript. The original parchment appears to have been cut especially in order to produce an abacus table that could be displayed in a teaching situation or used as a board for calculating. The letters are all capitals, written clearly in a strong black ink. The columns of the abacus table were originally drawn vertically across the whole sheet. A little later another scribe, using a lighter coloured ink, reused the bottom half of the sheet in order to draw a representation of the Boethian monochord. Perhaps at the same time, other scribes used the verso of the sheet to represent the whole of logic in the form of schemata. Both these additions, of course, took place before the sheet was bound into the Riesenbibel, and the logical schemata are difficult to read because they are on the side of the sheet that was pasted onto the binding. The *mensura monochordi* does not obscure any writing on the abacus board, but only shortens the columns, giving less room for making calculations. A scholar may simply have wished to adapt the sheet into a compendium of information on logic, geometry (to which the abacus was regarded as belonging) and music. That this is an adaptation of the original plan is clear. For the full-sheet abacus was obviously meant to be a display copy or to be laid out on a table in class; the music and logic, on the other hand, are written in small letters, and the music includes some continuous text that cannot be read at a distance, which suggests that both these additional elements were meant for individual reading. The priority of the abacus

[6]The only description of this manuscript sheet up to now is J. Leclercq, 'Un nouveau manuscrit d'Echternach à Luxembourg', *Scriptorium*, 7, 1953, pp. 219–25. Leclercq briefly mentions the paste down which was still attached to the bible when he saw it (p. 225), but he refers only to the *mensura monochordi* and the schemata of logic written on it, and not to the abacus. The paste down was detached and then mislaid some time after Leclercq's visit to the library.

[7]See section 3 ii below. The original dimensions were probably ca. 680 x 440 mm, as is also indicated by the vestige of a fold 340 mm from the left hand side (i.e. it is probable that the sheet was once folded in half). I owe this observation to Luc Deitz.

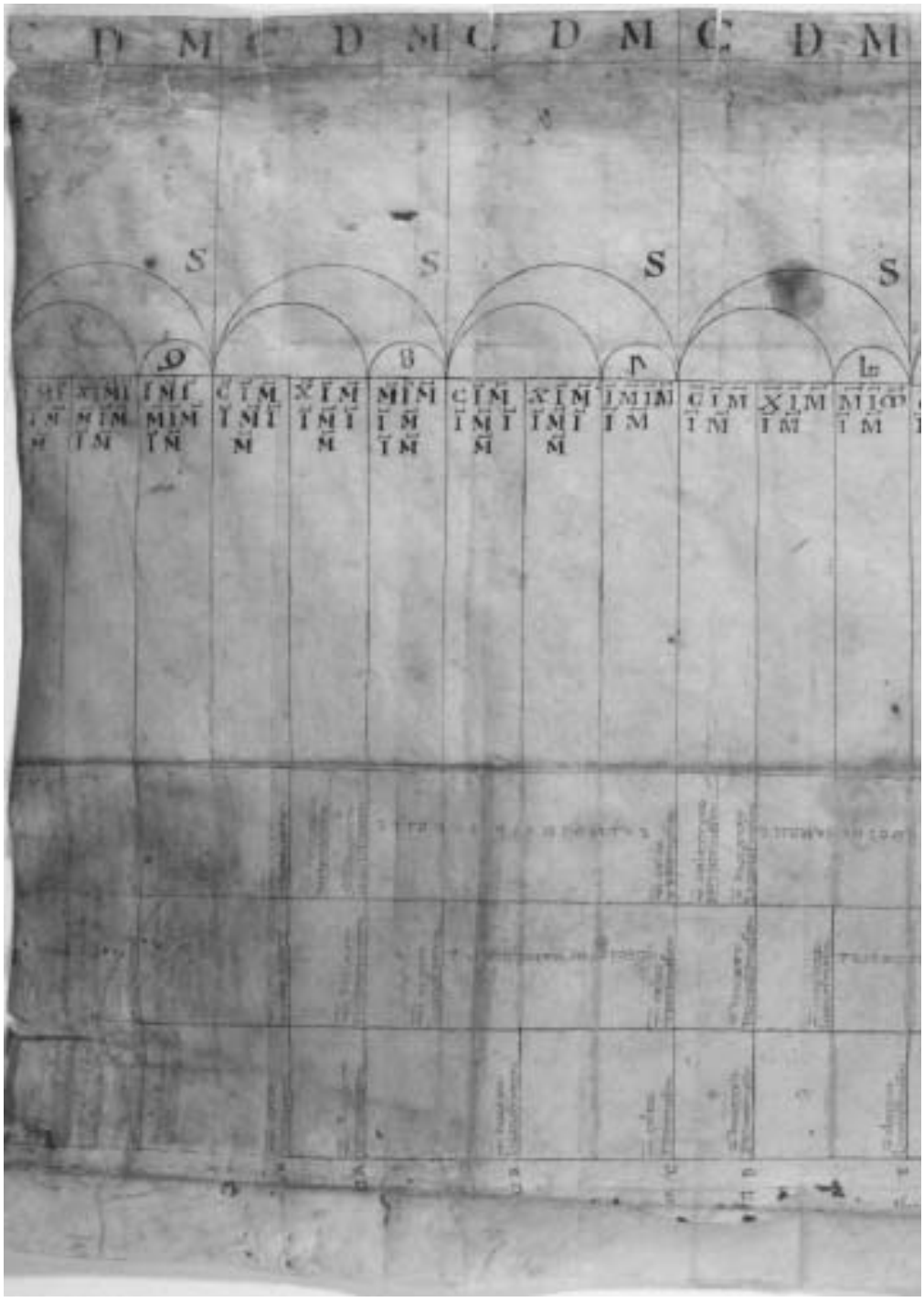
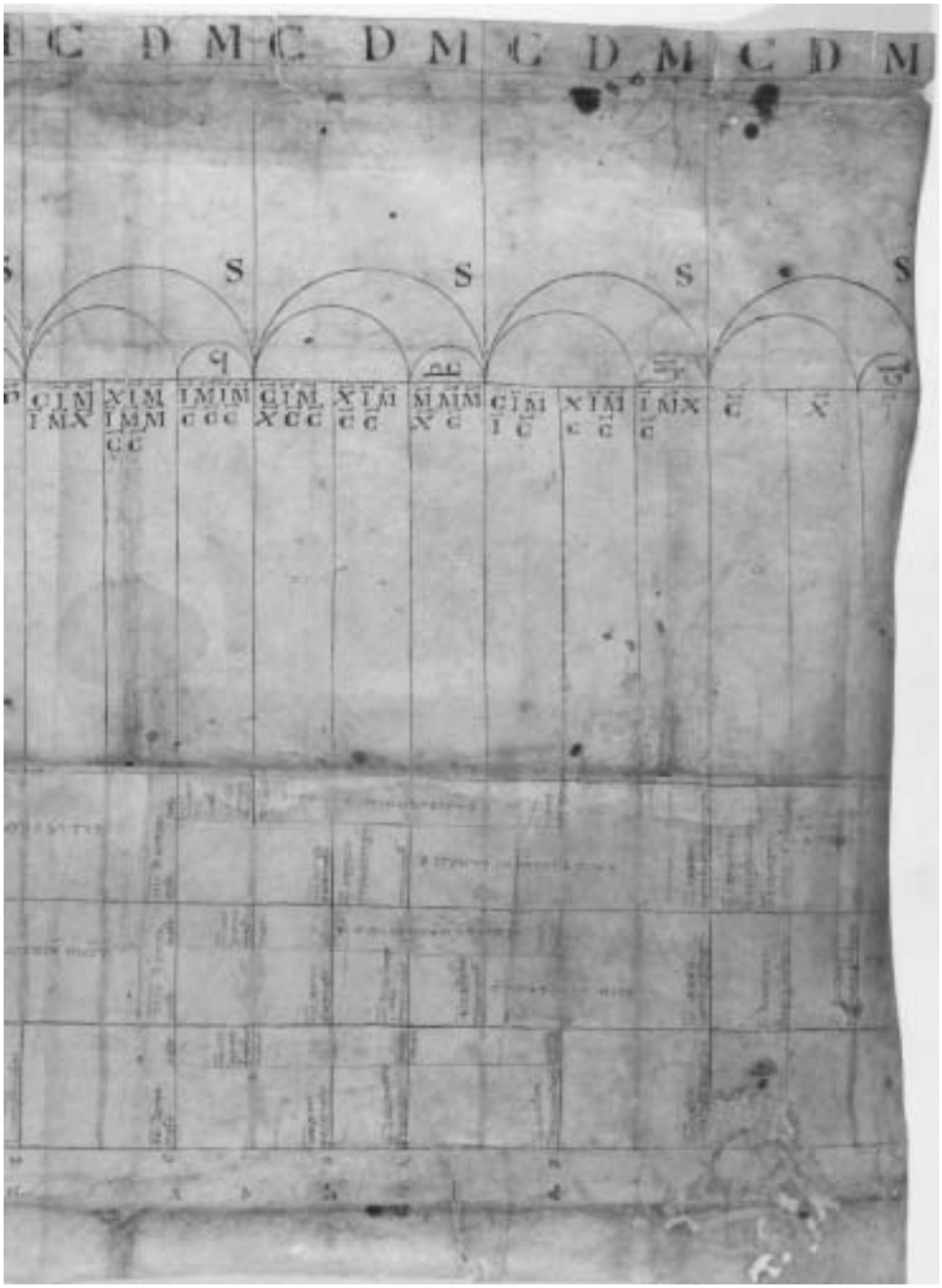


Plate 1. The Echternach abacus board (MS Bibliothèque nationale de Luxembourg, 770; reproduced with permission)


C D M C D M C D M C D M
S S S S

text to the other items is also suggested by the two folds still visible on the sheet. The latitudinal fold has been made to separate off the *mensura monochordi*. The longitudinal fold, however, crosses through the middle of an abacus column, damaging the letters in the colues through the middle of an abacus column, damaging the letters in the column, whereas the scribe of the *mensura monochordi* has scrupulously avoided writing where the fold occurs.

3 The Echternach manuscript sheet as an example of Gerbert's own abacus board

We may compare the Echternach example, point by point, with the earliest descriptions of abacus boards: (a) the description of Gerbert d'Aurillac's own abacus by his pupil Richer in his *Historiae*, III, ch. 54 (= Rich.), [8] and (b) a fuller description of the abacus in the earliest set of instructions for calculation that contains such a description, namely the *Liber abaci* of Bernelinus of Paris, who attributes his knowledge to Gerbert's writings (= Bern.).[9]

i) Rich.: 'Abacum, id est tabulam dimensionibus aptam, opere scutarii effecit' ('He <Gerbert> employed the offices of a shield-maker to make an abacus, i.e. a board fitted with divisions of measurement'). The usual medieval shield was made of wood covered with leather.[10] The Echternach table consists of only one sheet of parchment and hence is not as sturdy as the example described by Richer. It does, however, have the length and width of a breastplate.

ii) Rich: 'cuius longitudini in .xxvii. partibus diductae novem numero notas omnem numerum significantes disposuit' ('He divided its length into 27 columns of which he marked nine with a numeral, these signifying all possible numbers'[11]); Bern.: 'Tabula ... diligenter undique prius polita, per .xxx. dividatur lineas, quarum tres primas unciarum minutiarumque dispositioni reservamus'

[8] Richerus, *Historiarum libri quattuor*, ed. G. Waitz, Monumenta Germaniae Historica, Scriptores rerum germanicarum in usum scholarum, Hannover, 1877.

[9] Bernelin, élève de Gerbert d'Aurillac, *Libre d'Abaque*, ed. B. Bakhouche, comm. J. Cassinet, Collection Istoría Matematíca Occítana 3, Montpellier, 2000, p. 21: 'abaci rationes persequar diligenter, negligentia quidem apud nos iam paene demersas, sed a domino papa Gerberto quasi quaedam seminaria breviter et subtilissime seminatas' ('I shall carefully pursue the rules of the abacus, now almost sunk beyond trace amongst us because of negligence, but, as it were, sown again in nurseries (seminaries) briefly and most subtly by our lord Pope Gerbert'). One may infer from how Bernelinus refers to Gerbert in this text that it was written during Gerbert's papacy (999–1003).

[10] C. Blair, *European Armour*, London, 1958, p. 18.

[11] Grammatically, this translation seems to be more correct than the usual interpretation: '... on which he arranged the nine numerical symbols which signify all possible numbers'. I have taken 'notas' as a past participle passive referring to 'partes', rather than as a noun referring to the

('A board, which has first been thoroughly polished, should be carefully divided into 30 columns, of which we reserve the first three for fractions'). Only 24 columns of the Echternach table survive, but, since the first column (marked with the Arabic numeral '2') is for the thousands (10^3), it is evident that originally it consisted of at least three more columns (for 10^0, 10^1, and 10^2), of which the first would have been marked with an Arabic '1'. The missing columns are due to the sheet being cut down to fit the binding of the Riesenbibel. That it did not include the three columns for fractions mentioned by Bernelinus is indicated by the following facts. (1) Arabic numerals,marking every third column, fit a 27-column abacus, but not a 30-column abacus,[12] and indeed are found on extant early depictions of 27-column abacuses,[13] but not in Bernelinus's description, which is confirmed by a depiction of a tenth-century 30-column abacus.[14] (2) From the evidence of the longitudinal fold, if we assume the table was folded in half, there is exactly the right space for three columns only.[15]

iii) Bern.: '... reliquarum vero .xxvii., per ternas et ternas, haec certa mensurandi proveniat regula: primam de tribus lineam circinus in hemisperii modum teneat amplexam; maior autem circinus duas reliquas amplectatur;sed easdem tres maximus complectatur pariter ... per omnem abaci tabulam omnes eaedem praescribantur litterae, id est C, D, M, S, hoc modo: C super centenum, D super decenum, M super unitatem, sive ut monos designet, sive ut mille significet; cui supponatur S singulare significans' ('the other 27 columns should be grouped in threes, linked in the following way: a semicircle should embrace the first of the three, a larger semicircle the other two, but an even larger semicircle should embrace all three.... The same letters should be repeated through the columns: namely, C, D, M, S, in this way: C over the hundreds, D over the tens, M over the units, whether because it indicates "monos" or "mille". Below it is placed an 'S' indicating "singulare"'). These details, which are not mentioned by Richer, are all present on the Echternach table.

iv) Bern.: 'Prima linea...unitate signetur ita: I, secunda, X, id est deceno; tertia

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numeral form (for which 'c(h)aracter' or 'figura' is the usual term in the abacus literature).

[12]The Arabic numerals at the heads of columns (unlike those on the abacus counters) do not have an arithmetical function, but, presumably, are added to show symbolically what a huge range of numbers they can represent. Moreover, as can be seen from paragraph vi below, they give examples of the symbols to be written on the counters.

[13]MSS **P** and **J** (see n. 5 above).

[14]MS **B** (see n. 5 above). This abacus agrees in almost every respect with Bernelinus's description. MS **Q**, however, which otherwise is identical to MS **B**, does include the numerals, and the 'S' below the 'M'.

[15]See n. 7 above. Further confirmation will be given below.

C, id est centeno; quarta elemento I, superaddito titulo, id est milleno; quinta elemento X, superaddito titulo, id est deceno milleno; sexta elemento C, superaddito titulo, id est centeno milleno; septimam vero haec elementa praesignent, additis titulis, M.I., id est mille milia. Octavam haec X.MI, additis tantum titulis super M et I, id est decies mille milia. Idque in omnibus

column number and value	Bernelinus	Echternach table
1. 1 (10^0)	I	<I>
2. 10 (10^1)	X	<X>
3. 100 (10^2)	C	<C>[16]
4. 1,000 (10^3)	$\bar{\text{I}}$	$\bar{\text{I}}$
5. 10,000 (10^4)	$\overline{\text{X}}$	$\overline{\text{X}}$
6. 100,000 (10^5)	$\overline{\text{C}}$	$\overline{\text{C}}$
7. 1,000,000 (10^6)	$\overline{\text{M.I}}$	$\overline{\text{IM}}$
8. 10,000,000 (10^7)	X.$\overline{\text{MI}}$	X$\overline{\text{IM}}$
9. 100,000,000 (10^8)	C.$\overline{\text{MI}}$	C$\overline{\text{IM}}$
10. 1,000,000,000 (10^9)	$\overline{\text{M.MI}}$	$\overline{\text{MMM}}$
11. 10,000,000,000 (10^{10})	X.$\overline{\text{MMI}}$	$\overline{\text{XIM}}$
12. 100,000,000,000 (10^{11})	C.$\overline{\text{MMI}}$	$\overline{\text{CIM}}$
13. 1,000,000,000,000 (10^{12})	$\overline{\text{M.MMI}}$	$\overline{\text{IMIM}}$
14. 10,000,000,000,000 (10^{13})	X.$\overline{\text{MMMI}}$	X$\overline{\text{IMIM}}$
15. 100,000,000,000,000 (10^{14})	C.$\overline{\text{MMMI}}$	C$\overline{\text{IMIM}}$[17]
16. 1,000,000,000,000,000 (10^{15})	$\overline{\text{M.MMMI}}$	$\overline{\text{MIMIM}}$
17. 10,000,000,000,000,000 (10^{16})	X.$\overline{\text{MMMMI}}$	$\overline{\text{XIMIM}}$
18. 100,000,000,000,000,000 (10^{17})	C.$\overline{\text{MMMMI}}$	$\overline{\text{CIMIM}}$
19. 1,000,000,000,000,000,000 (10^{18})	$\overline{\text{M.MMMMI}}$	$\overline{\text{IMIMIM}}$
20. 10,000,000,000,000,000,000 (10^{19})	X.$\overline{\text{MMMMMI}}$	X$\overline{\text{IMIMIM}}$
21. 100,000,000,000,000,000,000 (10^{20})	C.$\overline{\text{MMMMMI}}$	C$\overline{\text{IMIMIM}}$
22. 1,000,000,000,000,000,000,000 (10^{21})	$\overline{\text{M.MMMMMI}}$	$\overline{\text{MIMIMIM}}$
23. 10,000,000,000,000,000,000,000 (10^{22})	X.$\overline{\text{MMMMMMI}}$	$\overline{\text{XIMIMIM}}$
24. 100,000,000,000,000,000,000,000 (10^{23})	C.$\overline{\text{MMMMMMI}}$	$\overline{\text{CIMIMIM}}$
25. 1,000,000,000,000,000,000,000,000 (10^{24})	$\overline{\text{M.MMMMMMI}}$	$\overline{\text{IMIMIMIM}}$
26. 10,000,000,000,000,000,000,000,000 (10^{25})	X.$\overline{\text{MMMMMMMI}}$	X$\overline{\text{IMIMIMIM}}$
27. 100,000,000,000,000,000,000,000,000 (10^{26})	C.$\overline{\text{MMMMMMMI}}$	C$\overline{\text{IMIMIMIM}}$

cavendum volumus ut, his elementis X et C <exceptis>, posthac superaddantur tituli omnibus ceteris[18] ...' ('The first line should be marked in the following way: I; the second, X, i.e. a ten; the third, C, i.e. a hundred; the fourth, the letter I with a tilde ontop, i.e. a thousand; the fifth, the letter X with a tilde on top, i.e. a ten thousand; the seventh is marked with the following letters, with tildes added: $\overline{\text{M}}.\overline{\text{I}}.$, i.e. 1,000,000; the eighth, with these: X.$\overline{\text{MI}}$, with tildes over the M and the I only, i.e. 10,000,000. We should warn the reader that in every case after this there are no tildes on the letters X and C, but tildes on all the other letters'). These indications of the powers of ten are followed in the Echternach table, except that, in the latter, (a) the two ways of indicating '1,000' — I with a tilde and M with a tilde[19] — alternate, and (b) '$\overline{\text{XM}}$' and '$\overline{\text{CM}}$' are always used instead of '$\overline{\text{XMM}}$' and '$\overline{\text{CMM}}$'.

According to the editor, there are errors in all the manuscripts of Bernelinus.[20] In the Echternach table, however, there is not a single mistake here.

v) Bern.: 'Quibus praesignatis, a prima priori linea usque ad primae vicesimam septimam quatuor trahantur lineae aequali spatio differentes inter se, quarum prima primus trames, ultima vero quartus nuncupabitur, duarum autem mediarum secunda secundus, tertia tertius nominabitur. Quod quare statuerim, diligenter intuentem non latebit cum ad divisiones venerit' ('When this has been done, four lines equally distant from each other should be drawn from the line before the first <column> to the line 27th in respect to the first <column>, of which the first is called the "first cross-line", the last, the "fourth cross-line", and the two middle ones respectively "the second" and "the third cross-lines". The careful observer will see why I have arranged them in this way when it comes to describing how to do division.'[21]). Here the Echternach sheet contains

[16] The first three columns on the Echternach sheet, now missing, can be presumed to have had these headings.

[17] A tilde on 'C' has been erased.

[18] Bakhouche's edition (n. 9 above, p. 24) gives 'his elementis X et C posthac superaddantur tituli omnibus ceteris generaliter superadditis', which appears to be corrupt. I have emended the text to make it conform to the examples that Bernelinus goes on to give.

[19] It seems irrational to add a tilde to M, when it already means '1,000', but Bernelinus states that this is done, and Bernelinus's prescription is confirmed both by the Echternach manuscript sheet and by most early abacus depictions. An exception is MS **V** (see n. 5 above) in which M is always written without a tilde (In MS **J** M without the tilde is used for '1,000' in the fourth column, but M with a tilde is used in all other cases).

[20] Bernelin, *Libre d'Abaque*, ed. B. Bakhouche (n. 9 above), p. 24, note 7.

[21] As far as I can see, Bernelinus never returns to describing the purpose of these cross-lines, which can only be inferred from MS **B** and **Q** (see n. 5 above) in which the second cross-line contains

a serious mistake. Numbers that should have been on the second cross-line have been added to the ends of the numbers on the first one, as follows:[22]

column and numerals in the first cross-line	numerals from the second cross-line
7. $\overline{\text{IM}}$	$\text{X}\overline{\text{C}}$ i.e. 10 x 100,000
8. $\text{X}\overline{\text{IM}}$	$\text{C}\overline{\text{C}}$ i.e. 100 x 100,000
9. $\text{C}\overline{\text{IM}}$	$\overline{\text{IC}}$ i.e. 1,000 x 100,000
10. $\overline{\text{MMM}}$	$\overline{\text{XC}}$ i.e. 10,000 x 100,000
11. $\overline{\text{XIM}}$	$\overline{\text{CC}}$ i.e. 100,000 x 100,000
12. $\overline{\text{CIM}}$	$\overline{\text{XCC}}$ i.e. 10 x 100,000 x 100,000
13. $\overline{\text{IMIM}}$	$\overline{\text{CCC}}$ i.e. 100 x 100,000 x 100,000
14. $\text{X}\overline{\text{IMIM}}$	$\overline{\text{MCC}}$ i.e. 1,000 x 100,000 x 100,000
15. $\text{C}\overline{\text{IMIM}}$	$\overline{\text{X}}$ (incomplete)

vi) Rich.: 'Ad quarum etiam similitudinem mille corneos effecit caracteres, qui per .xxvii. abaci partes mutuati, cuiusque numeri multiplicationem sive divisionem designarent' ('In the likeness of which <i.e. the columns marked with Arabic numerals> he (Gerbert) made 1,000 counters of horn, which, by being moved around through the 27 columns of the abacus, could show the multiplication or division of any number you like'); Bern.: 'His igitur expeditis, ad ipsos caraceres veniamus et quibus figuris praenotentur adscribere properemus. Unitas, quae primus caracter dicitur, sic figuratur : 1, sive per Graecum A alpha. Binarius...' ('Having done this, let us turn to the characters themselves, and let us hurry to describe the shapes by which they are represented. The one, which is called the first character, is the following shape: 1 or represented by the Greek A = alpha. The two...'). Both Richer and Bernelinus are describing the counters that are marked with each of the nine digits. Richer does not give the forms of

numbers which by being added (in columns 1 to 6 — disregarding the first three columns of fractions) and multiplied (in columns 7–27) produce the number at the head of the column; the third cross-line contains the halves of those numbers; and the fourth cross-line lists the names, symbols and equivalents in roman numerals, for fractions. The fact that no numerals are included in the first three fraction columns of MSS **B** and **Q** suggests that these columns were added at a later stage than the four cross-lines.

[22]The values from the second cross-line are exactly those in MSS **B** and **Q**, and it may be significant that the scribe has started to copy them from the point where they change from being additive to multiplicative (see previous note).

> the Arabic numerals; Bernelinus gives the forms of all of them, as well as the alternative Greek letters that can be used. In the Echternach table the counters are, of course, not depicted. One must presume that they existed separately and were moved over the table for the purpose of making calculations. The columns are wide enough to accommodate such counters. We can infer from Richer's passage that they would have had, written on them, the same forms of the Arabic numerals as have been written over every third column in the Echternach table. These shapes are remarkably similar to those in the text of Bernelinus, but are common to other early abacus texts.[23] Bernelinus gives Latin names to these counters: 'unitas', 'binarius', 'ternarius' etc. ('the one', 'the two', 'the three', etc.), but other abacus texts have the distinctive names (some of Arabic origin): 'igin, andras, ormis, arbas, quimas, calctis, zenis, temenias' and 'celentis'.[24]

The Echternach manuscript sheet, then, has no features that are not mentioned in Richer or Bernelinus's descriptions of the abacus board, nor do either of these descriptions include features absent from the Echternach table, with the exception of the four cross-lines. However, the fact that some of the numbers which belong to the second of these cross-lines have also been copied onto the Echternach sheet (but in the wrong place) indicates that the cross-lines had been present in a previous version. What we appear to have, then, on the Echternach sheet, is the abacus board as devised by Gerbert, but before the modification of Bernelinus, which consisted in extending the number of columns from 27 to 30, and dropping the Arabic numerals as headings to columns.

4 A second Echternach abacus

Virtually a facsimile of the abacus on the Echternach manuscript sheet can be found in another Echternach manuscript, namely Trier, Stadtbibliothek, 1093/1694. The depiction of the abacus fills fol. 197r (see Plate 2). At 510 x 345 mm it approaches the size of the Echternach sheet more closely than any other known abacus table, and it is written in the same bold capital letters. Moreover it is complete, and so confirms the assumption that the Echternach sheet, before being cut down in size, consisted of 27 columns and not 30. The first three columns of the Trier abacus,

[23] A table of forms is given in M. Folkerts, *"Boethius" Geometrie II, ein mathematisches Lehrbuch des Mittelalters*, Wiesbaden, 1970.

[24] The most complete account of the occurrence of these names is in M. Folkerts, 'Frühe westliche Benennungen der indisch-arabischen Ziffern und ihr Vorkommen' in *Sic itur ad astra: Studien zur Geschichte der Mathematik und Naturwissenschaften. Festschrift für den Arabisten Paul Kunitzsch zum 70. Geburtstag*, ed. M. Folkerts and R. Lorch, Wiesbaden, 2000, 216–33.

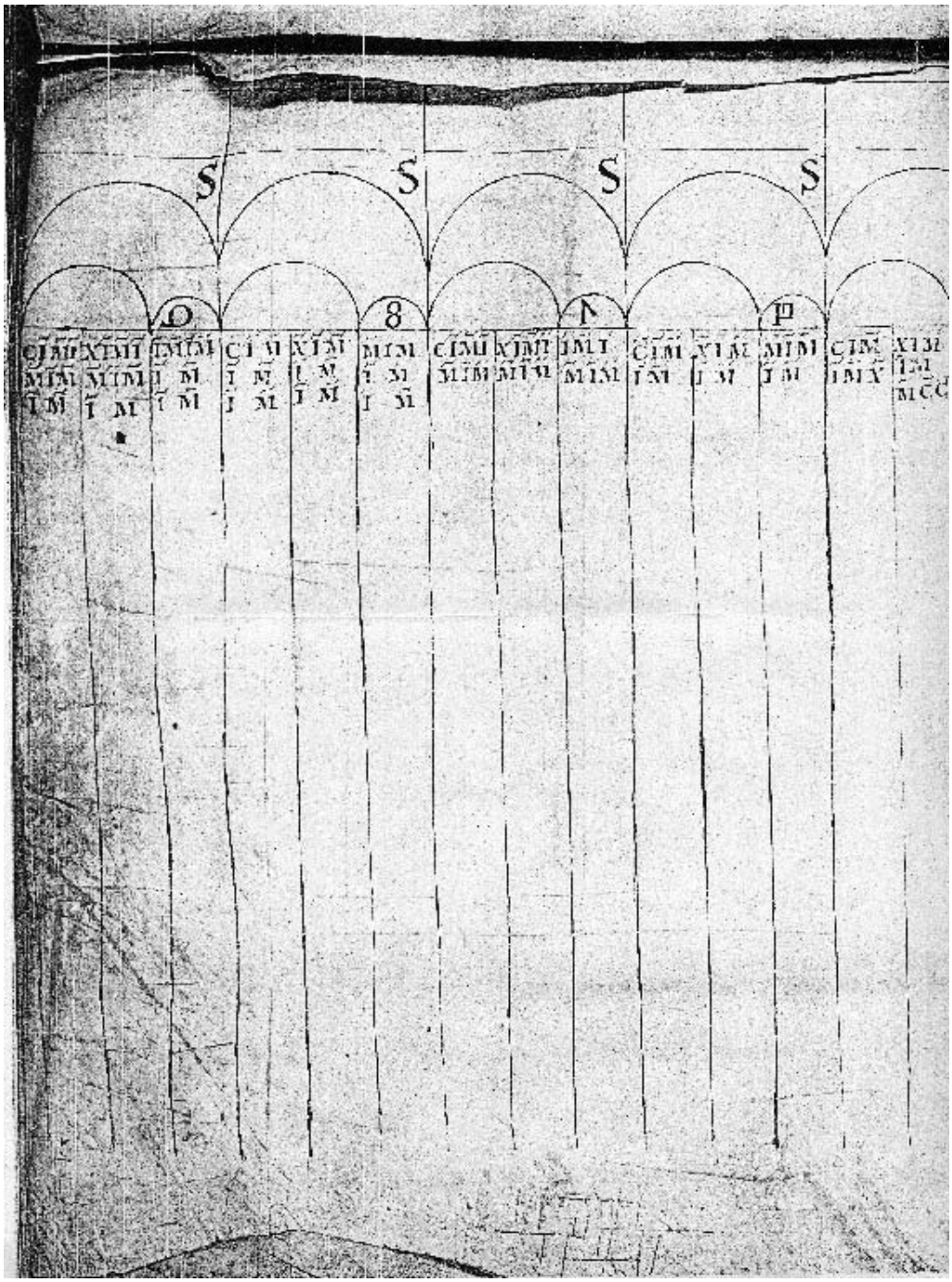
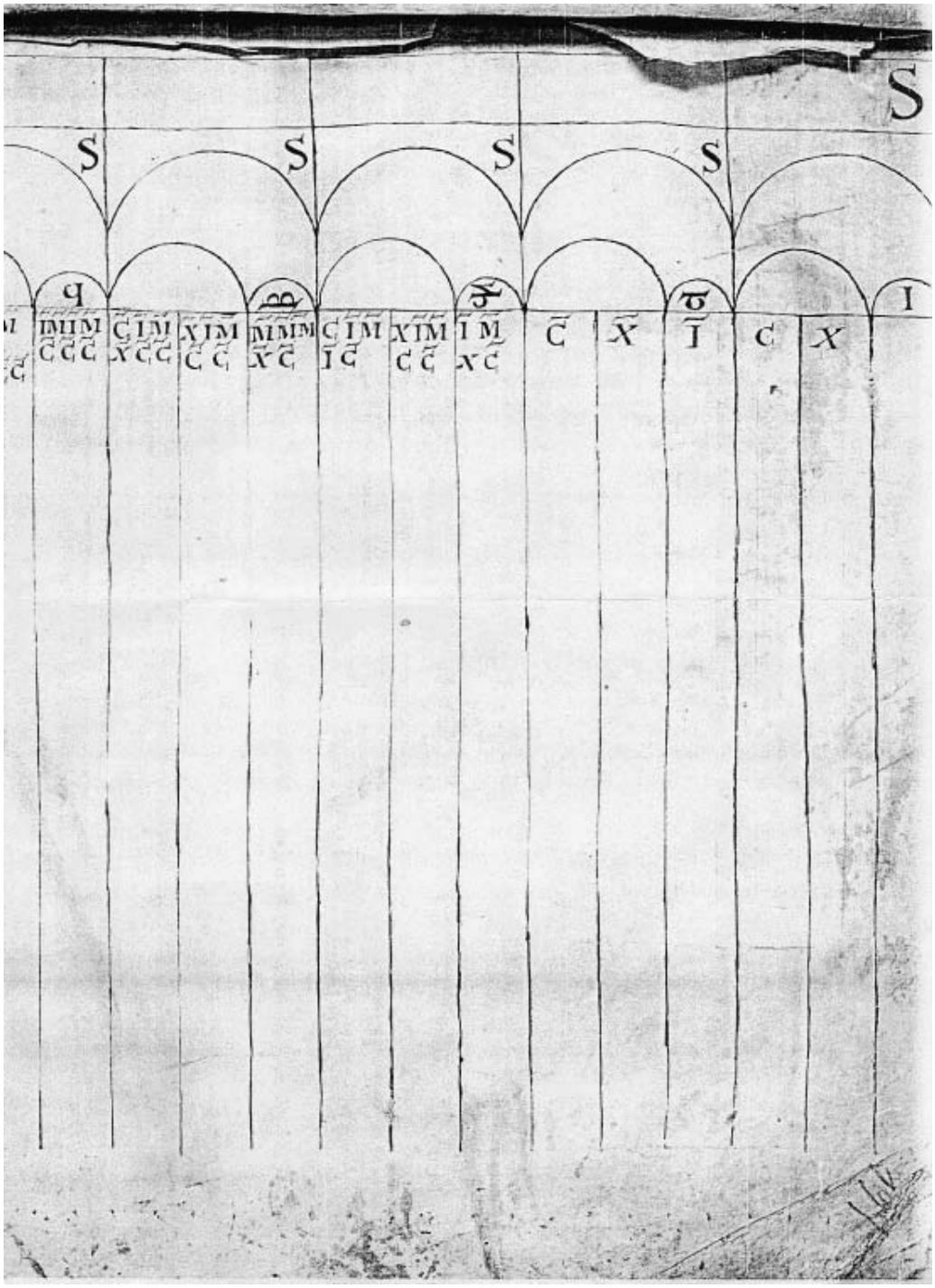


Plate 2. The second Echternach abacus (MS Trier, Stadtbibliothek, 1093/1694, fol. 197r; reproduced with permission)


S
S
S
S
S
I
C
X

as one would have expected, are devoted to the units, tens and hundreds, but there are two features which could not have been guessed from the Echternach sheet: i) a blank is left where one would have expected the upper-case roman 'I' for the units; instead, the Arabic 'I' (which is of course identical in shape) in the small semicircle above the first column serves as the only heading; ii) again, above the first column, the large 'S', which in every other case is placed over the first of each set of three columns to the right of the large semicircle, is doubled in size and put at the top right-hand corner of the sheet. Both these features enhance the visual aspect of the table, and show the reader that the table begins from the right. In the parts of the table that the Trier manuscript and the Echternach manuscript sheet share in common, the only differences to be observed are the following:

i) the letters M D C, repeated eight times at the heads of the columns, are missing in the Trier version, though two horizontal lines have been drawn (absent in the Echternach sheet) which may have been expected to contain them. The space between these lines is occupied only by the above-mentioned large 'S' over the first column. The depictions of the abacus in MSS **P** and **J** (see n. 5 above) show that such an S replaces the M in the first column, while S and M appear together in all other cases.
ii) In the case of the Arabic numerals in the Trier version the lower loop of the '3' is written as a separate stroke rather than as a continuation of a curved line passing through the central horizontal stroke, and the horizontal stroke of the '6' leaves the vertical stroke at the mid-point rather than at the base. In the latter case the numeral is more similar to its equivalent in Arabic script.
iii) While the same numbers from the second cross-line follow immediately after the numbers of the first cross-line in both abacus tables, in the Trier version they always begin on a new line, except in the case of the last, incomplete, number ('X') which is tagged onto the end of the first number. Here the Trier version could be said to be more accurate, but the inclusion of only a few numbers from the second cross-line, and their placement immediately after those of the first cross-line, would still seem to be a mistake, shared by both versions.
iv) There are trivial differences in the line-arrangement of the higher numbers.

The Trier table, then, is marginally more correct and may be closer to an archetype. The absence of the headings M D C, however, show that it is not the exemplar from which the Echternach sheet was copied.

But the Trier version has another feature which is not present on the Echternach sheet: namely, it adds some miscellaneous texts on calculation, fractions and Arabic numerals. These texts do not reveal anything further on the construction of the abacus, but they are clearly meant to be complementary to the depiction of the instrument. They consist of the following:

1) Fol. 197v. Rules for multiplication and division. These begin with instructions on how to determine in which columns the product of a multiplication falls ('Singularis multiplicator quemcumque multiplicaverit, in eundem pone digitum quem multiplicat, in ulteriorem articulum ...). These resemble most closely the rules for the abacus given by Heriger of Lobbes.[25] They are followed by miscellaneous rules for multiplying and dividing numbers ('Si multiplicare vis quemlibet numerum et semis ...'; 'Omnis divisor sive simplex sive compositus denominationem (?) semper in prima ponit linea et quot lineas rediens divisor descenderit, tot subposita denominatio ascendet ...'; 'Si compositus est divisor solus, qui dux vocatur ...'; 'Dividendus autem, sive sit simplex sive compositus, semper erit per denominationes et divisores, facta multiplicatione, reformandus ...'; 'In simplici divisori simplici dividendo et in simplici divisori composito dividendo "quotiens" dicendum est semper ...'). Here 'linea' is used for decimal place.
2) Fol. 197v. Rules for 'equating', apparently in regard to finding the quotients in long division ('De eo aequandis haec prima servatur ratio. Primus superior semper debet maior esse reliqui minores inferioribus ...').
3) Fol. 197v-198r. The problem of what to do with the remainder in a division leads without a break into a discussion of Roman fractions ('Omne quod remanet de maioribus indivisum per uncias primum erit dividendum. As enim .xii. habet uncias et ideo per duodenarium habes invenire quot uncias remanentia maiorum possint reddere. Unciis autem ita inventis, per maiorum divisores iterum dividere curabis. Si quid autem de unciis remanserit, per scripulos dividendum erit...'). The scribe describes the divisions of the Roman *as* in detail, and includes a table of the 22 different divisions, with their symbols and their values in *uncie* and *scripuli*. The table is very similar to that associated with the 'Incertus abacista saeculi decimi' edited by Bubnov,[26] while the textual account of the relationship between the various fractions is similar to that found in the *Calculus* of Victorius in the version used by Bernelinus[27] and with the commentary by Abbo of Fleury.[28]
4) Fol. 198r. A detailed account of the etymology of each of the Roman fractions ('Libra vel as sive assis xii untiae. Libra dicta a librando quasi libera, eo quod liberaliter pondera adaequat ...'). This does not correspond to any

[25] Bubnov, *Gerberti opera* (n. 4 above), pp. 224-5.

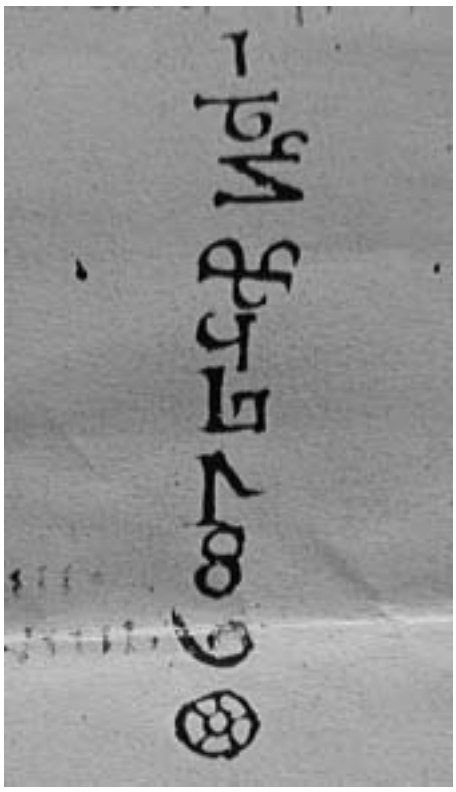
[26] Ibid., pp. 228-9.

[27] Bernelinus, *Liber abaci* (n. 9 above), Book IV.

[28] See Abbo of Fleury and Ramsey, *Commentary on the Calculus of Victorius*, ed. A. Peden, London, 2002, Appendix 8.

text that I know of, but quotes 'Comminianus',[29] Augustine,[30] Priscian,[31] and Ambrosius.[32]

5) A poem on the names of the nine Arabic numerals and the zero ('Ordine primigeno nomen iam possidet ygin').[33] One line is devoted to each numeral and each line is followed by the relevant roman and Arabic form of that numeral (See illustration).



The forms differ a little from those on the abacus table on the previous folio: the '3' is upside down, the '4' has an extra loop below the horizontal line, the

[29]'Assis secundum antiquos nominativus fuit, teste Comminiano, sed hodie genitivus'. Comminianus is a fourth-century Latin grammarian.

[30]'Augustinus: scripulus per diminutionem, a lapillo breve qui scrupus vocatus est, scripulus translato de littera nomine appellatur ...'.

[31]'Unde Priscianus dicit: semina sex alii siliquis latitantia curvis'.

[32]'Ambrosius quoque in expositione Lucae dicit siliqua est humillimum genus ligni nascens in littoribus maris'.

[33]This poem has been edited by Menso Folkerts in 'Frühe westliche Benennungen' (n. 24 above), but without the use of this manuscript. The readings of the manuscript conform to those of the edition.

horizontal stroke of the '6' leaves the vertical stroke at the base (agreeing here rather with the Echternach manuscript sheet), and the '9' gives the appearance of lying on its back. The zero is a real wheel (the 'rota' in the poem), composed of two concentric circles linked to each other by five short lines.

6) The beginning of another account of the divisions of the *as* ('Assis enim libra est .xii. untiarum, ex quibus si subtracta fuerit una .xi., quae remanent deunx vocatur ...'). This breaks off in the middle of a sentence.
7) The etymology of 'digit' ('Digitos ideo appellari puto quod sicut digiti corporales a brevibus iuncturis incipientes in longiores extenduntur, sic et innumerabiles a finita[te] unitatis quantitate inchoantes in infinitum'[34]).
8) A depiction of the portable sundial (labelled as the 'horologium viatorum').

These miscellaneous notes do not correspond consistently with any known source. They appear to be written by a single scholar, at different times, who has made frequent erasures and corrections.

Conclusions

MS Trier, Stadtbibliothek, 1093/1694 belongs to a group of fine manuscripts of a large format, copied in Echternach towards the end of the tenth century, by a hand (known as 'hand B') which also completed a copy of a manuscript dedicated to the monastery of Echternach by Leofsin.[35] The English-born Leofsin had fled to Echternach in 993 from Mettlach, where he had been abbot. Thus the Trier manuscript, including the abacus and notes on calculation, is likely to have been written soon after this date, and therefore is probably contemporary with the Echternach manuscript sheet.[36] In Leofsin's time at Mettlach learning greatly flourished, due, it seems, to a previous abbot, Ruotwic (941–ca. 975) who had sent two of his monks to Gerbert as students. One of these students, Nizo or Nithard, succeeded Ruotwic as abbot, and kept in touch with Gerbert by letter. The Trier manuscript includes a poem by Gerbert concerning Boethius and texts that reflect the teaching curriculum established by Gerbert in Reims. Thus the information provided by the Trier

[34] The sentence is incomplete; the following words have been erased.

[35] J. Schroeder, *Bibliothek und Schule der Abtei Echternach um die Jahrtausendwende*, Inaugural-Dissertation, Freiburg i. B., 1975, pp. 39–43 and 78–81. The historical information in this paragraph is summarised from Schroeder's book. The detailed investigation of the Echternach manuscripts currently being carried out by Thomas Falmagne may necessitate a slight revision of the argument.

[36] A later hand ('X') also appears in the manuscript and was responsible for adding the calendar which has a starting date of 1049. But even if the abacus and notes on calculation were added by hand X, this would not necessarily place them later than the writing of the Echternach manuscript sheet.

manuscript corroborates the connection with Gerbert d'Aurillac which we deduced in the first half of this article from the similarity of the abacus of the Echternach manuscript sheet to the instrument devised by Gerbert himself. Bernelinus of Paris, while ascribing the description of this kind of abacus to Gerbert (as has already been noted[37]), singled out the Lotharingians as those contemporaries of his who were experts in the use of the instrument.[38] It may not be by chance, then, that a depiction of an abacus, and what is probably an abacus board itself, surviving from Bernelinus's time, should come from the monastery of Echternach, which lies in the heart of medieval Lotharingia.

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[37]See n. 9 above.

[38]Bernelinus, *Liber abaci* (n. 9 above), p. 16: 'Quod si tibi taedium non esset, harum fervore, Lotharienses expetere, quos in his, ut cum (= quam?) maxime expertus sum, florere...' ('If, because of your passion for these (rules) it were not too difficult for you to seek out the Lotharingians, whom, as I have especially experienced, excel in these matters, <I would not have written this treatise>').