SCIAMVS 3 (2002), 109–125

A New Reading of *Method* Proposition 14 : Preliminary Evidence from the Archimedes Palimpsest (Part 2)

Reviel Netz
Stanford University

Ken Saito
Osaka Prefecture University

Natalie Tchernetska
Trinity College, Cambridge

I Introduction[1]

In the first part of this article (*SCIAMVS* 2(2001): 9–29) we published an edition of The Archimedes Palimpsest, coll. 110v–105r, containing about half of the text of what Heiberg called '*Method* proposition 14'. The text was especially interesting in that it included a previously unsuspected application of Lemma 11 of the *Method* (= *Conoids and Spheroids*, prop. 1) with a proportion involving infinitely many objects. We briefly touched on the implications of this new reading for the history of Greek ideas concerning infinity, and for Archimedes' position in the history of the pre-calculus.

The full text of proposition 14 begins before, and ends after, the text published in the first part. The columns 110v–105r constitute together one side (the second one) of a single Archimedes folio, 105–110. The proposition apparently begins at the very beginning of that folio, and ends on another folio (158–159). A year ago, when the first part was published, we had available to us digital images for 110v–105r alone. Now that we possess the digital images for the entire proposition we complete and revise the edition accordingly. In terms of columns of text as defined by Heiberg, we now publish five new columns, which, together with the four previously published, cover the entire proposition. The new columns are printed as bold in the sequence:

110r. col. 1 – 105v. col. 1 – 110r. col. 2 – 105v. col. 2 – 110v. col. 1 – 105r. col. 1 – 110v. col. 2 – 105r. col. 2 – **158r. col. 1**

[1]The same words of thanks from the first part of the article hold here as well. This study would have been impossible without the crucial contribution of many people: William Noel, curator of manuscripts, Cathleen Fleck, assistant curator, Erin Loftus, Conservation technician, and Abigail Quandt, senior conservator of manuscripts, all of the Walters Art Museum; Roger Easton and Keith Knox of the Rochester Institute of Technology; William Christens-Berry of Johns Hopkins University; Michael Toth of R.B. Toth Associates; and the owner of the Archimedes Palimpsest.

The main reason we find it necessary to publish the remainder of the proposition is that Heiberg's edition for it was, in all probability, mistaken even at the elementary sense of geometrical configuration. As a consequence, the first part of this article referred to the wrong diagram. This does not touch on any of the conceptual issues raised in the first part since, in fact, the central argument of proportion theory is so abstract as to involve almost no reference to the labeled diagram. Thus the historical and mathematical discussions of part 1 stand without need of any correction. Only now, however, are we in a position to offer an edition and translation of Method proposition 14.[2] This is offered in section 2. Section 3 offers brief textual and mathematical remarks on the new readings from the earlier part of the proposition.

II *Method* Proposition 14: Translation

In this article, we give the translation of the whole of proposition 14 together with the revised diagram, replacing the one we gave in the first part of this article, and the Greek text (in appendix 2) of the columns which were not included in the first part The changes are discussed in the next section.

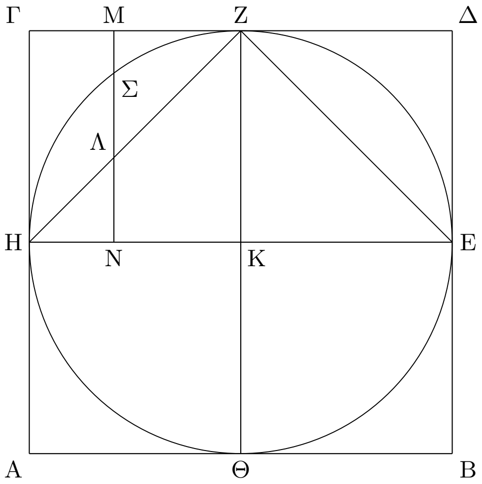
Our conventions are as follows. We use '()' for our own glosses clarifying the text; '⟨⟨⟩⟩' for the re-insertion of material that was lost through textual corruption (in this we usually follow Heiberg); '[ ]' for what was considered spurious by Heiberg.

For ease of reference, we insert Latin letters to enumerate steps of construction, and Arabic letters to enumerate steps of argument.

We provide litteral translation, which might seem somewhat less natural as English. For example, we try to use the same English words for the same Greek words ('and so' for the particle δή, etc.); we adhere to the difference of present and future tenses, so that we translate the Greek expression for proportion of the type "ἐστὶν (ἔσται) ὡς Α πρὸς Β οὕτως Γ πρὸς Δ" "it is (shall be): as A to B, so Γ to Δ.".

---

[2]To prevent the further accumulation of confusion due to proliferation of interim versions, we do not revise the edition of the four previously edited columns (with the exception of removing a single manifest error in 110v. col. 1:10, see p. 124 below), since everything will be revised in the complete publication of the Archimedes Palimpsest. The main change we might have introduced at this point would have been to 'upgrade' some readings from conjecture to fact, based on growing familiarity with the script, and, even more important, improvements in digital image processing.



(a) Let there be a right prism having a square base, (b) and let one of its bases be 110r.col.1
the square ΑΒΓΔ, (c) and let a cylinder be inscribed inside the prism, and let the base of the cylinder be the circle ΕΖΗΘ, touching ⟪the square, (d) and through its (=the circle's) centre, and⟫ (through) the side of the square in the plane opposite the (square) ΑΒΓΔ, (namaly, through) the side above ΓΔ, let a plane be drawn, (1) and so it shall cut, from the whole prism, ⟪another prism, which⟫ shall be a fourth part of the whole prism.[3] (2) This, (other) prism shall be contained by three parallelograms, and two triangles opposite each other. (e) And so, let a section of a right-angled cone[4] be drawn in the semi-circle ΕΖΗ, (f) and let its diameter be ΖΚ,
(g) and let the (line) ΖΚ itself also be that, (applied) on which, the (lines) drawn in 105v.col.1
the section are equal in square,[5] (h) and let some (line), (namely) ΜΝ, be drawn in the parallelogram ΔΗ, being parallel to ΚΖ. (3) And so it shall cut the circumference of the semi-circle at Σ, and the section of the cone at Λ. (4) And the (rectangle contained) by the (lines) ΜΝΛ is equal to the (square) on ΝΣ. (5) For this is clear.[6] (6) And so, because of this, it shall be: as ΜΝ to ΝΛ, so the (square) on ΜΝ to the

---

[3] *Elements* I. 41, XI. 32.

[4] What we call a 'parabola'.

[5] A formulaic expression, well-known from Apollonius but also attested in Archimedes (*Conoids and Spheroids* prop. 3, Heiberg [1910] 272:16–17), for what we call the 'latus rectum' of a conic section. In the case of a parabola and in terms of the diagram at hand, this is the line L satisfying the property that, for every line on the segment such as HK, $\mathrm{r}(\mathrm{KZ}, L) = \mathrm{q}(\mathrm{HK})$ ($\mathrm{r}(\mathrm{A}, \mathrm{B})$ and $\mathrm{q}(\mathrm{AB})$ represent rectangle contained by lines A and B, and square on AB, respectively). Archimedes' definition provides an elegant way of making the parabola cut the circle at H, E ($\mathrm{r}(\mathrm{HK}, \mathrm{ZK}) = \mathrm{q}(\mathrm{ZK})$).

[6] The claim of 'clarity' at Step 5 is misleading. It has, indeed, misled Heiberg: he was unable to read Step 4 so that, based on Step 5 (which he was able to read) he restored a 'clear' Step 4 — which did no more than state the property of the parabola. Step 4 is in fact the main theorem in geometrical proportion theory developed by Archimedes for this proof, not so 'clear' except to Archimedes himself. This step could be justified, for example, in the following way: since $\mathrm{MN} = \mathrm{K}\Sigma$,

(square) on NΣ.[7] (i) And let a plane be set up on MN, perpendicular to the (line) EH. (7) And so the plane shall make, in the prism cut off from the whole prism, a section, (namely,) a right angled triangle, of which one of the (sides) around the 110r.col.2
right angle shall be MN, while the other (shall be) the (line) drawn up from M in the plane on ΓΔ, perpendicular to the (line) ΓΔ, equal to the axis of the cylinder, and the hypotenuse (shall) be in the cutting plane itself;[8] (8) and so it shall also make, in the segment cut off from the cylinder by the plane that was drawn through EH and (through) the side of the square opposite ΓΔ, a section, (namely,) a right-angled triangle of which one of the (sides) around the right angle shall be NΣ, and the other (shall be) in the surface of the cylinder drawn up from Σ, perpendicular to the plane ΔH, and the hypotenuse (shall be) in the cutting plane. (9a)[9] And the triangles are 105v.col.2
similar. (9b) And since the (rectangle contained) by MN, NΛ is equal to the (square) on NΣ, (10) for this is obvious, as has been said, (11) it shall be: as MN to NΛ, so the (square) on MN to the (square) on NΣ. (12) But as the (square) on MN to the (square) on NΣ, so the triangle on MN in the whole prism cut off to the triangle on NΣ, taken away, in the (segment) cut off, from the cylinder; (13) Therefore as MN to NΛ, so the triangle to the triangle. (14) And similarly it shall be proved also that if any other (line) is drawn in the parallelogram ΔH, parallel to KZ, and a plane is set up on the drawn (parallel line), perpendicular to the (line) EH, it shall 110v.col.1
be: as the triangle made in the prism to the triangle in the segment cut off from the cylinder, so the (line) drawn in the parallelogram ΔH, being parallel to KZ, to the (line) taken by the section of the right-angled cone HZ and (by) the diameter EH. (15) Now, this parallelogram ΔH being filled by the (lines) drawn parallel to KZ, (16) and the segment contained by both: the section of the right-angled cone, and (by) the diameter EH, (being filled) by the (lines) in the segment, (17) and also the prism being filled by the triangles made in it, (18) as well as the segment cut 105r.col.1
off from the cylinder, (19) there are certain magnitudes equal to each other — the triangles in the prism; (20) and there are other magnitudes, which are lines in the parallelogram ΔH, being parallel to KZ, which are both equal to each other (21) and equal in multitude to the triangles in the prism; (22) and other triangles, in the

---

$q(MN) = q(K\Sigma)$; hence $r(MN, N\Lambda) + r(NM, M\Lambda) = q(ZM) + q(N\Sigma)$ (*Elements* II–2, I–47), of which $r(NM, M\Lambda) = q(MZ)$ because of the parabola; therefore the remaining $r(MN, N\Lambda)$ is equal to the remaining $q(N\Sigma)$ (*Elements* I–common notion 3).

[7]This inference can be explained as follows (though it must have been almost straighforward for Archimedes): from Step 4, MN, NΣ, NΛ are proportional (*Elements* VI.17); therefore MN:NΛ :: q(MN):q(NΣ) (*Elements* VI.19 Cor.).

[8]That is, the hypotenuse is in the plane drawn in Step (d), which cuts a slice of the prism (rather then the new plane, introduced in Step (i)).

[9]This Step was not read by Heiberg at all. To keep our numbering of Steps consistent with that of the first part, we label this Step (9a) and the next one (9b).

segment cut off, shall also be equal in multitude to the triangles made in the prism, (23) and other lines taken away from the lines drawn parallel to KZ between the 110v.col.2 section of the right-angled cone and EH, shall be equal in multitude to the (lines) drawn in the parallelogram ΔH parallel to KZ , (24) it shall be, as well: as all the triangles in the prism to all the triangles taken away in the segment cut off from the cylinder, so all the lines in the parallelogram ΔH to all the lines between the section of the right-angled cone and the line EH. (25) And, from the triangles in the prism, is composed the prism; (26) while, from the (triangles) in the segment cut off from the cylinder, (is composed) the segment; (27) and, from the lines in the 105r.col.2 parallelogram ΔH, parallel to KZ, (is composed) the parallelogram ΔH; (28) and, from the lines between the section of the right-angled cone and EH, (is composed) the segment [of the parabola]; (29) therefore as the prism to the segment (cut off) from the cylinder, so the parallelogram ΔH to the segment EZH contained by the section of the right-angled cone and (by) the line EH.[10] (30) But the parallelogram ΔH is half as much again as the segment so contained by the section of the right-angled cone and (by) the line EH (31) for this has been proved in the (treatises) published 158r.col.1 previously[11]; (32) therefore the prism, too, is half as much again as the segment taken away from the cylinder; (33) therefore, of such (parts) that the segment of the cylinder is (composed) of two, ⟪the prism is (composed) of three, (34) but, of such (parts) that⟫ the prism is (composed) of three, the whole prism around the whole cylinder is (composed) of 12, (35) because of (the fact that) the one being 4 (times) the other. (36) Therefore, of such (parts) that the segment of the cylinder is (composed) of two, the whole prism is (composed) of 12; (37) so that the segment cut off from the whole cylinder is a sixth part of the whole prism.

III Comments

III.1 Comments on the Reading

It is clear that Archimedes moves in the first part of the proposition towards the proportion which in the first part of the article we have described as:

$$\triangle_{\mathbf{pr}}\mathbf{:}\triangle_{\mathbf{cyl}} \mathbf{::} \mathbf{l}_{\mathbf{rect}}\mathbf{:}\mathbf{l}_{\mathbf{segm}}$$

On this basis, Archimedes develops the second, 'indivisible'-like argument in proportion theory, discussed in the first part of the article.

[10]The structure of the argument in Steps 15–29 is analyzed in the first part of this article.

[11]An equivalent result (one also requires *Elements* I.41) is proved in the first proposition of the *Method* itself, but Archimedes clearly refers to the more rigorous proof in the treatise *Quadrature of the Parabola*.

Since the first part of the argument is dedicated to 'normal' geometrical proportion theory, it naturally makes repeated reference to the diagram. Heiberg failed to reconstruct the diagram correctly and so his reconstruction of the text was faulty as well. In the first part of the article our translation, for that earlier section of the proposition, followed Heiberg's text, and the reader can see that Archimedes' argument is totally different from Heiberg's. This is a surprising result, not only because we tend to trust Heiberg, but also because we tend to trust the power of mathematical reconstruction. We may usually assume that the constraints of mathematical reasoning are such that, when reading large parts of a text, the reconstruction of the remainder becomes nearly certain. This assumption is rarely tested, and it is proved false in this particular case.

Nor should we assume that the text printed here is beyond doubt. We try to make clear what is conjectural, or doubtful, in our new reading. Steps 8–12, in particular, contain many gaps. Not that the situation overall is as difficult as it is regarding the other side of that folio where, particularly in the columns 105r. col. 1 – 110v. col. 2 Heiberg was unable to read any of the text. Here, while many of the individual words are fragmentary, they can usually be reconstructed from the visible characters. This can be seen if we print, for Steps 8-12, only those words whose reconstruction is reasonably certain:

(8) ποιήσει δὴ 110r. col. 2
κ⟨αὶ⟩ ἐν τῶι τμήματι τῶι ἀποτμη
θέντι ἀπὸ τοῦ κυλίνδρου ὑπὸ τ⟨ο⟩ῦ
ἐπιπέ̣δου τ⟨οῦ⟩ ἀχθέντος διὰ τῆς
ΕΗ̣ καὶ τῆς τοῦ τετραγ⟨ώνου⟩ πλε̣υ̣ρ(ᾶς)
τ̣ῆ⟨ς κα⟩τεναντίον τῆ̣ι̣ ΓΔ τομὴν
τρίγωνον ὀρθογώνιο̣ν οὗ ἔ̣σ̣τ̣αι μί
⟨α τῶ⟩ν περὶ τὴν ὀρθ̣ὴν γων⟨ί⟩αν ἡ
ἐν τ̣⟨ῆι ἐπιφ⟩ανείαι
⟨το⟩ῦ κ̣υ̣⟨λ⟩ίνδρου ἀνα̣⟨γο⟩μέν⟨η⟩
το⟨ῦ⟩ Σ̣ ὀρθὴ (πρὸς) τὸ ΔΗ ἐπίπεδον
⟨ὑπο⟩τείνους⟨α⟩
⟨ἐπ⟩ι̣π̣⟨έ⟩δω⟨ι⟩ **(9a)** ⟨τ⟩ρ̣⟨ίγω⟩ 105v. col. 2
να ὅμοια **(9b)**κα̣ὶ̣ ἐπεὶ ἴσον ἐστὶν τὸ
ὑπὸ ΜΝ ΝΛ τ̣ῶι ἀ̣πὸ ΝΣ **(10)** (γὰρ)
(11)
ΜΝ̣ (πρὸς) ΝΛ ο̣ὕ̣⟨τως⟩ τ̣⟨ὸ ἀ⟩πὸ ΜΝ (πρὸς) τὸ
(12) ⟨ἀ⟩πὸ ΜΝ (πρὸς) τὸ ἀπὸ
ΜΝ̣ ⟨τ⟩ρί̣⟨γ⟩ω⟨(ν)
ον⟩ ῶι ἀπο⟨τμη⟩θέντι πρ(ίς)
ματ̣ι̣ ⟨τρί⟩γω̣νο̣ν

10 τὸ ἐν τῶ⟨ι⟩
τοῦ κυ̣λί̣νδρου ἀ̣φ⟨η⟩ρ̣⟨η⟩μένο̣ν̣

It can be seen that most of the sense of the text can be recovered. Step 8 describes a triangle whose properties are very well understood with the crucial exception of the letters of the diagram. Step 9a clearly speaks of similar triangles; Step 9b is readable as a whole. Step 10 supports Step 9b in some enigmatic way. Finally, Steps 11 and 12 each define a proportion only some of whose terms are certainly given by the reading.

We also know that the text should somehow lead on to Steps 13–14 with the central geometrical proportion of the proposition; we also assume that each of the proportion statements made is correct (always having in mind, of course, the possibility of textual corruption). Thus we need to find out the reference of the letters to the points in the diagram so that, based on them, we can recover the correct proportions leading on to Steps 13-14.

The essential problem Heiberg faced was that, with the letters referring to the diagram, contextual reconstruction is no longer possible: if a letter is invisible, the adjoining character is not throwing any light on its identity. We face the same problem and simply have, thanks to digital imaging, a few more pieces of data than Heiberg had. Note that the diagram itself (which, unlike Heiberg, we attempt to print based on the manuscript evidence) is not everywhere legible and, in one place, we believe it must be emended.

Having said all that, we do believe that we have reconstructed, finally, a correct text. The main argument is the following.

The central object of the proposition is the line MN, drawn at random inside the parallelogram ΔH parallel to KZ, cutting the two inner lines — the circle and the parabola — at two points. Thus the identity of four points determines most of the remainder of the proposition. We need to answer three questions: which of M, N is which (which is on the diameter of the circle, and which is on the side of the square)?[12] How is the intersection with the circle labeled? How is the intersection

[12]In the terms of Netz [1999] 20ff., the line MN is under-specified by the Step (h) at which it is introduced. It has been noted by the same work that parallel lines in Greek geometry tend to have the same 'direction' so that the mention of MN being parallel to KZ points out a suggested preference for M being on the diameter, and N being on the side of the square. (It is very likely that Heiberg's labeling was influenced by this consideration). However, this is no more than a tendency. It might be worthwhile to notice that the text flouts another, more strictly observed tendency of Greek geometrical figures: the square ABΓΔ is not defined by a continuous circuit going through its vertices (Once again, Heiberg took it for granted that the text observes such conventions, so that we had to permute Heiberg's letters A and B).

with the parabola labeled?

The last question is the easiest to answer: the text at 105v. col. 1:8, where the label for the intersection with the parabola is introduced, is partly obscured by the Euchologion text, yet the shape of a Λ is easy to distinguish. Λ is also very clearly marked in the diagram itself on this very intersection, and nothing later on in the text is inconsistent with this specification of Λ. We may take it then as well founded. (So far, indeed, we follow Heiberg himself).

M and N usually appear as the single entity MN, but there are three ways to tell them apart. This will also bring in the point Σ. First, Step 7 makes reference to a triangle rising into space above the point (either M or N) on the side of the square. Heiberg had printed it as a dotted N. In fact the manuscript at 110r. col. 2:5 has a very clear M. (The letters are of course not radically different from each other and our improvement over Heiberg is directly due to our ability to 'zoom in' on the digital image).

Second, the diagram seems to have M on the side of the square (the diameter of the circle, on the other hand, suffered so much by being folded in the gutter of the Euchologion text that, at this point, it had become illegible).

Third and most significant, two important relations are legible with very great probability:

Step 6: MN : NΛ :: q(MN) : q(NΣ) (105v. col. 1: 11–12)
Step 9b: r(MN, NΛ) = q(NΣ) (105v. col. 2: 2–3)

The two relations are of course equivalent, giving further credence to the reading. (Notice also that in these two cases our improvement over Heiberg takes the form of emending characters he had printed with a dot: both also come from f. 105 for which Heiberg had no photograph).

NΣ is thus the mean proportional between MN and NΛ. Of the three, the geometrical significance of MN is fully known to us (regardless of which of M, N is which) — it is the line drawn inside the parallelogram. As for NΛ, it is also nearly determined, possessing only two possible meanings. It is a segment of the same line MN taken from the intersection with the parabola either to the side of the square or to the diameter.

We expect the point Σ to coincide with one of the straight lines of the figure and, most likely, to be on the line MN itself. Since, in fact, we have some reason to think that the point M is the one on the side of the square (so that the point N is on the diameter), this mathematically determines NΣ to be the line segment of MN intercepted between the diameter and the circle.

This interpretation has two main further supports. First, the character Σ is probably read (not with certainty, however) five more times in the proposition, where it would be at least consistent with being the intersection of MN with the circle. In particular, while it is almost completely hidden by a Euchologion character there, Σ is one of the few characters consistent with the visible pattern in 105v. col. 1:7 where the label is explicitly introduced (the only alternatives would be some other 'rounded' characters such as E, Θ or O). In particular, the reading in Heiberg [1915] 496,2 — a Ξ, with no dot to signal doubt concerning the reading (!) — is certainly wrong. The character Ξ is clearly ruled out by the visual evidence for 105v. col. 1:7 and, in fact, we do not find this character anywhere else in the proposition. (This is also the moment to mention that Heiberg's line in his own diagram, ΛΣ, has no basis in the figure of the manuscript).

Second, our interpretation of the relations stated in Steps 6, 9b brings out the important geometrical proportion required by the further development of the proposition. In particular, by the time we have reached Step 9b we already need this proportion: the development in Steps 10–14 ties this proportion of line segments to the relation between triangles, so that there is no room for the proportion to be developed any later in the proposition. Given all of this evidence, then, we feel confident in establishing M, N on the side of the square and on the diameter, respectively, and assigning Σ to the intersection of MN with the circle. Once this identification has been made, the remainder of the text does indeed follow almost wholly as a matter of logic.

There are two difficulties with our interpretation. First, the figure does not have Σ where we expect it. Instead, the character visible there is certainly Γ. We do not consider this a major difficulty: this Γ must be a textual corruption, since it is clear the text never refers to this point as Γ, while the letter is introduced as one of the vertices of the square. A slightly rotated half-circle Σ could easily come to be read as a Γ, explaining the corruption.

Second, the resulting text is strange on two counts. It presents the main geometrical proportion of the proposition very early on, without any argument, claiming that it is 'clear'; and then it repeats itself — down to repeating the claim of clarity. In the next section we discuss the anomalous character of the geometrical argument of Method 14.

III.2 General Comments

Put in extreme terms: the main reason why Heiberg was wrong in reconstructing the geometrical argument at the beginning of Proposition 14 is that he had honestly tried to reconstruct one. In the text as it seems to emerge, there is no geometrical argument at all.

- Steps 1–3 are descriptive and no more than clarify the geometrical configuration arising from the construction.
- Step 4 contains the central geometrical observation of the proposition: the segment intercepted by the circle is a mean proportional between the two other

segments — that intercepted by the parallelogram, and that intercepted by the parabola. (This is stated by Archimedes in terms of an equality between the rectangle on the extremes and the square on the mean). This statement is left without any argument, merely said to be 'clear' at Step 5. Step 6 then re-states the claim of Step 4 in what is a very standard manipulation of proportions. From the equality (the rectangle contained by the first and the third) = (the square on the second), we derive the proportion: (square on the first):(square on the second) :: (first):(third) (see note 7 above). Acquaintance with such equivalent ways of stating mean proportions is so deeply ingrained in the mind of a Greek mathematician that the Steps 4 and 6 hardly constitute an argument. They nearly constitute a mere notational variation.

- Steps 7–9a then move on to enrich the geometrical configuration, once again being descriptive in character. Step 9a adds explicitly the useful information (that, in Heiberg's reconstruction, was left implicit) that the two triangles, on the parallelogram-segment, and on the circle-segment, are similar. As Heiberg's implicitness itself suggests this is a very elementary and obvious observation.
- Steps 9b–11, incredibly, then exactly repeat the statements of steps 4-6.
- Steps 12–14, using the banal fact that similar plane figures are to each other as the squares on their sides, then derive from Steps 11 (providing the proportion) and 9a (providing the similarity of the triangles) a proportion involving any two triangles in the prism and in the nail-form figure, and the respective parallelogram and parabola segments:

$$\triangle_{\mathbf{pr}} \mathbf{:} \triangle_{\mathbf{cyl}} \mathbf{::} \mathbf{l}_{\mathbf{rect}} \mathbf{:} \mathbf{l}_{\mathbf{segm}}$$

From which the second part of the proposition (published in the first part of this article) proceeds, using the argument based on an extension into infinity of Lemma 11 of the Method.

In other words, we see that Steps 1–14 — the first part of the proposition — do not so much produce an argument, as create a framework for such a possible argument. This framework is produced with great care. Clarification of the construction is the only function of Steps 1–3, 7–8. An important premise of the argument is made explicit, however obvious it might appear — this is the function of Step 9a. The basic geometrical relation of the proposition is brought to the precise form required by the argument — and for this purpose the equality of a rectangle and a square (Steps 4, 9b) is transformed into a proportion involving squares and lines (Steps 6, 11). The very repetition of Steps 4–6 as Steps 9b–11 makes some sort of sense, in this perspective: the proportion involving the lines is first stated as soon as their construction is in place; following that, when the similar triangles are set up above the lines, the proportion is re-stated so that it can be seen in the richer framework, where the triangles are present as well. Everything is in place — except for an argument supporting the one remarkable fact stated by Archimedes.

Now it should be said that this basic structure — rich and careful statement of the framework for the argument, with the argument itself merely suggested — is not impossible for Archimedes. Indeed, it may be said to characterize the second half of Archimedes' argument as well. We have argued in the first part of this article that Archimedes derives a proportion of solids and areas from the proportion of areas and lines, based on a rule of summation of proportion — Lemma 11. We did not have anything explicit in Archimedes' text to support this statement. All we could note is that Archimedes, very carefully, observes the existence of several relations that, taken together, amount, as we can see, to the conditions of Lemma 11, before finally moving to stating the proportion of solids and areas. Nothing in Archimedes' text, however, is designed to show that Lemma 11 holds in this particular case. His enumeration of geometrical relations does not follow in any obvious way the structure of Lemma 11 and appears, at first glance, to be totally unmotivated (which is probably why Heiberg had failed to recover it from the Palimpsest). We thought hard of the proposition, and we are now convinced that it is based on Lemma 11: yet the important fact is that we, as readers, had to think *hard*. Archimedes had carefully set up the framework for the application of Lemma 11 — and then has left the main argument implicit.

We note one further example of the same style of argumentation, this time global in character: the axiomatic structure of Archimedes' *Sphere and Cylinder* I. The axiomatic introduction to this work sets up a rich framework. It provides a very precise definition of the concept of concavity, and adds the deep insight that one has to postulate that when two objects are concave to the same direction, and one contains the other, the container is greater than the contained. Following this careful setting up of axiomatic foundations, Archimedes then moves on, inside the treatise itself, to state, time and again, that A is bigger than B (when as a matter of fact both are concave to the same direction and A contains B), almost without making any use of his own explicit conditions. It is left to the reader to identify the arguments relying on Archimedes' axiomatic foundations. Once again, then, a detailed framework for argumentation is laid out, and then the argumentation itself is merely sketched or is simply left for the reader to complete.[13]

The argument as recovered is thus not impossible in principle. It does remain strange. We now return to discuss the remarkable, double anomaly of the text.

The first strange feature of the proposition is that it states, twice, at Steps 5/10, that the claim of Steps 4/9b is 'clear' or 'obvious'. This is almost disingenuous of Archimedes. To utter *ambulando* a beautiful result is one thing; to state then that the result is 'clear' or 'obvious' is almost dishonest. Heiberg — no mean master of the Greek mathematical form — was at some pain, reconstructing an argument

[13]We owe this observation to Henry Mendell.

leading to the claim of Step 4/9b. We believe our own reconstruction is a little easier, but it is still not very obvious.

The second strange feature of the proposition is the repetition itself: that Steps 4–6 get repeated at Steps 9b–11, down to the repetition of the meta-statement of Steps 5/10. This is extraordinary because, by the time Archimedes gets beyond Step 9a, he no longer requires Step 4 at all. What he requires is Step 6. Thus there is no reason to repeat Step 4. One could of course have accepted a brief recalling of Step 6 immediately after Step 9a, e.g. the hypothetical Step

> (*9b) and it is: as MN to NΛ, so the (square) on MN to the (square) on NΣ.

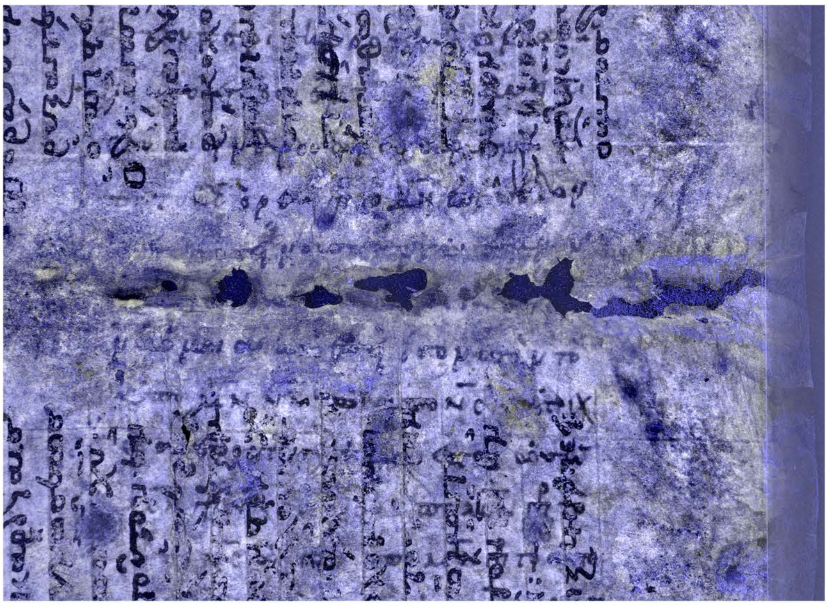
Step (*9b) is perfectly well imaginable instead of the current sequence of Steps 9b–11. Yet our text repeats not merely the end-result of Step 6, but also the whole sequence leading there of Steps 4–6. (This is particularly bizarre if we accept the tentative reading in our edition, as Step 11 then repeats the future tense of Step 6 —'it shall be: as MN to NΛ, so the (square) on MN to the square on NΣ.' While the αι at the end of ἔσται is not visible, the visual evidence seems to suggest it rather than the ιν of ἐστίν. A future tense — for a relation that is already known to hold!) In short, the repetition of Steps 4–6/9b–11 is simply inelegant.

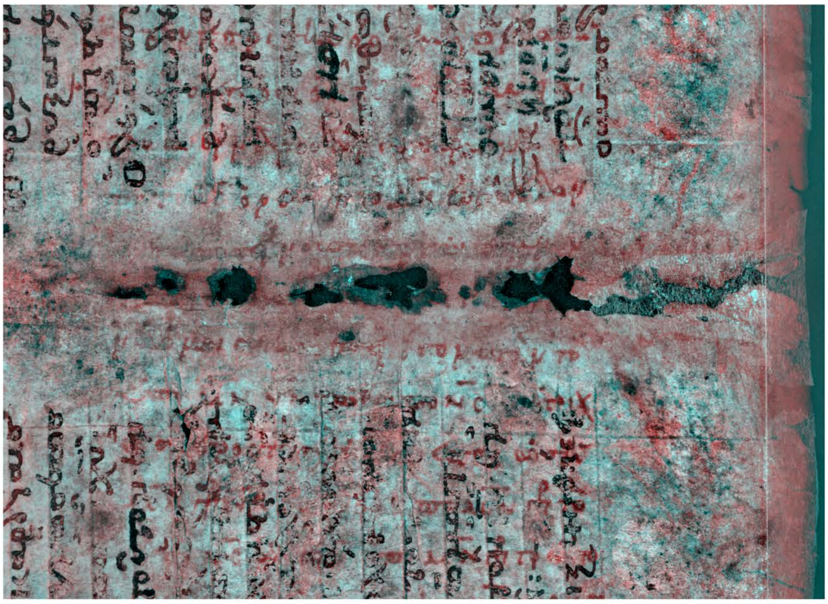
There is something jarring in this combination of brevity, in providing no argument for Step 4, and verbosity, in repeating the same Step twice. Perhaps Archimedes had produced a jarring text. However, as editors, we cannot help thinking that the most natural way to account for such a jarring combination is to assume that our text is the produce of a collaboration: Archimedes himself, as well as a less inspired scholiast. For instance, imagine the following scenario. Suppose Archimedes did not have Steps 9b–11 as we have them, but have had instead the hypothetical Step (*9b) above. A scholiast, then, realizing how central this claim was and wishing to provide it with some kind of basis, was looking for some gloss. However, all he could find was that this was indeed equivalent to Archimedes' Step 4. He thus had inserted Step 4, once again, this time as an argument to support what we hypothesize as Archimedes' Step (*9b), possibly reverting to the tenses in the original statement. Following that, the argument would have looked as follows:

> (*9b) And since the (rectangle) contained by MN, NΛ is equal to the (square) on NΣ (*10) it shall be: as MN to NΛ, so the (square) on MN to the square on NΣ.

At this point our hypothetical scholiast might have realized that he did not really understand why the rectangle is equal to the square, either. But then this was said by Archimedes to be 'clear' which, at any rate, was good enough authority to cite!

This page is left blank





Hence the final form of the text:

> (9b) And since the (rectangle) contained by MN, NΛ is equal to the (square) on NΣ, (10) for this is obvious, as has been said, (11) it shall be: as MN to NΛ, so the (square) on MN to the square on NΣ.

Obviously, many other accounts might be given, ascribing more or less of the text we have to Archimedes himself, and the above is offered merely by way of introducing the kind of textual criticism the text calls for.

Two conclusions stand:

1. The 'standard' geometrical argument of Proposition 14 — just like the special argument based on 'indivisibles' — combines a detailed build-up of geometrical framework, together with an extraordinarily brief explicit argument.
2. Whether due to an interpolator or to Archimedes himself, the text we have is anomalous: once again, we find that the more we know of the Method, the more enigmatic it becomes.

Bibliography

Heiberg, J. L. 1910–1915. *Archimedes / Opera Omnia.* Leipzig: Teubner (Vol. 1, 1910; Vol. 2, 1913; Vol. 3, 1915).

Netz, R. 1999. *The Shaping of Deduction in Greek Mathematics.* Cambridge, Cambridge University Press.

Appendix 1: Images of the Archimedes Palimpsest

The images represent roughly 110r. col. 2 l.14 – 105v. col. 2 l.6. The reproduction can not capture all the grounds for our readings, since we rely on our ability to 'zoom in' on the details of the characters. Even so, the images brings out the reading of Step 9b, in 105v. col. 2 ll.2–3 — from which, to a large extent, follows our re-identification of the labels of the diagram.

Plate 1 (facing page). Images of the Archimedes Palimpsest; *top to bottom:* picture taken by ordinary strobe light (as it looks to naked eye); ultraviolet photography (the Archimedean text is more readable); "pseudocolor" image (digitally processed image with Archimedean text appearing in red).

Appendix 2: New text of *Method* Proposition 14, 110r–105v, 158r

The following text is the parts of proposition 14 before and after the one published in the first part of this article, to which we also afford one correction.[14]

H494,5 ἔστω πρίσμα ὀρθὸν τετράγων⟨(ον)⟩ 110r. col. 1
ἔ̣χο̣ν βάσ⟨(ιν)⟩, καὶ ἔστω α̣ὐ̣του μ⟨ι⟩α τῶ(ν)
⟨β⟩άσεων τὸ ΑΒ̣ΓΔ τετ̣ράγωνον,
⟨(καὶ)⟩ ἐ̣γγεγράφ⟨θω⟩ εἰς τὸ πρίσμα κύ-
⟨λινδ⟩ρ⟨ο⟩ς, καὶ ἔστω τοῦ κυλίνδρου
βάσις ὁ ΕΖ ΗΘ κύκλος ἐφαπτό-
H494,10 μενος ⟪τοῦ τετραγώνου, διὰ δὲ τοῦ κέντρου
αὐτοῦ καὶ⟫ τῆς τοῦ τετραγώνου πλευ-
ρᾶς τῆς ἐν τῶι κατενα̣ντι̣ον ἐπ⟨ιπέ⟩-
δωι τοῦ ΑΒ ΓΔ τῆς κατὰ τὴν ΓΔ
ἐπίπεδον ἤχθω· ἀποτεμεῖ {ἀ-
ποτεμεῖ} δὴ τοῦτο ἀπὸ τοῦ ὅλου
πρίσματος ⟪ἄλλο πρίσμα, ὃ⟫ ἔσται τέτ̣αρτον μέρ(ο̣ς)
τοῦ ὅλου πρίσματος. αὐτὸ δὲ το̣ῦ̣το
ἔσται περιεχόμενον ὑ̣πὸ τριῶν
πα̣ραλληλογράμμων καὶ δύο τρι-
H494,20 γώνων κατεναντίον ἀλλήλο̣ις.
γεγράφθω δὴ ἐν τῶι ΕΖ̣Η ἡμικυ-

[14]The principle in transcription is essentially the same as in the first Part. We take this opportunity to clear a confusing statement in the first part, as if our conventions were roughly 'papyrological' in character: they are in fact quite distinct from them. This is because we wish to make our text directly comparable with Heiberg. Heiberg had used the '[ ]' symbolism for text read in the manuscripts, and considered by him to be interpolated. He thus used '⟨ ⟩', instead, as symbolism for text which ought to have been in the manuscript, but was not visible. We follow Heiberg in this usage and therefore our '[ ]' and '⟨ ⟩' differ from those used by papyrologists. A special convention we introduce here, as a result, is to use '⟪ ⟫' for text which is not read in the manuscript, but which we consider to have been omitted through textual corruption. Finally, we also use the standard conventions: subscript dot for characters visible but read without certainty, '()' for scribal abbreviations, and the '{ }' symbolism for text which is read in the manuscript but which ought to be bracketed as obvious scribal error. Note that, to avoid clutter, the critical apparatus does not note as variants the places where the new edition conflicts with Heiberg's. We add breathings, accents and punctuation. We print iota adscriptum following the manuscript, rather than modern conventions, since our readings sometimes depend upon such characters.

κλίωι ὀρθογωνίου κώνου τομῇ̣,
(διά)⟨μ⟩ετ̣ρος δὲ αυτῆς ἔστα̣ι̣ ἡ ⟨ZK⟩,

7 τοῦ τετραγώνου διὰ δὲ τοῦ κέντρου αὐτοῦ καὶ] add. Heiberg (additio ipsius longior: τῶν τοῦ ΑΒΓΔ πλευρῶν κατὰ τὰ Ε, Ζ, Η, Θ, διὰ δὲ τοῦ κέντρον αὐτοῦ καὶ). **8** ἐν τῶι] corr. Heiberg ex E. **10–11** {ἀποτεμεῖ}] exp. Heiberg. **12** ἄλλο πρίσμα ὃ] add. Heiberg. **13** τοῦ] corr. ex τὸν.

ἔστω̣ δὲ καὶ π̣α̣ρ̣ ἣν δ⟨ύ⟩ν̣αν⟨ται αἱ⟩ 105v. col. 1
καταγόμεναι ἐν τῆι τομῆι α̣ὐ̣-
τὴ ἡ ΖΚ καὶ ἤχθω τις ἐν τῶ
ΔΗ παραλληλογράμμωι ἡ ΜΝ
H496,1 ⟨πα⟩ρ̣ά̣λλ⟨η⟩λ(ος) οὖ⟨σ⟩α τῆι ΚΖ· τεμεῖ
δὴ αὕτη τὴν μὲν τοῦ ἡ̣μικυκλί⟨(ου)⟩
περιφέρειαν κατὰ τὸ Σ̣ τὴν δὲ τ⟨(οῦ)⟩
κώνου τομὴν κατὰ τὸ Λ. καί ἐστι⟨(ν)⟩
ἴσον τὸ ὑπὸ ΜΝΛ τῶ⟨ι⟩ ἀπὸ τῆ⟨ς⟩
ΝΣ̣· τοῦτο γὰρ ἐστι σαφ̣ές̣· (διὰ) τοῦ-
το δὴ ἔστ(αι), ὡς ἡ ΜΝ (πρὸς) ΝΛ οὕτ⟨ως⟩
τὸ ἀπὸ ΜΝ (πρὸς) τὸ ἀπὸ ΝΣ. καὶ ἀ-
πὸ τῆς ΜΝ ἐπίπεδον ἀνεστά-
τω ὀρθὸν (πρὸς) τὴν ΕΗ· ποιήσει δὴ
τὸ ἐπίπεδον ἐ̣ν τῶι πρίσμα̣τ̣ι̣
τῶι ἀποτμηθέντι ἀ̣π̣ὸ τοῦ ὅλ⟨(ου)⟩
πρίσματος τομὴν ⟨τρ⟩ίγωνον

9 τῆς] corr. ex τὴ̣ν̣

H496,10 ὀρθογώνιον οὗ ἔσται μία τῶν (π̣ε̣ρ̣ὶ̣) 110r. col. 2
τὴν ὀρθὴν γωνίαν ἡ ΜΝ, ἡ δὲ ἑ-
τέρα ἡ ἐν τῶι ἐπιπέδωι τῶι ⟨ἀπὸ⟩ τ̣ῆ̣⟨ς⟩
ΓΔ ὀρθὴ (πρὸς) τὴν ΓΔ ἀ⟨ναγομ⟩ένη
ἀπὸ το̣ῦ̣ Μ ⟨ἴσ⟩η τῶι ἄξονι τοῦ κυλί(ν)
δρου, ἡ δὲ ὑποτείνουσα ἐ⟨ν⟩ α⟨ὐτ⟩ῶ
τῶι τέμνοντι ἐπιπέδω· ποιήσει δὴ
κ⟨αὶ⟩ ἐν τῶι τμήματι τῶι ἀποτμη-
θέντι ἀπὸ τοῦ κυλίνδρου ὑπὸ τ⟨ο⟩ῦ
ἐπιπ̣έ̣δου τ⟨οῦ⟩ ἀχθέντος διὰ τῆς

ΕΗ̣ καὶ τῆς τοῦ τετραγ⟨ώνου⟩ πλε̣υ̣ρ(ᾶς)
τ̣ῆ̣⟨ς κα⟩τεναντίον τῆ̣ι̣ ΓΔ τομ̣ὴ̣ν
τρίγωνον ὀρθογώνιο̣ν̣, οὗ ἔσ̣τ̣αι μ̣ί-
⟨α τῶ⟩ν περὶ τὴν ὀρθ̣ὴ̣ν γων⟨ί⟩αν ἡ̣
⟨ΝΣ, ἡ̣ δ⟩ὲ̣ ἑ̣τ⟨έρα⟩ ἐν τ̣⟨ῆι ἐπιφ⟩ανείαι
⟨το⟩ῦ κ̣υ̣⟨λ⟩ίνδρου ἀνα̣⟨γο⟩μέν⟨η⟩
H496,20 ⟨ἀ⟩π̣⟨ὸ⟩ το⟨ῦ⟩ Σ̣ ὀρθὴ̣ (πρὸς) τὸ ΔΗ ἐπίπεδον,
⟨ἡ δὲ ὑπο⟩τείνους⟨α ἐ⟩ν̣ τ̣⟨ῶι τέ⟩μ̣ν̣⟨ο⟩ν̣

2–3 ἡ̣ δὲ ἑτέρα ἡ̣] corr. ex α̣ἱ̣ δὲ ἑτέραι (ἡ̣ δὲ ἑτέρα Heiberg). **3–4** ἀπὸ τῆς ΓΔ ὀρθὴ̣] Heiberg (quod ipse dubitavit: ἀ̣π̣ὸ̣ τ̣ῆ̣ς ΓΔ ὀρθὴ̣), sed legimus (π̣ρ̣ὸ̣ς̣) τη̣(ν) ΓΔ ὀρθὴν.

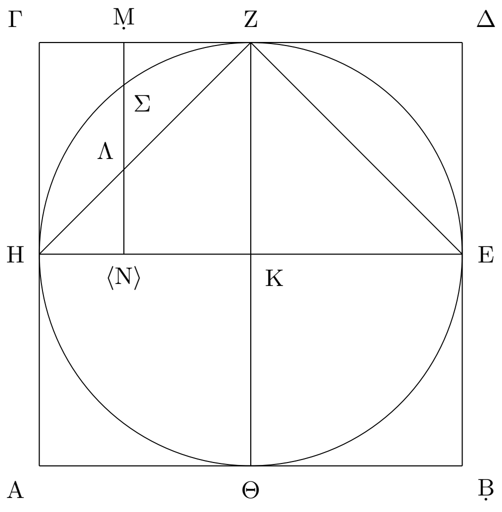
⟨τι ἐπ⟩ι̣π̣⟨έ⟩δω⟨ι, καὶ⟩ ἔσ̣τ̣ι̣⟨ν τὰ τ⟩ρ̣⟨ίγω⟩- 105v. col. 2
να ὅμοια. κα̣ὶ̣ ἐπεὶ ἴσον ἐστὶν τὸ
ὑπὸ ΜΝ, ΝΛ τ̣ῶι ἀ̣πὸ ΝΣ· ⟨τοῦ⟩το̣ (γὰρ)
φ⟨αν⟩ε̣ρ̣ό̣⟨ν, ὡ⟩ς ⟨εἴρ⟩η̣⟨ται⟩, ἔσ̣τ̣α̣ι̣ ὡς ἡ̣
ΜΝ̣ (πρὸς) ΝΛ ο̣ὕ̣⟨τως⟩ τ̣⟨ὸ ἀ⟩πὸ ΜΝ (πρὸς) τὸ
⟨ἀπὸ Ν⟩Σ̣, ὡ̣⟨ς⟩ δ̣⟨ὲ⟩ τ̣⟨ὸ ἀ⟩πὸ ΜΝ (πρὸς) τὸ ἀπὸ
⟨Ν⟩Σ̣, ⟨οὕτως τὸ⟩ ἀ̣⟨πὸ τ⟩ῆ̣⟨ς⟩ ΜΝ̣ ⟨τ⟩ρί̣⟨γ⟩ω⟨(ν̣)-
ον ἐν τ⟩ῶι ⟨ὅλωι⟩ ἀπο⟨τμη⟩θέντι πρ(ίς)-
ματ̣ι̣ (π̣ρ̣ὸ̣ς̣) ⟨τὸ ἀ⟩π̣ὸ̣ τ̣ῆ̣⟨ς⟩ Ν̣⟨Σ τρί⟩γ̣ωνο̣ν̣
τὸ ἐν τῶ⟨ι⟩ ἀ̣⟨ποτ⟩ε̣⟨μνομέ⟩νω̣⟨ι τὸ⟩ ἀ̣-
π̣ὸ̣ τοῦ κ̣υλί̣νδρου ἀ̣φ⟨η⟩ρ̣⟨η⟩μένο̣ν̣.
ὡς (ἄρα) ἡ̣ ΜΝ (πρὸς) ΝΛ ο̣ὕ̣⟨τως τὸ⟩ τρίγ⟨ω⟩ν̣(ο̣ν̣)
(πρὸς) τὸ τρ⟨ί⟩γ⟨ω⟩ν⟨ο⟩ν̣. ὁμοίως δὲ δειχθή-
σ̣εται (καὶ) ἐὰν ἄλλη τις ἀχθῆ ⟨ἐν⟩
H498,1 ⟨τῶ⟩ι̣ ΔΗ̣ π̣α̣ρ̣α̣λ̣λ̣η̣λόγραμ̣μωι π(αρὰ)
τὴν Κ̣Ζ, ⟨καὶ ἀ⟩πὸ τῆς ἀχθείσης
ἐπίπ⟨ε⟩δο⟨ν ἀνε⟩στάτ̣⟨ω⟩ ὀ̣ρθὸ⟨ν⟩ (πρὸς) τ(ὴν)

11 post ἀφηρη̣μένον add. Heiberg ὑπὸ τῆς τοῦ κυλίνδρου ἐπιφανείας adnotans ‘τῆ̣ς] om.’ quod mirandum est; lineam enim omnem scripsit ipse. **14** ἐὰν] corr. Heiberg ex ἐν.

Here comes 110v–105r, edited in the first part of this article. We have one correction to the previous edition; the apparatus to 110v. col. 1:10 is suppressed, and the text τοῦ ΔΗ reads τού(του) ΔΗ.

H:500,1 δειϰται γὰρ τοῦτο ἐν τοῖ⟨ς⟩ πρότερο(ν) 158r. col. 1
ἐϰδ⟨εδομ⟩έ̣ν̣⟨οι⟩ς· ἡμιόλ̣ιον ἄρα (ἐστὶ)
ϰ̣⟨αὶ τὸ πρίσ⟩μ̣⟨α⟩ τ(οῦ) ἀπ̣⟨ο⟩τμήματ(ος)
τοῦ ⟨ἀ⟩φηρημένου ἀπὸ τοῦ ϰυλ̣ί̣ν̣-
δρου· οἵων (ἄρα) ἐστὶ τὸ ἀπότμημα
το⟨ῦ ϰυ⟩λίν⟨δ⟩ρου δύο, τοιούτων (ἐ̣σ̣τ̣ὶ̣ν̣)
⟨⟨τὸ πρίσμα τριῶν. οἵων δὲ⟩⟩ τὸ
πρίσμα τρι̣ῶν τοιούτων ἐστὶν τὸ
ὅλον πρίσμα τὸ περὶ ὅλον τὸν
ϰυλίνδρον ΙΒ (διὰ) τὸ Δ (εἶναι) τὸ ἕτερον
⟨τ⟩ο̣ῦ̣ ἑτέρου· οἵων (ἄ̣ρ̣α̣) ⟨τ⟩ὸ ἀπ̣⟨ό⟩τμημα
τοῦ ⟨ϰυ⟩λί⟨νδρου⟩ δύο, τοιούτων ἐστὶ(ν)
τὸ ὅλον πρίσμα ΙΒ. ὥστε τὸ τμῆ-
H:500,10 μα τὸ ἀποτμηθὲν ἀπὸ τοῦ ὅλ⟨(ου)⟩
ϰυλίνδρου ἕϰτον μέρος ⟨(ἐστὶ)⟩ τ⟨(οῦ) ὅλου⟩
πρίσμα⟨τος⟩.

6 τὸ πρίσμα τριῶν. οἵων δὲ] add. Heiberg.



Σ] corr. ex Γ.

(Received: June 13, 2002)