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The Geometrical Problems of Nu^{c}aim ibn Muḥammad ibn Mūsā (ninth century)

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Introduction

In 2001, the Institute for the History of Arabic-Islamic Science in Frankfurt published a facsimile edition [24] of a medieval Arabic manuscript in Istanbul. This manuscript contains, in addition to a number of astronomical texts, a collection of geometrical problems[1] from a book by "Nu^{c}aim ibn Muḥammad ibn Mūsā the astronomer." No other manuscripts of this work have been found. In this paper I present these *Geometrical Problems* of Nu^{c}aim ibn Muḥammad ibn Mūsā in a critical Arabic edition with literal English translation and notes. My main aim is to make this interesting text accessible to the reader. Although some footnotes have been added to the translation, I have not attempted to write an extensive commentary.

The author's name, "Nu^{c}aim ibn Muḥammad ibn Mūsā the astronomer," is not mentioned in the standard medieval Arabic biographical works. He was probably the son of Muḥammad ibn Mūsā ibn Shākir, one of the three Banū Mūsā ("sons of Mūsā"),[2] who were distinguished mathematicians, astronomers and translators of Greek works in the ninth century AD. Al-Bīrūnī mentions observations of the solar solstices by Muḥammad ibn Mūsā and his brother Aḥmad in or near Bagdad between the years AD 857 and 868 [7, 64-65]. Muḥammad ibn Mūsā ibn Shākir died in Rabī^c I 259 H./jan.-feb. 873 [13, 316:5]. Thus it is likely that Nu^{c}aim compiled his *Geometrical Problems* in the late ninth century AD.

An implicit reference to Nu^{c}aim may be found in the following passage in a list of works by Thābit ibn Qurra (836-901 AD),[3] who was a close collaborator of the Banū Mūsā:

"Among his (i.e., Thābit's) works is a number of summaries in astronomy and geometry, which I saw in his own handwriting, and their translations, in his handwriting; what Thābit made (i.e., wrote) for the boys (*al-fityān*), may God protect them, and I think that he meant the children of Muḥammad ibn Mūsā ibn Shākir."

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[1]This collection of problems was first mentioned by Sezgin in 1979 [23, VII:404].

[2]On the Banū Mūsā see [23, V:246-252, VI:147-148].

[3]On Thābit ibn Qurra see [23, V:264-272. VI:163-170].

The "I" in the quoted passage is Abu ^{c}Alī al-Muḥassin ibn Ibrāhīm ibn Hilāl al-Ṣābī, who was married to a granddaughter of Thābit.[4]

The *Geometrical Problems* is a disorganized collection of 42 geometrical propositions. The level of the work is higher than that of the *Book of Assumptions* (kitāb al-mafrūḍāt) by Thābit ibn Qurra, but lower than that of the the *Selected Problems* (al-masā'il al-mukhtāra) by Thābit's grandson Ibrāhīm ibn Sinān (906-947).[5] These three works are all about plane geometry of lines and circles, and they contain problems that can be constructed by means of ruler and compass. Some of the geometrical problems of Nu^{c}aim were also solved by other geometers, including Thābit ibn Qurra, Ibrāhīm ibn Sinān, and the late-tenth century authors Abū Sahl al-Kūhī and Aḥmad ibn Muḥammad ibn ^{c}Abd al-Jalīl al-Sijzī.[6]

The text of Nu^{c}aim is interesting because it dates back to an early period in the history of Islamic geometry. Some of his techniques seem archaic. For example, Nu^{c}aim solves most problems which involve an arbitrary given ratio only for numerical ratios such as 1 : 2.

A major problem in the history of Islamic geometry is the identification of the context of the surviving medieval Arabic geometrical works. It is clear that some geometrical problems were motivated by ancient Greek works or by teaching purposes, and that others were studied because of possible applications in astronomy surveying, architecture and optics. However, it is usually impossible to determine the context of a given geometrical problem. Thus we do not really know for what purpose Nu^{c}aim composed his collection of geometrical problems. Perhaps the veil can be lifted by future research of the Arabic geometrical and astronomical literature of the early period.

In the unique manuscript of the *Geometrical Problems* it is stated that the text was copied from the handwriting of Naṣīr al-Dīn al-Ṭūsī (1201-1274).[7] Al-Ṭūsī was a famous mathematician and astronomer, who produced editions of many Arabic mathematical texts and Arabic translations of Greek mathematical texts.[8] He also revised the *Book on the Measurement of Plane and Spherical Figures* by the Banū

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[4]The list was preserved by Ibn al-Qifṭī (died 1248), for the quoted passage see [13, 120:1-2]. Although Nu^{c}aim may have been an astronomer himself, the "astronomer" was probably meant to refer to his father Muḥammad ibn Mūsā, in order to distinguish him from the algebraist Muḥammad ibn Mūsā al-Khwārizmī (see [13, 358:3]). Al-Khwārizmī was also an astronomer but not of the same calibre as Muḥammad ibn Mūsā ibn Shākir.

[5]On Ibrāhīm ibn Sinān see [23, V:292-295, VI:193-195]. His *Selected Problems* have been edited in [21] and also in [4] with French translation.

[6]On al-Kūhī and al-Sijzī see [23, V:314-321, 329-334].

[7]This statement does not imply that the extant manuscript was *directly* copied from a text in al-Ṭūsī's handwriting.

[8]On al-Ṭūsī see [16, II:392-408].

Mūsā [29], and the *Book of Assumptions* by Thābit ibn Qurra [30]. In [8], Yvonne Dold has compared the original text of the *Book of Assumptions* with the revision by al-Ṭūsī. She concludes that al-Ṭūsī rewrote the text and changed the proposition numbers, but left the contents essentially unchanged.

In the case of Nu^{c}aim's text, al-Ṭūsī added a brief introduction in which he says that he had one corrupted manuscript of the *Geometrical Problems*, and that he corrected the text where he could, but left the passages that he could not understand unchanged. The text we have contains a few additions by al-Ṭūsī in props. 7, 35, 36, 42, including an alternative proof of prop. 36 by his colleague-astronomer al-^{c}Urḍī.[9] Al-Ṭūsī began his introduction and his additions to the *Geometrical Problems* by *aqūlu*, "I say," just as in his revision of the Arabic version of Archimedes' *On the Sphere and Cylinder* [15, 98].[10] Thus it appears that the rest of the text of the *Geometrical Problems* contains the original argumentation (if not the wording) of Nu^{c}aim.

Most of the propositions in the text are provided with proposition numbers, and the missing numbers 28-33 and 37-39 can be restored unambiguously. The reconstructed numbering is interesting because the text of propositions 28 and 29 is repeated almost literally as 30 and 31. I conclude that the numbering was introduced not by Nu^{c}aim but by a later editor, with al-Ṭūsī as the most likely candidate. Al-Ṭūsī's other editions show that he was a careful editor (cf. [14, 54]), who would certainly have deleted the useless propositions 30 and 31 if he had worked through Nu^{c}aim's text a second time. Thus our manuscript seems to be based on some very preliminary form of a revision of Nu^{c}aim's *Geometrical Problems* by al-Ṭūsī. It is therefore likely that our manuscript containt the *Geometrical Problems* of Nu^{c}aim ibn Mūsā essentially in their original form, apart from the lacuna in prop. 27 which al-Ṭūsī indicated, and with the possible exception of propositions that al-Ṭūsī decided not to copy.

The next section contains a brief summary of the propositions in the extant version of the *Geometrical Problems*. This is followed by the edition and translation, with some mathematical commentary in footnotes. Because most of the proofs are clear, I have not paraphrased them in my summary.

Summary of the Geometrical Problems

Here is a summary of the 42 geometrical problems of Nu^{c}aim ibn Muḥammad ibn Mūsā in modern notation. A notation such as ΔABC means the area of triangle

[9] On al-^{c}Urḍī see [16, II:414-415] and also [22, 111-116].

[10] Compare also al-Ṭūsī's revision of the *Lemmas* of pseudo-Archimedes, where he used *qāla* ("he said") for the words of Archimedes and the earlier commentator al-Nasāwī, and *aqūlu* ("I say") for his own comments. See [31, 3:19, 11:20, 18:12].

ABC, $[ABCD]$ means the area of quadrilateral $ABCD$. The reader may want to refer to the figures in the translation, which bear the number of the proposition in question.

If a problem is also solved by one or more other authors, these authors are mentioned in the summary but the detailed references can be found in footnotes to the translation.

1. Theorem: If the four sides of a quadrilateral are known and the angles at the base are equal (but not known), then the two diagonals are known.
2. Geometrical construction of a right-angled triangle with sides a, b, c in such a way that $a : b = b : c$.
3. Let E be a given point inside triangle ABG. Required to construct a straight line HEZ which intersects AB and AG at Z and H respectively such that (1) $ZE = 2EH$, or (2) $ZE : EH$ is a known ratio.
4. Solutions of the algebraic equations $x^2+ax = b$ and $x^2+b = ax$ for $a, b > 0$. For each equation, Nu^{c}aim presents (1) a solution by "application of areas" as in Book VI of Euclid's *Elements*; (2) an analysis using Book II of the *Elements*; (3) a geometrical construction somewhat like the algebraic solution in the *Algebra* of al-Khwārizmī [20, 8-11]. Unlike al-Khwārizmī, Nu^{c}aim fails to remark that the equation $x^2 + b = ax$ can only be solved if $b \leq (a/2)^2$ and has two roots if $b < (a/2)^2$.
5. Let DEB be a triangle such that $\angle E$ is a right angle, draw the altitude EG, and let H be a point such that $DE = DH = HB$. Theorem: $HG = GD$.
6. Let D be a given point on side AB of the given triangle ABG. Required to construct a straight line DME which intersects AG at M and BG extended at E in such a way that (1) $\Delta BDE = \Delta ADM$, or (2) $\Delta BDE = k \cdot \Delta ADM$, where k is a given number.
7. Required to construct a triangle ABD in such a way that AG and $AD+DB$ are known in magnitude and $BG : GD$ is a known (numerical) ratio. This problem is reduced to the following auxiliary problem: to divide a given segment a into two parts x and y in such a way that $a = x + y$ and $x^2 = c^2 + (\gamma y)^2$ for a given segment c and a given numerical ratio γ. Al-Ṭūsī complains that he does not understand the solution of this auxiliary problem and he therefore copies the corrupted text just as it stands. I have corrected the scribal errors in my critical edition. The solution of the auxiliary problem is also invoked in props. 28-31.
8. D is a given point on side AG of the given triangle ABG. Required to construct a straight line WDE which intersects side AB at W and side BG extended at E in such a way that $\Delta DGE : \Delta ADW$ is a given ratio. Nu^{c}aim only treats the case where $\Delta DGE = 2\Delta ADW$ and leaves the general case to the reader.
9. From point H on the centroid of triangle ABG we draw two lines HE, HZ to

two different points E, Z on the rectilinear extension of the base BG. Points E and Z are on different sides of the triangle in such a way that $BE = GZ$. Let HZ intersect AG at T and let HE intersect side AB at L. Theorem: $\Delta AHT = \Delta AHL$. The proof is complicated.

10. Required to construct a square circumscribing a given scalene triangle. The same construction is found in a work of Abu'l-Wafā' al-Būzjānī (940-ca. 998)
11. Required to construct a square inscribed in a given scalene triangle. This constructions also occurs in the same work of Abu'l-Wafā' as prop. 10.
12. Theorem: the three altitudes in a triangle pass through a single point.
13. Let triangle BDW be given. Required to construct Z on BD and H on BD such that $ZH//WD$ and (1) $ZH = HD + ZW$ or (2) $ZH = c \cdot (HD + ZW)$ for a given number c.
14. The same as in no. 13 with $ZH^2 = HD^2 + ZW^2$ instead of (1) or $ZH^2 = c \cdot (HD^2 + ZW^2)$ instead of (2).
15. BG is a given chord in a circle of given diameter. Point A is on arc BG such that $BA : AG$ is a given ratio. Theorem: BA and AG are given in magnitude. A more complicated solution of the same problem can be found in the *Book of Assumptions* by Thābit ibn Qurra.
16. D is a given point on the rectilinear extension of the base BG of triangle ABG. Required to construct a straight line DZE which meets side AG at Z and side AB at E in such a way that (1) $\Delta DZG = \Delta AZE$. The solution is easy.
17. The same as no. 16, but instead of (1), $n\Delta DZG = \Delta AZE$, with n a given number (the case $n = 2$ is treated in detail). This more difficult problem was solved in a different way by the late tenth-century Iranian geometer Abū Sahl al-Kūhī.
18. Let $ABGD$ be a given parallelogram. Required to construct a straight line $AHZE$ with H on GB, Z on GD and E on BD extended, in such a way that $\Delta GZH = \Delta ZDE$. It is likely that this problem is a variation on the "lemma of Archimedes" for the construction of a regular heptagon, see [12, 204-213].
19. T is a given point on the rectilinear extension of the base ZH of a given parallelogram $ZHBA$. Required to construct a straight line TLM which intersects the sides BH and AZ at points L and M in such a way that (1) $\Delta TMZ = [AMLB]$, or (1) $\Delta TMZ = q \cdot [AMLB]$, with q a given number.
20. A square $ABGD$ is divided by the given straight line EZ parallel to side AG into two areas, and one (square) ell of area $AEZG$ is worth five times as much as one square ell of the remaining area $BEZD$. Required to construct a straight line through A which divides the square into two area of equal value.
21. A is a given point, and lines HG, GL are two given straight lines intersecting at G. Required to construct a straight line AZS which intersects LG at Z and HG extended at S in such a way that ΔZGS is equal to a given area B. The same problem was solved in a more difficult way by Ibrāhīm ibn Sinān

(907-946), and in an easier way by Abū Sahl al-Kūhī.

22. A given parallelogram $ABGD$ is divided by a given line EZ parallel to sides AB and GD. Required to construct a straight line $AKML$ which intersects EZ at K, DG at M and BG extended at L in such a way that $\Delta LGM = [MKED]$.
23. A lemma to prop. 25.
24. Another lemma to prop. 25.
25. In the square $ABGD$ we consider given points E on side AD and Z on side AB such that EZ is parallel to the diagonal BD. Required to construct a straight line $ASLT$ which intersects EZ at S, BD at L and GD at T in such a way that (1) $\Delta LDT = \Delta AES$, or (2) $\Delta LDT = n \cdot \Delta AES$ for any given number n.
26. If the area of a right-angle triangle is known and its circumference is known, its hypotenuse is known.
27. If the area of a right-angle triangle is known and its circumference is known, the two other sides are also known. The proof breaks off after the first sentence. Al-Ṭūsī also presents Nu^{c}aim's figure, but I have not been able to reconstruct the proof.
28. Angle A of triangle ABG is a right angle. We draw the altitude AD. If AD and $AG + GB$ are given in magnitude and $BD : DG$ is a given ratio, the sides of the triangle can be found. The solution is based on the auxiliary problem in prop. 7.
29. In the same triangle as prop. 26, if AD and BG are given in magnitude and $AB : AG$ is a given ratio, the sides of the triangle can be found. The solution is also based on the auxiliary problem in prop. 7.
30. The same problem as prop. 28, with the same solution.
31. The same problem as prop. 29, with the same solution.
32. From point H on a circle we draw two perpendiculars HZ, HT onto two perpendicular diameters EA, EG of the circle. If AZ and GT are known in magnitude, the diameter is also known in magnitude. The solution is based on prop. 15.
33. Triangle ABG is given and $\angle B$ is a right angle. Required to construct point Z on side AB in such a way that $GA + AZ = GZ + ZB$. Thābit ibn Qurra solves the same problem in his *Book of Assumptions*. Abū Sahl al-Kūhī solves the problem for an arbitrary triangle ABG.
34. Triangle ABG is isosceles. We draw the perpendicular AD. If AB, AG and $AD + BG$ are known in magnitude, the triangle is known. The text states a diorismos (necessary and sufficient condition for the existence of a solution) but the discussion is incomplete.
35. In any circle, the sum of the chord of an arc of 60^{o} plus the chord of an arc of 36^{o} is equal to the chord of an arc of 108^{o}. (This theorem is equivalent to $\sin 18^{o} + \sin 30^{o} = \sin 54^{o}$.)
36. An alternative proof of prop. 35 (probably by Nu^{c}aim), and an alternative proof of *Elements* XIII:9. Al-Ṭūsī added another proof of prop. 35 from a work

entitled *Fawā'id* ("Useful things") by his colleague, the astronomer Mu'ayyad al-Dīn al-^{c}Urḍī.

The remaining propositions show how to divide a given segment a into two segments x and y according to certain conditions. Notations: γ is a given number, c_1 and c_2 are given segments. One condition is of course $a = x + y$, and the other condition is listed after the proposition number. Nu^{c}aim provides two types of solutions. In one type of solution, which I call the rectangle-type, he constructs a certain rectangle by Euclid's *Elements* VI:28-29. In the other type of solution, which I call the triangle-type, he constructs an isosceles right-angled triangle of given area. While the rectangle-type of solution is standard in the medieval Islamic tradition, the triangle-type of solution does not seem to have been much used after the ninth century. The two types of solutions will be briefly introduced below. Here are the propositions with the second conditions and the solution methods:

37. $\gamma ax = y^2$; rectangle.
38. $ax + c_1^2 = y^2 + c_2^2$; rectangle.
39. $\gamma ax = y^2$; triangle.
40. $2ax + c_1^2 = y^2$; triangle.
41. $2ax + c_1^2 = y^2 + c_2^2$; rectangle.
42. $\gamma xy + c_1^2 = y^2 + c_2^2$; rectangle. The text begins with a solution by al-Ṭūsī of case $\gamma = 2$, and it then continues with Nu^{c}aim's solution for arbitrary γ, which al-Ṭūsī may not have understood.

Finally, al-Ṭūsī adds a proposition which he found at the end of an unspecified book of the Banū Mūsā:[11]

43. If segment a is divided in extreme and mean ratio into x and y (i.e., $a = x + y,\ a : x = x : y$), then $ax + y^2 = 2x^2$ and $a^2 + y^2 = 3x^2$.

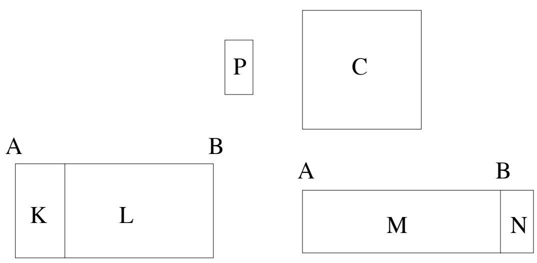

Figure 0

[11]Note that they were the father and two uncles of Nu^{c}aim.

I now introduce some terminology which may be unfamiliar to the modern reader. Nu^{c}aim often uses the Euclidean technique of "application of areas." In *Elements* VI:28-29, Euclid explains how to apply to a given line AB a parallelogram equal to a given area C and deficient by (or exceeding by) an area similar to a given parallelogram P. Figure 0, left side, illustrates the "deficient" case: the defect, namely parallelogram K, is similar to P, and L is equal in area to C. Figure 0, right side illustrates the "excessive" case: the excess N is similar to P and $M+N$ is equal in area to C. Nu^{c}aim uses the proposition for the case where P is a rectangle, so the applied parallelogram is also a rectangle. In most cases P is a square, so the problem boils down to the solution of the quadratic equations $AB \cdot x - x^2 = C$ ("defective" case) and $AB \cdot x + x^2 = C$ ("excessive" case), as Nu^{c}aim explains in proposition 4. Nu^{c}aim would say that we apply the known area C to the line AB in such a way that it exceeds (or falls short of) its completion by a square.

In propositions 39-40 Nu^{c}aim does not use the Euclidean application of areas, but rather the construction of a right-angled isosceles triangle equal to a given area C. This construction is based on *Elements* V:25 or *Elements* V:13. Algebraically, the side of the required triangle can be found as the square root of $2C$. In props. 37-38 and 39-40 he solves the same problem in both ways. In anachronistic algebraic terms, Nu^{c}aim reduces the solution of an arbitrary quadratic equation in props. 37-38 to the extraction of a square root in props. 39-40.

The Manuscript and Editorial Procedures

The manuscript Istanbul, University Library A.Y. 314 is a collection of nine mathematical and astronomical treatises. A description of its contents can be found in [24, v-x]. The text by Nu^{c}aim ibn Muḥammad ibn Mūsā are found on f. 122b-136b of the manuscript, which pages have received the numbers 246-274 in the facsimile edition [24]. The text by Nu^{c}aim is undated but the previous texts are copies of texts written in A.D. 1297/98 (cf. [24, v]). The manuscript is clearly written but unfortunately the quality of the text is rather imperfect. In the beginning of the extant manuscript of Nu^{c}aim's *Geometrical Problems* it is stated that the text was copied from al-Ṭūsī's handwriting. This implies that an archetype of the extant manuscript (not necessarily the extant manuscript itself) was a direct copy of the manuscript in al-Ṭūsī's own hand.

In my edition of the Arabic texts I have tried to restore the wording of Nu^{c}aim's original as far as is possible, assuming that al-Ṭūsī's revision consisted in the correction of scribal errors. I have corrected some random mathematical mistakes which were caused by scribal error and which al-Ṭūsī may not have noticed. Most of the times these mistakes can be explained by the assumption that one or two words were omitted from the text by mistake. There is no reason to assume the presence of these mistakes in the original text by Nu^{c}aim. In the passages in prop. 7 and 42 which al-Ṭūsī did not understand, I have attempted to restore the original text of Nu^{c}aim.

In the Arabic text I have provided some punctuation, and I have made a few adaptations in the orthography without noting this in the apparatus. Thus I have changed ثلثة and شينا to ثلاثة and شئنا. I have added passages between angular brackets < > in order to restore the original text. Passages between square brackets [ ] should be deleted from the manuscript in order to restore the original text by Nu^{c}aim. References to new pages of the facsimile edition are in parentheses ( ).

I have redrawn the figures but have tried to stay close to the figures in the manuscript, which can of course be consulted in the facsimile edition [24]. Letters in geometrical figures have been transcribed according to the following system (the letters have been listed in the order of increasing numerical values): ا = A, ب = B, ج = G, د = D, ه = E, و = W, ز = Z, ح = H, ط = T, ي = I, ك = K, ل = L, م = M, ن = N, س = S, ع = O, ف = F, ص = C, ق = Q, ر = R, ش = J, ت = t, ث = h, خ = J, ذ = V, ض = d, ظ = Y, غ = g. In most propositions, Nu^{c}aim labels the points in the order of the numerical values. Propositions in which this "natural" order is not used may be adaptations by Nu^{c}aim from other sources.

In the translation, words in in angular brackets < > are translations of words or passages that I have added to the Arabic text in order to restore the original text by Nu^{c}aim. Passages in parentheses ( ) or square brackets [ ] are my own additions, which do not occur in the text by Nu^{c}aim. The footnotes contain some mathematical commentary. In order to keep the footnotes limited in number, I have included in the text a few references to Euclid's *Elements* [9] and *Data* [10], [11] in the form [*El.* VI:28]. These references do not occur in the Arabic manuscript text. In the translation, the page numbers in the facsimile edition [24] also appear in square brackets.

Acknowledgements. I began working on the text by al-Nu^{c}aim in 2001 during a sabbathical leave at the Department of Mathematics of the University of Virginia in Charlottesville (USA). I am grateful to Professors Karen Parshall and Jim Howland for their hospitality and for creating excellent working conditions. It is a pleasure to thank Professor J. Len Berggren (Burnaby BC, Canada) and an anonymous referee for comments and corrections to a preliminary version of this paper.

Translation

Copied from the handwriting of the blessed master, his excellency Naṣīr al-Dīn, may God sanctify his soul.

These are geometrical problems from the book of Nu^{c}aim ibn Muḥammad ibn Mūsā the astronomer, which I copied from an extremely corrupted text. I have corrected what I understood of it, and I have copied what I did not understand as it was in the corrupted version in the text. I ask God for help.

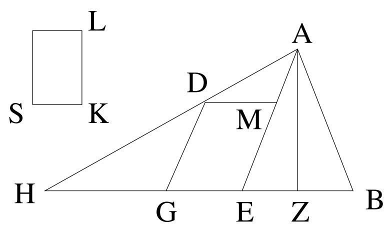

Figure 1

1. (Fig. 1) If the sides of the quadrilateral $ABGD$ are known and its angles ABG, DGB are equal, and we draw the perpendicular AZ [and let it be known],[12] and (if) we want to know the two diagonals of the quadrilateral, then let us draw AE parallel to DG. We extend AD, EG to meet at H. We draw DM parallel to line BGH, then it is equal to EG. Since AB is equal to AE and ME is equal to DG, AM and ME are known. [247] But the ratio of AM to ME is as the ratio of AD to DH, and AD is known, so DH is known.

The known square AH is equal to the (sum of the) squares of AZ, ZH, that is the (sum of the) squares of AE, EH and the area[13] BE times EH, that is the (sum of the) square of AE and the area BH times HE. But the square of AE is known, so by subtraction, the area BH times HE is known. But BG is known.

Let the ratio of EG to GH, which is known since it is equal to the ratio of AD to DH, be the ratio of one to two. We construct the area KLS with right angles such that LK is < one and a > half times KS. If we apply to the known line BG the known area BH times HE in such a way that it exceeds its completion by an area similar to the area KLS, and in such a way that the (segment) GH which is

[12] The solution does not assume that the length of the perpendicular AZ is known. I therefore take this passage as an addition by a later writer, possibly Naṣīr al-Dīn. If the length of AZ is known, the problem is overdetermined.

[13] The author almost always uses *area* (saṭḥ) in the sense of rectangle.

produced corresponds to KS, then GH is known, and the whole BH is known.[14] After this we know the two diagonals of the quadrilateral.[15] That is what we wanted.

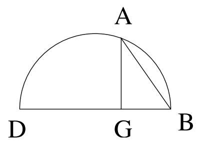

Figure 2

2. (Fig. 2) We want to construct a right-angled triangle in such a way that the ratio of the shorter side to the intermediate (side) is as the ratio of the intermediate (side) to the longer (side). Thus let us assume a straight line BD. We describe on it the semicircle BAD. We divide the line at G in extreme and mean ratio [*El.* VI:30]. We draw the perpendicular GA and we join AB. Then the area DB times BG is equal to the square of GD, that is the square of AB, and the ratio of BG to GA is as the ratio of GA to GD, that is AB. That is what we wanted.[16]

3. (Fig. 3) If there are a triangle ABG and an arbitrary line AD in it, and at AD a point E, and we want to pass a line through it which ends at AB, AG in such a way that the part intercepted between AB, AD is twice the part intercepted between AD, AG, or three times (it), or any ratio we want, and let it (i.e., the part between AB, AD) for example be twice it (i.e., the part between AD, AG),

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[14]Using *El.* VI:29, the author constructs a rectangle BHX with HX perpendicular to HB in such a way that (1) $BH \cdot HX = AH^2 - AE^2$, which is a known area, and (2) $HX : HG = HE : HG = HA : HD$, which is a known ratio. In the terminology of Euclid, condition (2) is expressed in the following form: the "exceeding" rectangle GHX is similar to an auxiliary rectangle LKS with $LK : KS = HA : HD$. If $AD : HD = 1 : 2$ as in the text, we must have $LK : KS = 1.5 : 1$, as in my emendation of the text.

[15]The construction presupposes familiarity with the *Data* of Euclid. Nu^{c}aim assumes that the sides AB, BG, GD, DA are known in magnitude but not necessarily in position, and that the angles ABG and DGB are equal (but not known). He then supposes without loss of generality that B and G are known in position, and he assumes by way of analysis that points A and D have been found. He then argues that AM, ME, AD, DH are "known", but the word means "known in magnitude", not position. Then he finds that the area BH.HE is known in magnitude, so he can construct the position of H. Because he knows the magnitudes of AB and AH, he can construct the position of A. Then AH is known in position so he can find the position of D.

[16]By $AG^2 = DG \cdot GB$, hence $AB^2 = AG^2 + GB^2 = DG \cdot GB + GB^2 = DG \cdot GB$ as in the text. Putting $BD = 1$, $\phi = \frac{1}{2}(\sqrt{5} - 1)$, and assuming $GB < GD$, we have $GD = AB = \phi, AG = \phi^{\frac{3}{2}}, GB = \phi^2, \angle BAG = 38^o 10'$. Triangles AGD and DAB also have the required property.

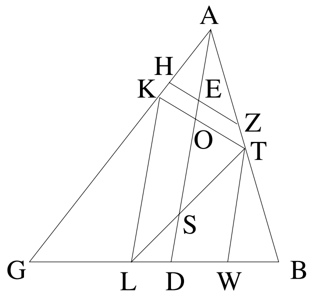

Figure 3

then we mark on AB the point T arbitrarily and we draw TW parallel to AD. We make WD twice DL. We join TSL. Then TS is twice SL. We draw LK parallel to AD and we join TK. Then TO is twice OK. We draw from E a line parallel [248] to line TK to meet the two sides at the points Z, H. Then ZE is twice EH. That is what we wanted.

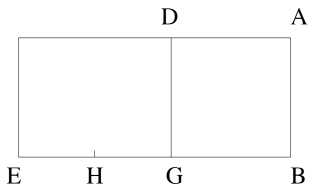

Figure 4a

4. If (the sum of) a property (x^2) and a number of roots (ax) is equal to a known number (b, i.e., $x^2 + ax = b$), we make the property the square $ABGD$ (Fig. 4a). We apply to side GD of it an area equal to the number of roots (ax),[17] namely the area GDE. Then GE is the number of the roots (a), and it is known. The whole area AE is known, since it is the sum of the property and the roots which are equal to the known number. If we apply the known area (b) to the known (line) GE (a) in such a way that it exceeds its completion by a square [*El.* VI:29], the exceeding square is the property.

There is another proof of it (Fig. 4a). We bisect GE at H. Then the known (line) GE has been bisected at H and GB has been added to it, so the known (area) EB times BG together with the known square GH is equal to the square of BH [*El.* II:6]. So it is known, and BH is known. But GH is known, so BG is known, and it is the root.[18]

[17]The manuscript has: "equal to the number of the roots" (a).

[18]This "proof" does not explain the construction of the root, but it is an analysis corresponding to the preceding proof.

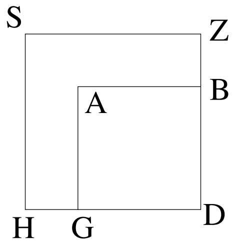

Figure 4b

There is a third proof of it (Fig. 4b). Let AB be half of the number of the roots; it is known. We draw on it the square $ABGD$. Then we construct an area similar to the above-mentioned square and equal to <the above-mentioned square together with> the known number [*El.* VI:25], namely (square) SD. Then its root HD is known. But GD is known, so the remainder GH is known, and it is the root, because the gnomon is equal to the square plus the roots and the two complements are the roots, because they are (together) the area of the number of the roots times the root.[19]

And if it is said: I have a property and a known number, and near me (?) are some roots among the roots of the property (i.e., $x^2 + b = ax$):[20] We make the property (x^2) the area $MLGE$ (Fig. 4c). We apply to one of its sides, namely GE, an area equal to the number (b), which has been mentioned above in connection with the square, and let it be $EGAB$. Then the line AL is known because the number of what is in it is equal to the number (a) of roots equal to the square < plus the number >. Thus we apply to the line AL an area which is deficient from its completion by a square, and the applied (area) is equal to the number (b) which has been mentioned above in connection with the square [*El.* VI:28]. Then the deficient square is the property.

[19]If $GD = DB = a/2, [DHSZ] = b + a^2/4, GH = x$, then the "gnomon" $GHSZBA$ consists of a square x^2 plus two complements with areas $xa/2$, so the area of the gnomon is $x^2 + ax$. Therefore x is a solution of $x^2 + ax = b$.

[20]The author tacitly assumes $b < a^2/4$ and he considers the root $x = (a/2) - \sqrt{(a/2)^2 - b}$. He shows no awareness of the second root $x = (a/2) + \sqrt{(a/2)^2 - b}$.

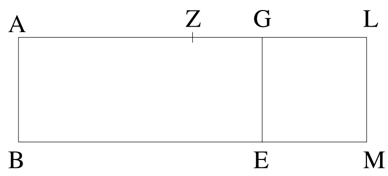

Figure 4c

There is another proof for it (Fig. 4c).[21] We bisect AL [249] at Z, and it has been divided into two unequal parts at G. So we have subtracted the product AG times GL, that is the known area AE, from the square of <half> the known line, and the remainder is the square of GZ [*El.* II:5], $<$ so the square of $GZ >$ is known, and its root GZ is known. Then the remainder LG is the root of the property.

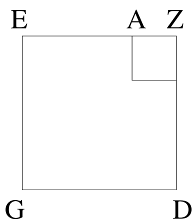

Figure 4d

There is a third proof of it (Fig. 4d).[22] We bisect the number of the roots, and we make the number of what is in GD equal to the number of half of the roots. We construct the known square GZ. We subtract from the known square GZ the known number which has been mentioned together with the square, and we construct from the remainder a square similar to the square GZ, and it (i.e., the new square) is known, and the line AZ is known, so the line AE is known, and it is the root of the property.

[21] The manuscript seems to render this figure twice, the first time on f. 248 bottom of page, without point Z, and the second time on f. 249, top of page, in a rotated form with point Z.

[22] Fig. 4d is my reconstruction based on a corrupt figure in the text, f. 249 lines 7-8 to the left. There is another corrupt figure on f. 249 in the middle of lines 7-8. The purpose of this figure is unclear to me so I have not rendered it in the translation. The text does not discuss the third type of equation $x^2 = ax + b$, or $b = x(x - a)$. If one interprets the equation as an equality between squares and rectangles in a figure, it is natural to consider the segment $y = x - a$, hence the equation reduces to $b = ay + y^2$ which has been treated above.

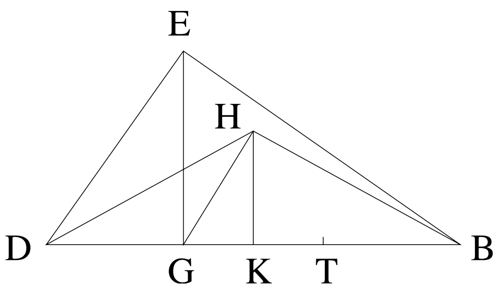

Figure 5

< 5. > (Fig. 5) In triangle BED angle E is (a) right (angle), and perpendicular EG is drawn from it to BD. From the points B, D two lines BH, DH are drawn, each of which is equal to ED, and HG is drawn.[23] I say that it is equal to GD.

Proof of this:[24] We drop the perpendicular HK and we cut off TK equal to KG. Then, by subtraction, BT is equal to GD. The square of DE is equal to the area BD times DG,[25] so the square of DH is equal to BD times DG, that is the square of GD plus the area BG times GD. Since the angle DGH is obtuse, the square of DH is equal to the square of GD plus the square of GH plus the area TG times GD [*El.* II:12], that is, equal to the square of GH plus the area BG times GD. We subtract the common area BG times GD, then the square of GD is equal to the square of GH, so GH is equal to GD, and that is what we wanted.

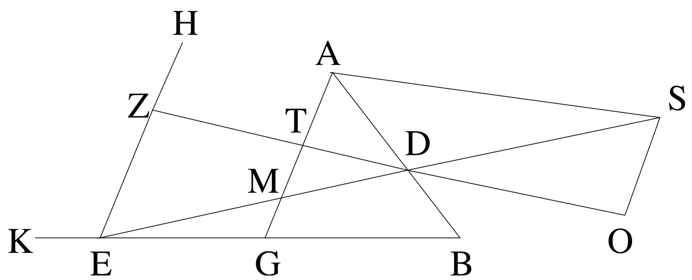

Figure 6

6. (Fig. 6) (In) triangle ABG, side BG is extended in the direction of K and point D is marked arbitrarily on side AB. We want to draw from it an line which ends at BGK in such a way that [a] the triangle produced by it on line BGK is

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[23]Point H only exists if $DB \leq 2DE$, that is to say $\angle D \leq 60^o$. If $\angle D = 60^o, H = K$, if $\angle D = 45^o, H = E$, if $\angle D =< 45^o$, point H is outside triangle EGB.

[24]The following proof is simpler: since $DH^2 = BD \cdot DG$, we have $BD : DH = HD : DG$. Because $\angle D$ is common, it follows that the triangles BDH and HDG are similar. From $BH = DH$ we conclude $HG = DG$.

[25]See the proof of *El.* I:47.

equal to [b] the triangle which is cut off by it and which remains of the triangle ABG, or ([a] is) two times it ([b]), or any known amount we want of it.

Thus let it first (be required to) be two times it. Let us draw in triangle ABG from point D the line DT [250] arbitrarily. We extend it towards O, and we make DO two times DT. We extend ODT in the direction of Z and we make the ratio of OD to DZ as the ratio of BD to DA. We let pass through the two points O, Z two lines OS, HZ parallel to the side AG, and we extend HZ to meet BG at E. We extend ED to meet OS at S. We draw SA.

Then it is clear that SD, DM are in the ratio of OD, DT, so SD is twice DM, and triangle SAD is twice triangle ADM. But triangle SDA is equal to triangle BDE, since the ratio of SD to DE, which is equal to the ratio of OD to DZ, is as the ratio of BD to DA in reciprocal proportionality, and angle SDA is equal to angle BDE [*El.* VI:14]. So triangle BDE is twice triangle ADM, and that is what we wanted.

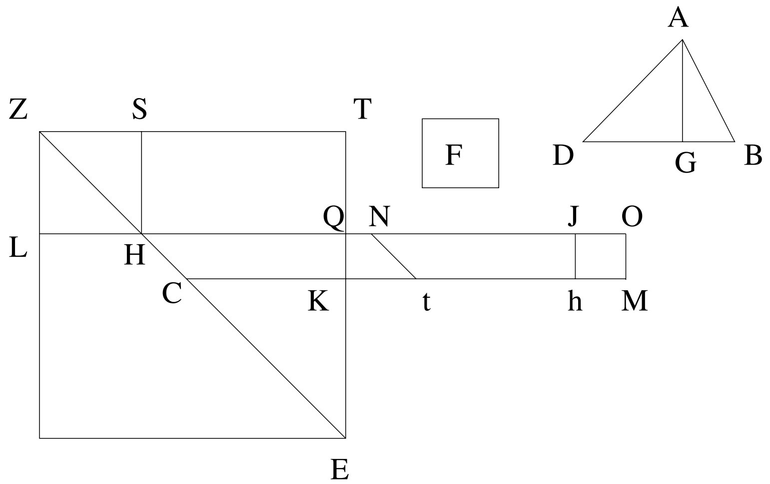

Figure 7

7. (Fig. 7) In triangle ABD, the sides AD, DB are together known,[26] and its perpendicular AG has been drawn and it is known, and the ratio of BG to GD is known. We want to know each of the lines AD, DB.

So for this we draw the line ZT and we make it equal to the two lines AD, DB together. We construct on it the square ZE, and we draw the diagonal ZE, and we construct the square ZH equal to the square of AG. And let in this problem GD be twice GB. Then the square of BD is twice the square of DG and a quarter of the square DG together, and line ZT is equal to the lines AD, DB together. So if we divide ZT into two parts in such a way that the square of one of the parts is equal to (the sum of) the square ZH and the square of two-thirds of the other part,[27] we

[26] The text means that the sum of AD and DB is known.

[27] We have $AD^2 = AG^2 + GD^2 = [ZH] + (\frac{2}{3}BD)^2$, and $ZS = SH = AG$.

have constructed what we wanted.

Proof [28] of this: We extend LH towards Q and we make HQ equal to (the sum of) LH, HS. We make QN equal to < a quarter of > LH. We draw Nt parallel to the diagonal EZ and we make NJ equal to HQ. We draw Jh at right angles. We make the area F such that its length is one and one-eighth[29] of its width, and we apply to HJ the area $MCHQO$ which exceeds its completion by the area JM, such that JM is similar to the area F, and such that the applied area ($MCHQO$) is equal to triangle EHQ. If we do this and drop the area $JNth$, which is equal to $HCQK$, then by subtraction area KO is equal to triangle ECK, and twice KO is equal to the square of EK. Line QJ is equal to (the sum of) LH, HS and QN < a quarter of > LH, and area JM is equal to the square of hJ and one-eighth of it, and that is what we wanted.

What I have seen here is different from this proposition and this figure, and I did not understand anything of it, so I have copied it like this.[30]

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[28]The text continues with the construction of the auxiliary problem just mentioned, but the proof is missing. At the end of the construction, al-Ṭūsī says that he did not understand the argument. I have tentatively emended several scribal errors in order to reconstruct the division by Nu^{c}aim of the given segment $TE = ZT = AD + DB$ into the desired parts $TK = AD$ and $KE = BD$.

[29]The manuscript reads "such that its length is equal to the chord of its width." I have emended "equal to the chord" *mithl watar* to "(equal to) one and one-eighth" *mithl wa-thumn*. The emendation restores the mathematical sense, which may have escaped al-Ṭūsī.

[30]I now analyze the construction for the general case. We wish to construct a triangle ADB in such a way that $AD + DB = a, AG = c, DG = \gamma DB$ for given a, c, γ with $c < a, \gamma > 0$. Put $AD = x, DB = y$. Because $AD^2 = AG^2 + DG^2$ we have $x^2 = c^2 + \gamma^2 y^2$ (*). Thus we have to divide a segment of length a into two parts x and y such that (*) is satisfied. In props. 28-31, the text refers to a "previous proposition" for the solution of this problem for arbitrary c and γ. Thus Nu^{c}aim must have intended to solve this general problem, although he assumes $\gamma = 2/3$ in the present prop. 7.

Nu^{c}aim could have argued somewhat as follows: Assume the problem solved, and let $TK = AD, KE = BD$. Then $TK^2 = [ZH] + (\gamma KE)^2$. If we extend KC to meet ZL extended at P, we have $KE^2/2 = (\gamma^{-2}/2)(TK^2 - [ZH]) = \gamma^{-2}[LHCP]$. Adding $[HQKC]$ we obtain $QE^2/2 = [HQKC] + \gamma^{-2}[LHCP]$. The left term $QE^2/2$ is the area of triangle HQE, which is determined by the given lengths of ZT and ZS. Nu^{c}aim may have tried to change the right term into a trapezium, as follows. Extend SH to meet KC extended at R. Then if we put $QJ = \gamma^{-2}LH$ and $JO = \gamma^{-2}OM/2$, we have $\gamma^{-2}[LHCP] = \gamma^{-2}[LHRP] + \gamma^{-2} \cdot \Delta HCR = [QJhK] + [JOMh] = [QOMK]$, whence $QE^2/2 = [HQKC] + [QOMK] = [HOMC]$. Thus we have to apply to the given segment HJ a trapezium $HOMC$ in such a way that the excess rectangle $[JOMh]$ is similar to a rectangle F of which the length (corresponding to JO) is $\gamma^{-2}/2$ times the width (corresponding to OM).

In prop. 7, the construction is explained for the case $\gamma = 2/3$. Then $\gamma^{-2}/2 = 9/8$, so the length of rectangle F is $9/8$ times its width, in agreement with my reconstructed text. However, several

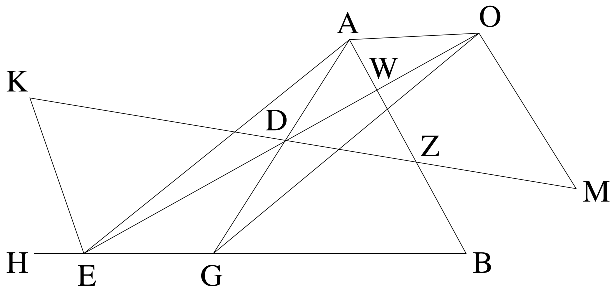

Figure 8

8. (Fig. 8)[31] Triangle ABG is known, and on AG the point D (is known), and BG has been extended towards H. We want to draw from point D a line which ends at the lines AB, BG in such a way that the triangle between AG, GH is, for example,[32] twice the triangle which it (the line) cuts off from the triangle ABG.

So we let pass through point D the line DZ arbitrarily, and we extend it towards M, K, and we make DZ equal to MZ. We let pass through M the line MO parallel to AB. We make the ratio of MD to DK as the ratio of GD to DA. We let pass through K the line KE parallel to AB. We draw from point E the line $EDWO$.[33] Then line WO is equal to line WD, and the ratio of GD to DA, which is as the ratio of MD to DK, [251] is as the ratio of OD to DE, in reciprocal proportionality.[34] We join OG, AE. Then they are parallel. We join AO. Then triangle ADO is equal to triangle GDE, and triangle AOD is twice triangle AWD, so triangle GDE is twice triangle AWD, and that is what we wanted.

problems remain. In the text, point Q is defined not as the intersection of LH extended and TE, but rather as the point on LH extended such that $HQ = LH + HS$. Point Q is on line TE only if $LH + HS + LH = ZT$, that is to say, $a = AD + DB = 3AG = 3c$. The text defines the positions of points N and J by $NJ = HQ$ and by $QN = LH$, which I have emended to $QN = LH/4$. Then $QJ = (9/4)LH$ only if $a = 3c$, so we find the same tacit assumption again. We note that line Nt and the equality of the trapezia $HCKQ$ and $NthJ$ are irrelevant but mentioned in the text. The construction of a trapezium $HOMC$ with given area is not one of the standard constructions of Euclid's *Elements*, although it can of course be reduced to *Elements* VI:29.

[31]From a modern point of view, propositions 6 and 8 are solutions of the same problem. In propositions 6 and 8, the required straight line through point D intersects the base BG on different sides of the triangle. Next to Fig. 8 the manuscript says: "Line OZG is straight." Line OG does not in general pass through point Z, although this happens to be true for the figure in the manuscript.

[32]The general problem and its solution are essentially the same as prop. 6. In prop. 8 Nu^{c}aim draws $OG//AE$, but in prop. 6 he does not draw the corresponding lines $SB//AE$.

[33]The lines MO and AB are assumed to be parallel.

[34]This terminology occurs in *El.* VI:14, which is used here to conclude $\Delta ADO = \Delta GDE$ from $GD : DA = OD : DE$ and $\angle ADO = \angle GDE$.

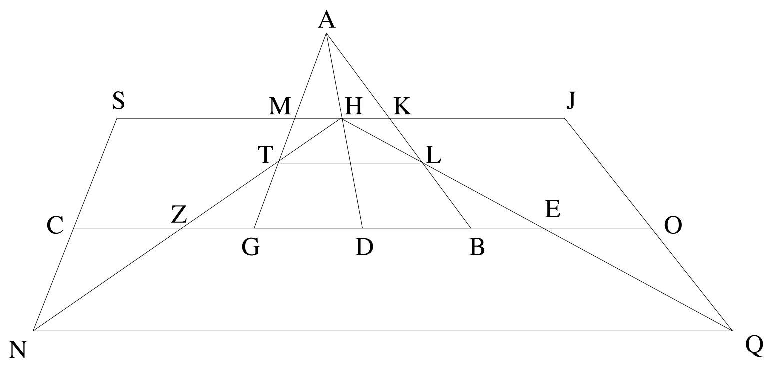

Figure 9

9. (Fig. 9) (Side) BG of triangle ABG is bisected at D, AD is drawn and H is marked on it. BG is extended in both directions and from it equal (segments) BE, GZ are cut off, and HLE, HTZ are drawn. I say: Then the triangles ALH, ATH are equal.

Proof of this: We let pass through H the line KHM parallel to BG. Then KH is equal to HM since BD is equal to DG, and the triangles AKH, AMH are equal.

We extend KM in both directions[35] and we cut off KJ, MS equal to the equal (segments) ED, DZ. We draw from the points J, S two lines parallel to the lines AB, AG, namely the lines JO, SC. We extend the lines HE, HZ to meet them (JO, SC) at Q, N.

We join QN. Then it is parallel to OC, because OE, ZC, which are equal to BD, DG, are equal to one another, and JH, SH are equal, so[36] triangle JQH is similar to triangle OQE,[37] and the ratio of OE to JH is as the ratio of QE to QH; in the same way, triangle SNH is similar to triangle CNZ and the ratio of CZ to SH is as the ratio of NZ to NH; therefore the ratio of HQ to QE is as the ratio of HN to NZ and QN is parallel to EZ. Again, triangle HJQ is similar to triangle HKL, so the ratio of JK to KH is as the ratio of QL to LH. Triangle HSN is similar to triangle HMT and the ratio of SM to MH is as the ratio of NT to TH.

So the ratio of QL to LH is as the ratio of NT to TH, and if we join LT, it is parallel to EZ.[38] So the triangles LKH, TMH are on equal bases and between parallel lines,[39] so the whole triangle ALH is equal to the whole triangle ATH, and

[35] Here begins an unnecessarily complicated part of the proof.

[36] The sentence "because OE, ZC, which are equal to BD, DG, are equal to one another, and JH, SH are equal," is mathematically irrelevant, and the word "so" is inappropriate.

[37] In the manuscript the label Q is often indistinguishable from the label W.

[38] That LT and EZ are parallel can also be proved by noting that $HL : LE = HK : BE = HM : GZ = MT : TG$. Then it is not necessary to define points S, C, N, J, O, Q. Prop. 9 is probably a lemma for a more complicated proposition in which these points played a role.

[39] By *El.* I:40, triangles LKH and TMH are equal in area.

that is what we wanted. [252]

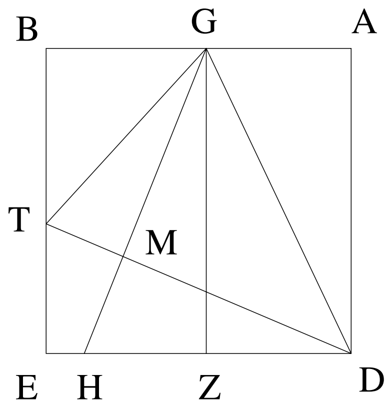

Figure 10

10. (Fig. 10) We want to construct on a scalene triangle, such as triangle GDT, a quadrilateral[40] with equal sides and angles which circumscribes the triangle if this is possible. Thus we draw the perpendicular GM of the triangle, and we extend it. We cut off GH equal to DT, and we join DH. We draw from T the perpendicular TE onto it, and we extend ET, and from G we draw onto it (ET) the perpendicular GB. We draw from D a line DA parallel to line ET, and we extend BG to meet it at A. I say: then AE is a square.

Proof of this: we draw perpendicular GZ onto DE. Then in (triangles) GZH, DMH angles M, Z are right (angles), and angle H is common, so triangles GZH, DMH are similar. In the same way we prove that triangles DMH, DET are similar. So triangle GZH is similar to triangle DET, so GZ is equal to DE. But GZ is equal to BE. So DE is equal to BE, so area AE is a square. That is what we wanted.[41]

[40]For quadrilateral the text uses the word *murabba*^{c} which normally means "square".

[41]The same construction is found in Chapter 7 of the treatise *On the Geometrical Constructions Necessary For the Craftsman* by Abu'l-Wafā' (940-ca. 998). Prop. 10 is the second of the three solutions which Abu'l-Wafā' presents. Abu'l-Wafā' labels the points differently. His proofs have not been transmitted in the extant Arabic manuscripts (compare [27]). See for the Arabic text [1, pp. 90-91] and f. 30-31 of the facsimile of MS. Istanbul, Ayasofya 2753 in [18].

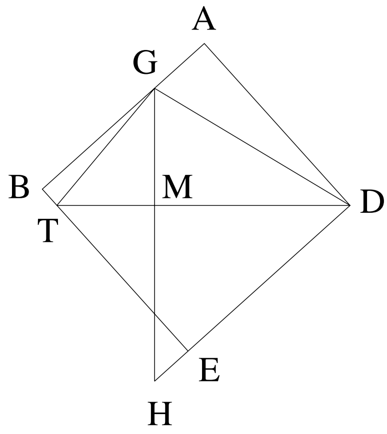

Figure 10a

Know that there are triangles which are not circumscribed by a square; this is to say that its perpendicular (i.e., its altitude) is equal to half the base or less than that.[42] For if GM is equal to MH and DM is common, and the angles on both sides of M are right (angles), then DG is therefore equal to DH, and this is absurd because DH is (then) necessarily longer than DE in the square. That is what we wanted.

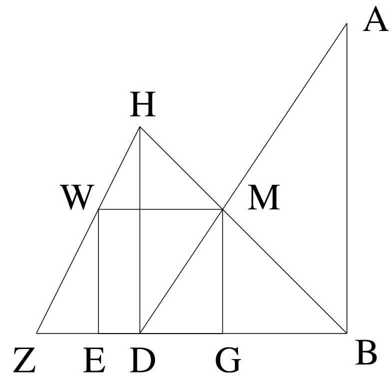

Figure 11

11. (Fig. 11) We want to construct in a scalene triangle a square which the triangle circumscribes. Let the triangle be HBZ. Then we draw from point H the perpendicular HD and from B a line parallel to DH, and we cut off from it (segment BA) equal to BZ. We join AD. We draw from point $< M >$ the perpendicular $< MG >$ and the line MW parallel to BZ, and from W the line WE parallel to MG. Then I say: ME is a square.

[42]The suggestion in the text is incorrect. In Figure 10a, which is not in the manuscript, the perpendicular GM is shorter than half the base DT, and DH is longer than DE.

Proof: Since AB is parallel to MG, the ratio of AB to MG is as the ratio of AD to DM. But the ratio of AD to DM is as the ratio of BH to HM, and this (ratio) is as the ratio of BZ to MW. So, alternately [*El.* V:16], the ratio of AB to BZ is as the ratio of MG to MW. But AB is equal to BZ, so MG is equal to MW, so ME is a square.[43] [253]

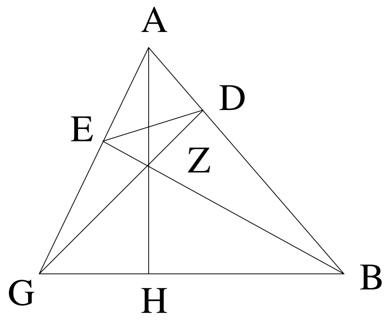

Figure 12

12. (Fig. 12) We want to prove that in any triangle, the three perpendiculars (i.e., altitudes), if extended, pass through one point. Thus let the triangle ABG first be acute-angled. We draw in it perpendiculars BE, GD to meet at Z, and we draw AZH. I say that it is perpendicular to BG.

Proof: we join DE. Then, since angles BEG, GDB are right (angles), a circle circumscribes trapezium $BDEG$, and angle GDE is equal to angle GBE. Similarly also, since angles ADZ, AEZ are right (angles), a circle circumscribes trapezium $ADZE$, and angle HAE is equal to angle GDE, that is angle GBE. So the angles GAH, GBE are equal, and angle G is common to them (triangles GAH, GBE), and by subtraction, angle AHG is equal to angle BEG, which is (a) right (angle), so it (angle AHG) is (a) right (angle).

But if the triangle is obtuse-angled, such as triangle BZG, and angle Z is obtuse, then if we draw two perpendiculars (to the opposite sides) from the points B, G to meet at A, the triangle ABG which is produced is acute-angled, and it is clear that if AZ is extended, it meets the base BG at a right (angle). So in every triangle, its perpendiculars meet at a point, and that is what we wanted.

[43] This construction is also found in Chapter 7 of the treatise *On the Geometrical Constructions Necessary For the Craftsman* by Abu'l-Wafā'. Our prop. 11 is the second of two solutions which Abu'l-Wafā ' presents. His proofs have not come down to us. See for the Arabic text [1, pp. 92-93] and f. 31 of the facsimile of MS. Istanbul, Ayasofya 2753 in [18].

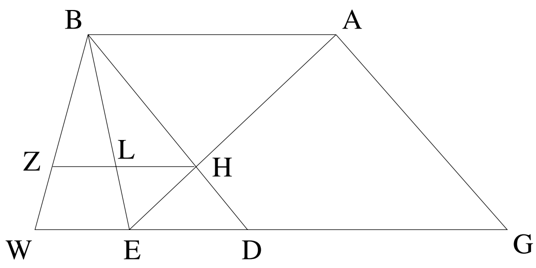

Figure 13

13. (Fig. 13) If triangle BDW has known sides, and (if) we want to draw in it a line parallel to DW in such a way that the drawn line is equal to (the sum of) the two lines which this line cuts off from the two sides on the side of D, W, then we divide line DW into two parts in point E in such a way that the ratio of DE to EW is as the ratio of DB to BW. We join EB and we draw from B line BA parallel to DW, and let BA be equal to BD. We extend WD in a straight line, and we draw from A line AG parallel to BD. It is clear that AB is equal to AG. We draw AE, which intersects BD at H and from H (we draw) line HLZ parallel to DW. Then I say: line HLZ is equal to (the sum of) the two lines HD, ZW.

Proof of this: The ratio of AB to LH is as the ratio of AE to EH since AB, HL are parallel [254]. But the ratio of AE to EH is as the ratio of AG to HD. So the ratio of AG to HD is as the ratio of AB to HL. Alternately, the ratio of AG to AB is as the ratio of HD to HL. But AG is equal to AB, so HD is equal to HL. But the ratio of HD to ZW is as the ratio of HL to LZ.[44] Alternately, the ratio of HD to HL is as the ratio of ZW to LZ. But HD is equal to HL, so LZ is equal to ZW, and line HLZ is equal to lines HD, ZW together. That is what we wanted.

And if we want that the parallel line which is drawn is equal to twice the (sum of the) lines which it cuts off from the triangle, we make AB twice BD, and in this way any magnitude we want.

[44] We have $HD : ZW = HL : LZ$ because $BD : BW = DE : EW$ by construction. Note that BE is the bisector of $\angle DBW$.

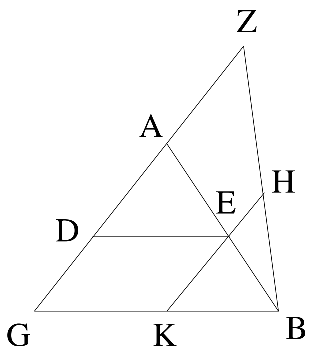

Figure 14

14. (Fig. 14) And if we want that the square of the parallel line which is drawn is equal to the (sum of the) squares of the two lines cut off from the triangle, then we make in triangle ABG, after extending GA, the square of GZ equal to the (sum of the) squares of GA, AB. We draw KEH parallel to GZ and equal to KG.[45] Then the square of KG, that is the square of ED, is equal to the (sum of the) squares of KE, EB, that is, the (sum of the) squares of GD, EB. And (we argue) by a similar reasoning, if we want that the square of DE is two times the (sum of the) squares of BE, DG, or any amount we want. That is what we wanted.

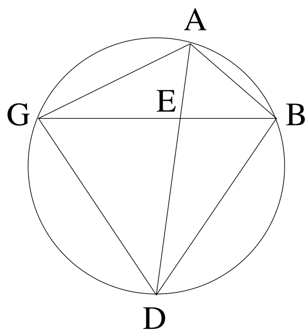

Figure 15

15. (Fig. 15) If in circle ABG chord BG is known, and its diameter is known, and (if) on chord BG two lines BA, AG are drawn[46] and the ratio between them is known, and we want to know each of them, then we bisect angle BAG by line AD and we draw BD, DG. Then they are equal and each of them is known. The ratio of BA to AG is as the ratio of BE to EG [*El.* VI:3], so each of BE, EG is known, and DE is known. [255] [By subtraction, AE is known.][47] The area of BE,[48] which

[45]Line GH bisects angle AGB.

[46]Point A is assumed to be on the circle.

[47]The passage [By subtraction, AE is known.] is incorrect because AD is not known in magnitude. In the original text, Nu^{c}aim may have referred to the "subtraction" $BD^2 - BE \cdot EG = DE^2$.

[48]The manuscript has BD.

is known, times BA (plus) AG is known,[49] so the sum of lines BA, AG is known, and the ratio between them is known, so each of them is known.[50]

In the same figure, if the diameter of the circle is known, and line BG is known, and the sum of lines BA, AG is known, and we want to know each of them, then each of BD, DG is known, and the product of BD times the sum of BA, AG is known, and it is equal to AD times BG, so AD is known, and BD is known, so AB is known.[51]

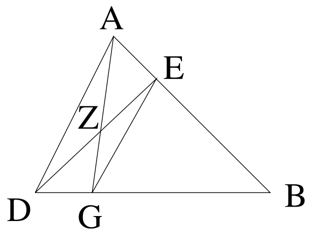

Figure 16

16. (Fig. 16) If triangle ABG is known, and side BG in it is extended to D, which is (a) known (point), and we want to draw from point D a line which ends at AB in such a way that the triangle which is produced between the drawn line and lines AG, GD is equal to the triangle which was produced by the line and lines BA, AG, then we join AD and we draw from G GE parallel to AD. We join ED. Then triangles AED, AGD are equal, because they are on the base AD and between parallels EG, AD [*El.* I:40]. We drop the common triangle AZD, then by subtraction

[49] Here Nu^{c}aim probably used Euclid, *Data* 93 to the effect that $ED\cdot(BA+AG) = BG\cdot BD$, which is also a known rectangle. Note that Naṣīr al-Dīn published an edition of Euclid's *Data*, see [28, p. 203]. According to *Data* 93, Euclid proves $BA+AG : BG = AD : BD$, hence $BD\cdot(BA+AG) = AD\cdot BG$. This could explain the scribal error BD for BE, see the previous footnote.

[50] The same problem is solved in the *Book of Assumed Things* (Kitāb al-Mafrūḍāt) by Thābit ibn Qurra, in prop. 34 in Thābit's original, cf. [8, p. 217] and prop. 11 in the revision by Naṣīr al-Dīn al-Ṭūsī, see [30, pp. 5-6]. Thābit's proof is simpler and based on the observation that the shape of triangle ABG is determined by $BA : AG$ and the (constant) angle BAG contained in the segment BAG. However, Thābit's proof cannot easily be used for a numerical determination of BA and AG. Nu^{c}aim's solution was more suitable for the practical determination of two Chords (or Sines) of arcs α and β from their sum $\alpha + \beta$ and the ratio between their Chords or Sines. Similar problems were studied in a trigonometrical context by the 11th-century Andalusian mathematician Ibn Mu^{c}ādh al-Jayyānī, see [32, pp. 153-164].

[51] In triangle ABD, sides AD and DB have a known length and angle BAD is known. Then the shape (and hence the size) of triangle ABD is known according to *Data* 44. AB is one of the two possibilities for the third side. (The other possibility is AG, because $BD = DG$ and $\angle BAD = \angle DAG$.)

triangles DZG, AZE are equal, and that is what we wanted.[52]

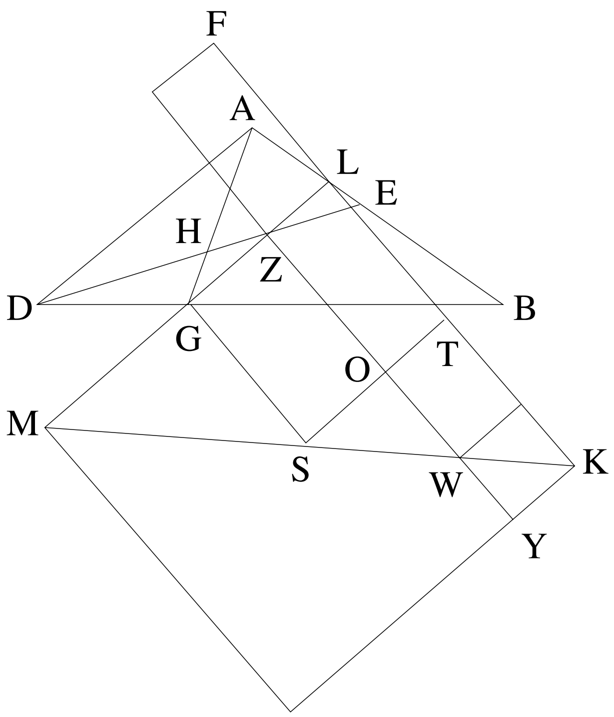

Figure 17

17. (Fig. 17) And if we want the triangle DHG to be half of triangle AEH,[53] we join DA and we draw from point G the line GL parallel to it. If we have drawn from D a line which intersects AG, LG and ends at AB in such a way that the line between AG, LG is equal to the line between LG, AB, we have constructed what we wanted.[54]

The drawing of this line is as we (now) explain. Let us put this line as line $DHZE$, which intersects AG, LG at the points H, Z. Then (by analysis) HZ is equal to ZE. The ratio of DE to EZ is as the ratio of DA to ZL, but the ratio of DE to EZ is as the ratio of DZ together with ZH to ZH, and the ratio of DZ together with ZH to ZH is as the ratio of AD together with twice ZG to ZG. So the ratio of AD together with twice ZG to ZG [256] is as the ratio of AD to LZ. We extend LG towards M and we make GM equal to half AD. Then the ratio of MZ to ZG is as the ratio of GM to LZ.[55] So area GM times GZ is equal to area ZM times ZL.

---

[52]In prop. 8 Nu^{c}aim solved the analogous problem in which point Z is given and point D is required.

[53]The manuscript has AEZ.

[54]We have $\Delta DHG = \Delta AHZ$ since DA and GZ are parallel. Therefore $HZ : HE = \Delta DHG : \Delta AHE = 1 : 2$.

[55]We have $MZ : ZG = (ZG + GM) : ZG = (2ZG + AD)/2 : ZG = AD/2 : LZ = GM : LZ$.

If we want to divide LM by this division, (i.e.,) in such a way that the product of the two parts is equal to the product of GM times the excess of LG over one of those parts, we construct on LM the square MK. We draw the diagonal KM. We draw from G to KM the line GS equal to GM. We draw TS parallel to LM. We draw LF equal to LT. Then we apply to KF a parallelogrammic area (FW) deficient from its completion by a square (WYK), and (FW being) equal to the oblong area $TSGL$. If we do that, by subtraction the area LW is equal to the area OG, which is the area SG times GZ. But the area WL is the area MZ times ZL.

And similarly if we want the magnitude of triangle AEH to be any magnitude we want (times the magnitude of triangle DGH), we reason in it (i.e., this problem) by this reasoning.[56]

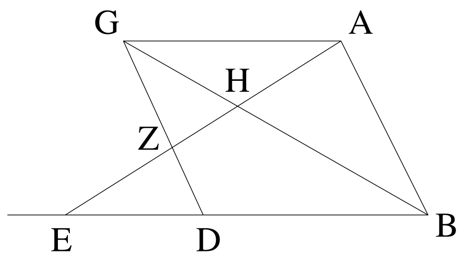

Figure 18

18. (Fig. 18) Let the parallelogrammic area $ABGD$ be known. The diagonal BG is drawn in it, and BD is extended indefinitely. We want to draw from A a line which intersects BG and ends at point E of line BD outside the area in such a way that the triangle which falls between the lines BG, DG is equal to the triangle produced outside the area, I mean the triangles GHZ, ZDE.[57] Then we make the remaining area of triangle BGD common, then triangle BGD is equal to triangle BHE. But triangle BGD is equal to triangle ABG. The ratio of triangle BHE to triangle ABH is as the ratio of EH to HA, that is the ratio of BH to HG. Again,

[56]If we want $\Delta AEH = c \cdot \Delta DGH$ for $c > 1$, we define $MG = AD/c$, and we obtain $ZM \cdot ZL = (c-1) \cdot GM \cdot GZ$. The same problem (for an arbitrary given ratio $\Delta DHG : \Delta AEH$) was the second of a set of six geometrical problems which were studied in Iran in the late tenth century AD. A solution by Abū Sahl al-Kūhī was preserved by his contemporary Aḥmad ibn Muḥammad ibn ^{c}Abd al-Jalīl al-Sijzī in the *Book on the Selected Problems Which Were Discussed by Him and the Geometers of Shīrāz and Khorāsān, and his (i.e., al-Sijzī's) Annotations*, see manuscript Dublin, Chester Beatty Library 3652, f. 46a:20-46b:10 [3, vol. 3, p. 59], [26, f. 42:4-43:11], [6, pp. 32-34].

[57]The author begins with an analysis, in which he assumes that line $AHZE$ has been found. The problem is a variation of the lemma in the construction of the regular heptagon by (pseudo?-) Archimedes which was translated into Arabic by Thābit ibn Qurra; in the lemma $ABDG$ is a square and one is asked to construct $AHZE$ in such a way that triangles AHG and DZE are equal in area [12, pp. 204-213].

triangle BHE is equal to triangle ABG, and the ratio of triangle ABG to triangle ABH is as the ratio of GB to BH. So the ratio of GB to BH is as the ratio of BH to HG and the area BG times GH is equal to the square of BH [257]. Thus, if we divide BG at H in extreme and mean ratio [*El.* VI:30] and draw through A line AHE, triangle HGZ is equal to triangle DZE, and that is what we wanted.[58]

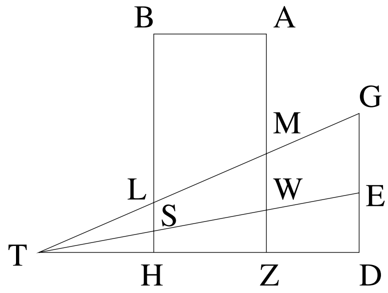

Figure 19

19. (Fig. 19) Let $AZHB$ be a parallelogram. ZH has been extended to T. We want to draw from T a line to (meet) AZ in such a way that the triangle which is produced is equal to the area above it, that is (the area) which it cuts off from area $AZHB$.

Thus we draw from T the line TE arbitrarily, and let it intersect the sides AZ, BH at points W, S. We make the triangle TED similar to the triangle TWZ and equal to (the sum of) the triangle TWZ and the area $WZHS$ [*El.* VI:25]. We extend DE. We make on DT the triangle DGT equal to the area $AZHB$ [cf. *El.* I:44]. We say: then line TM is what we wanted.

(Proof:) Since in triangle $< DGT >$ the lines ZM, HL are parallel to line GD, and WE has been drawn in it, and the area $EDZW$ is equal to the area $WZHS$, (therefore) the area $GDZM$ is equal to the area $MZHL$.[59] If we subtract the area $MZHL$, which is common to the area $AZHB$ and the triangle TGD, which are equal, the sum of the area $GMZD$ and the triangle TLH, that is the triangle TMZ, is equal to the area $AMLB$, and that is what we wanted.

---

[58]In the late tenth century AD, Abū Sa^{c}d al-^{c}Alā' ibn Sahl constructed line $AHZE$ in Fig. 18 in such a way that $\Delta GHZ : \Delta ZDE$ is a given ratio. Al-^{c}Alā' used a parabola and a hyperbola. The construction has been preserved in the form of a synthesis in an anonymous text entitled *Synthesis of the Problems of which al-^{c}Alā' ibn Sahl gave analyses.* See the edition of the anonymous text in [19, pp. 179-184]. In 1977, A. Anbouba [2, p. 373] identified the author of the anonymous text as the late tenth-century mathematician Abū'l-Jūd Muḥammad ibn al-Layth. R. Rashed tried to identify the author of the anonymous text as the tenth-century geometer Abū ^{c}Abdallāh al-Shannī [19, pp. cxxxiv-cxxxvi]. Compare, however, the evidence in [12, p. 260], not cited by Rashed, and supporting the identification by A. Anbouba.

[59]We have $[EDZW] : [WZHS] = (ED + ZW) : (WZ + HS) = (GD + ZM) : (MZ + HL) = [GDZM] : [MZHL]$.

And if we want that the triangle is half of the area, or two times it, or any amount we wish times it, then we construct what we want in the way we have described.[60]

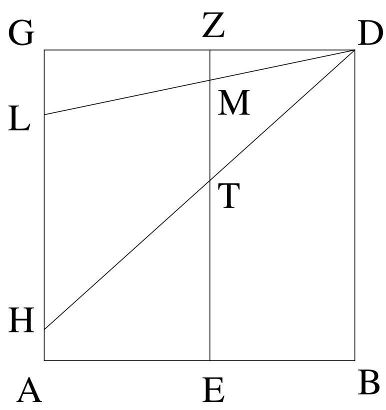

Figure 20

20. (Fig. 20) In square $ABDG$ line EZ is (drawn) parallel to AG. One (square) ell of the area AZ is equal in value to five (square) ells of the area ZB. We want to draw from point D a line to AG in such a way that it cuts the areas GE, ZB into two parts and the value of one part of the square $ABDG$ is equal to the value of the other part of it.

Thus we draw the line DML to (cut off) a [258] known (arbitrary) magnitude of AG, namely GL. Then the area $LMZG$ and its value are known, and similarly the triangle MZD and its value, and the value of half of the two areas AZ, ZB is known. The ratio of the value of triangle LGD to the value of triangle MZD, both of which are known, is as the ratio of the value of half of the two areas AZ, ZB, (the values of) which are known, to the value of what corresponds (in this ratio) to triangle MZD (in the first ratio), and that is triangle ZDT. So its value and its area are known. We extend DT to H, then the value of triangle DGH is equal to the value of half of the two areas AZ, ZB. That is what we wanted.[61]

[60]For $\Delta TMZ = c \cdot [AMLB]$, choose $\Delta TED = \Delta TWZ + c \cdot [WZHS]$ and $\Delta TDG = c \cdot [ABHZ]$.

[61]The problem always has a solution H between G and A, because the value of triangle DGA is greater than half the value of the whole square.

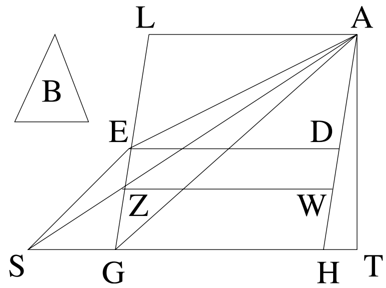

Figure 21

21. (Fig. 21) If the quadrilateral $ATGL$ with unequal sides is known, and angle GTA in it is (a) right (angle), and TG is extended indefinitely, and we want to draw from point A a line which intersects LG and ends at line TG in such a way that the triangle which is produced by the drawn line and by the lines LG, HG[62] is equal to the known triangle B: then we draw from point A line AH parallel to line LG. Then GH is known, and that is the case because TG and angle AHT, that is angle LGT, are known [*Data* 30].

Let the drawn line be line AS,[63] which intersects LG at Z. We draw ZW parallel to line GH, so it is equal to it, and (it is) known (in magnitude). We apply to WZ a parallelogrammic area equal to twice triangle B, namely area $DWZE$, which is twice B. We join AE, ES, AG. Then triangle AZE is equal to triangle B,[64] that is to say, equal to triangle ZGS, so line ES is parallel to line AG.[65] But EG was (assumed to be) parallel to AH. So triangle EGS is similar to triangle AHG, and triangle AHG has known sides. So the ratio of EG to GS is known, and line ZE is known, and triangle ZGS has a known area, so lines ZS, GS are known.[66]

By this reasoning we draw from point A a line such that triangle ZGS is equal to triangle B even if angle [259] T is not (a) right (angle).[67]

[62]Point H is supposed to be on line TG.

[63]The text begins with an analysis.

[64]$2\Delta AZE =$ area $DWZE$ because AH and LG are parallel.

[65]We have $\Delta AGE = \Delta AZE + \Delta AZG = \Delta GZS + \Delta AZG = \Delta AGS$.

[66]The analysis can be completed as follows. Since triangle ZGS has a known area and angle G is known, the rectangle $ZG \cdot GS$ is known by Euclid, *Data* 66. Since the ratio $EG : GS$ is known, it follows that the rectangle $ZG \cdot EG$ is known. Since ZE is known in magnitude, ZG can now be determined by application of areas, according to Euclid's *Data* 59.

[67]In the *Selected Problems*, Ibrāhīm ibn Sinān (907-946) solves the same problem by a more complicated analysis than Nu^{c}aim. See [21, pp. 237-238] (no. 30), [4, pp. 549-550, pp. 714-717] (no. 31). The same problem (in different wording) was the last of a set of six geometrical problems which were studied in Iran in the late tenth century AD. A simple solution by Abū Sahl al-Kūhī was preserved by his contemporary Aḥmad ibn Muḥammad ibn ^{c}Abd al-Jalīl al-Sijzī in the *Book on the Selected*

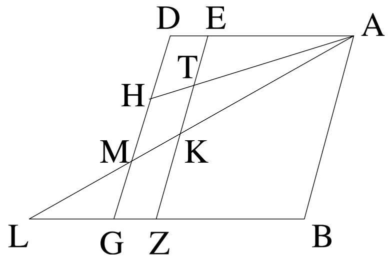

Figure 22

22. (Fig. 22) The area $ABGD$ is a parallelogram, side BG has been extended indefinitely, and line EZ has been drawn in it parallel to the sides AB, GD. We want to draw from point A a line which ends at line BGL and which intersects lines GD, ZE and such that the triangle which is produced outside the area $ABGD$ is equal to the triangle which is cut of by the drawn line from area $EDGZ$.

Thus we draw line ATH arbitrarily and we divide GD into two parts in M in such a way that the duplicate of the ratio of DM to MG is as the ratio of triangle AHD to area $ETHD$.[68] We draw AML. I say: the area $EKMD$ is equal to the triangle MGL.

Proof of this: The ratio of triangle ADH to area $ETHD$, which is as the ratio of triangle AMD to area $EKMD$, is equal to the duplicate of the ratio of DM to MG, that is to say that it is as the ratio of triangle AMD to triangle MGL [*El.* VI:19]. So the ratio of triangle AMD to area $EKMD$ and to triangle MGL is the same. So they are equal. That is what we wanted.

Problems Which Were Discussed by Him and the Geometers of Shīrāz and Khorāsān, and his (i.e., al-Sijzī's) Annotations, see the manuscript Chester Beatty 3653, f. 47a:13-25, [26, f. 45:9-22], [6, pp. 38-39].

[68]This can be done as follows. First construct segments p, q such that $p : q = \Delta AHD : [ETHD]$. Then construct segment r such that $p : r = r : q$ [*El.* VI:13], and divide segment DG at M in the ratio $p : r$ [*El.* VI:10].

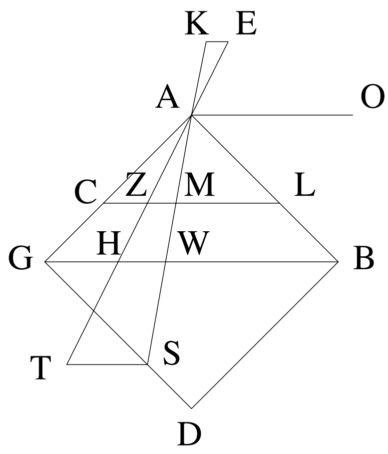

Figure 23

23. (Fig. 23) If there is a square $ABGD$ and its diameter is BG and in triangle ABG line LC has been drawn parallel to BG, and we want to draw from point A a line which ends at GD and intersects lines LC, BG in such a way that [a] the line (segment) of it which falls between point A and line LC is equal to [b] the line (segment) which falls between BG and GD, or (such that [a] is) two times it ([b]), or any amount we wish, then we first construct it on the condition that it ([a]) is equal to it ([b]). We draw line AZH arbitrarily, and we let pass through A line AO parallel to lines LC, BG. We extend HZA towards E and we make AE equal to ZH. We extend AZH towards T and we make HT equal to AZ. Since EA was (supposed to be) equal to ZH, thus EZ is equal to ZT. We let pass through point T a line parallel to line LC and the line which was (constructed) on point A, namely AO, and let it fall on GD at point S. We draw SA and produce it to K. Then, since EA is equal to ZH and ZA is equal to HT, thus AM is equal to WS, and that is what we wanted. In this way we do (the case where [a] is) any amount we wish (times [b]). In this proposition there were additions.[69] [260]

[69]These "additions" may have involved line AO and point K, which are not really used in the proof. In the figure, line KE and AO are parallel. Note that $EA = ZH$ is irrelevant in the proof.

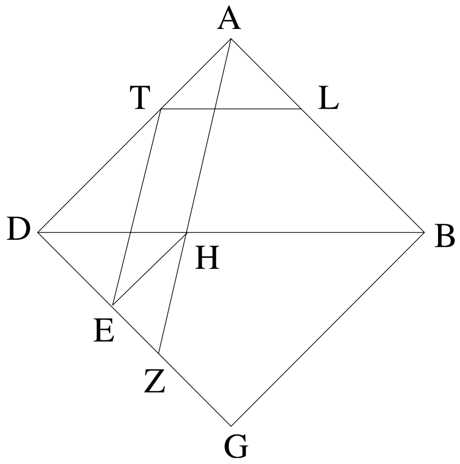

Figure 24

24. (Fig. 24) If there is a square $ABGD$ and its diagonal is BD and LT is drawn parallel to BD, and AZ is drawn (in such a way that) it falls on GD at Z and it intersects BD at H, and TE is drawn parallel to AZ, and we draw EH, then I say: so the ratio of AT to TD is as the ratio of triangle HZE to triangle EHD, and the ratio of AD to DT is as the ratio of triangle HZD to triangle HED.

Proof of this: TE is parallel to AZ, so the ratio of AT to TD is as the ratio of ZE to ED, that is, the ratio of triangle HZE to triangle HED, and the ratio of AD to TD is as the ratio of triangle HZD to triangle HED, and that is what we wanted.

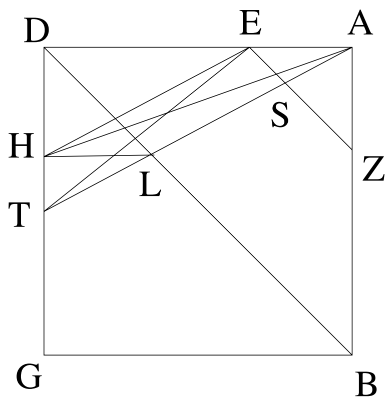

Figure 25

25. (Fig. 25)[70] (In) square $ABGD$ the diagonal is BD, and line ZE in it is parallel to BD and intersects AD according to a known ratio, and let it be a third. We want to draw from A a line (AT) which ends at GD in such a way that [a] the

[70] My figure is not drawn to scale.

triangle (DLT) between BD, GD is equal to [b] the triangle (ASE) which the drawn line cuts off from triangle AZE, or (in such a way that [a] is) two times it ([b]), or any amount we wish.

Thus let it ([a]) first be equal to it ([b]). We join AT as we wanted,[71] and EH parallel to it, and (we join) AH, LH. Then, since the ratio of AD to EA is known, and as the ratio of triangle LTD to triangle LTH (prop. 24), and DA is three times EA, therefore triangle LTD is three times triangle LHT. But we assumed triangle ASE equal to triangle DLT. So triangle ASE is three times triangle LHT, but they are between parallel lines, so AS is three times LT. Thus, if AE was (assumed to be) a third of AD, and we want to draw from point A a line which ends at GD in such a way that the triangle which is cut off from triangle AZE by the drawn line, namely triangle ASE, is equal to the triangle which is produced between BD, DG, then we draw from A a line which ends at GD in such a way that the (segment) of it which falls between A and line ZE is three times the segment of it which falls between BD, GD (prop. 23). And according to this example [261] which we have described, we proceed if we want ([a]) two times it ([b]) or any amount we wish.[72] That is what we wanted.

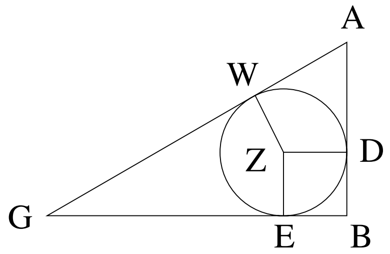

Figure 26

26. (Fig. 26) If triangle ABG is right-angled, and its area and the sum of its sides are known, and we want to know each of its sides, then we inscribe in it circle WED with center Z [*El.* IV:4], and we draw the perpendiculars ZD, ZE, ZW. We know that if we divide the area by < half > the perimeter, the result is half the diameter, so line DZ is known, but it is equal to BE and to BD. So lines DB, BE are together known (i.e., their sum is known). Line DA is equal to AW and line WG is equal to EG, so (the sum of) lines DA, EG is equal to AG, so AG, half the remainder,[73] is known, and after this we know each of AD, EG.[74]

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[71] The manuscript has "we wanted we assumed." The text now presents an analysis of the solution.

[72] If $AD : AE = \gamma$, we have $\Delta DLT = \gamma \Delta HLT$. If $\Delta DLT = c \cdot \Delta ASE$, then $TL : AS = \Delta HLT : \Delta ASE = c : \gamma$.

[73] AG is half the difference between the perimeter of the triangle and $BD + BE$.

[74] See prop. 27.

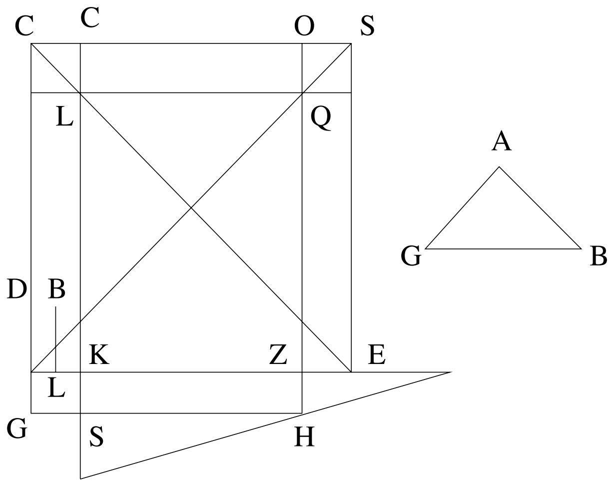

Figure 27

27. (Fig. 27) And if we want to know in such a triangle each of the sides, < ... > Proof: we draw DE with the magnitude of (the sum of) AB, BG, AG, it < ... >

and after this nothing was written in the copy, and the figure of it was like this (Fig. 27), so I have not understood from it what he wanted.[75]

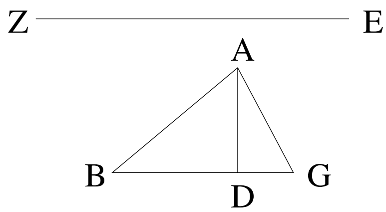

Figure 28

< 28 > (Fig. 28) If the triangle AGB has unequal sides, and the perpendicular AD is known, and it divides BG in a known ratio, and the (sum of the) two sides AG, GB is known, and we want to know each of the sides of the triangle, then we draw EZ and we make it equal to AG, GB together. The square of AD is known, and the square of AG is equal to the (sum of the) squares of AD, DG. So if we divide EZ into two parts in such a way that the square of one of the parts is equal to the square of AD and a ninth of the square of the other part, if it (DG) is a third

[75] The remark "and after this ... what he wanted." is evidently by Naṣīr al-Dīn al-Ṭūsī. I have not been able to find a plausible reconstruction of the figure; Fig. 27 is the faulty figure in the manuscript. The problem is not difficult: let s be the semiperimeter and A the area of the triangle, and call the hypotenuse c and the two other sides a, b. Prop. 26 boils down to $c = s - A/s$. Since $a^2 + b^2 = c^2$ and $ab = 2A$, a and b can be found from, for example, $a + b = 2s - c$, $(a - b)^2 = c^2 - 4A$.

of BG, we have constructed what we wanted. This problem[76] is divided into cases and known by the previous figure (i.e., prop. 7).[77]

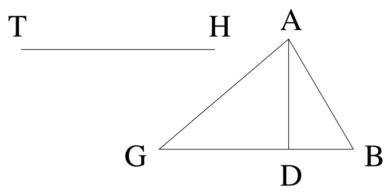

Figure 29

$< 29 >$ (Fig. 29) In triangle ABG, the perpendicular AD has been drawn from point A, and the perpendicular is known, and BG is known, and the ratio of AB to AG is known. We want to know each of AB, AG. Then, since the ratio of AB to AG is known, the ratio of the square [262] of AB to the square of AG is known. We make line HT equal to BG, and we want to divide it into two parts in such a way that the square of AD together with the square of one of the parts of HT is equal to a third of the square[78] of the other part together with the square of AD if the square of AB is a third of the square of AG, or a quarter of it if the square of AB is a quarter of the square of AG. In general, we divide HT into two parts in such a way that the ratio of the square of one of them together with the square of AD to the square of the other (part) together with the square of AD is as the ratio of the square of AB to the square of AG. If we do that, we know each of AB, AG,[79] and we know that from the previous figure,[80] and that is what we wanted.

[76] The manuscript has "triangle" instead of "problem." My emendation is inspired by the end of proposition 30.

[77] Proposition 7 includes the solution of the following problem: given a segment a, another segment $c < a$ and a ratio γ. Required to divide a into two segments x and y in such a way that $x^2 = c^2 + \gamma^2 y^2$. In the notation of prop. 28, $a = AG + GB, c = AD, \gamma = DG/BG$. We have $\gamma = 2/3$ in the example of prop. 28 and $\gamma = 1/3$ in the example in prop. 7.

[78] Instead of "a third of the square" the manuscript has incorrectly "the square of a third."

[79] The manuscript has AB, BG by scribal error.

[80] This problem can also be solved by means of (my reconstruction of) proposition 7: Suppose $AB < AG$, and choose $a = HT, \gamma = AB/AG, c = \sqrt{(1-\gamma^2)}AD$.

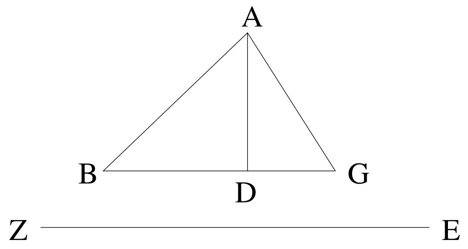

Figure 30

< 30 > (Fig. 30)[81] If the triangle AGB has unequal sides, and the perpendicular AD is known and it divides BG in a known ratio, and AG, GB are together known, and we want to know each of the sides of the triangle, then we draw EZ and make it equal to AG, GB together. The square of AD is known and the square of AG is equal to the (sum of the) squares of AD, DG. So if we divide EZ into two parts in such a way that the square of one of the parts is equal to the square of AD and a ninth[82] of the square of the other part, if DG is a third of GB, we have constructed what we wanted. This problem is divided into cases and known by the previous figure.

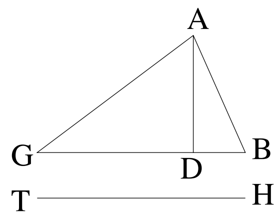

Figure 31

< 31 > (Fig. 31) (In) triangle ABG, the perpendicular AD issues from point A, and the perpendicular is known, and BG is known, and the ratio of AB to AG is known. We want to know each of AB, AG. We have said that the ratio of AB to AG is known, so the ratio of the square of AB to the square of AG is known. We make HT equal to BG and we divide it into two parts in such a way that the square of AD together with the square of one of the parts of line HT is equal to a third of the square[83] of the other part together with the square of AD, if the square of AB

[81]Props. 30 and 31 are the same as props. 28 and 29, and the texts are also practically the same. Various explanations are possible: al-Ṭūsī or the scribe copied the two versions from two different manuscripts; or prop. 28, 29 may be the revisions by al-Ṭūsī and prop. 30, 31 the originals by Nu^c^aim, or vice versa.

[82]Instead of "a ninth," the manuscript has "a quarter" by scribal error.

[83]Instead of "a third of the square" the manuscript has incorrectly "the square of a third," just as in prop. 29.

is a third of the square of AG, or a quarter of it if the square of AB is a quarter of the square of AG. In general, we divide HT into two parts in such a way that $<$ the ratio of $>$ the square of one of them together with the square of AD to the square of the other (part) together with the square of AD is as the ratio of the square of AB to the square of AG. If we do that, we know each of AB, AG, and we know that from the previous figure [263], and that is what we wanted.

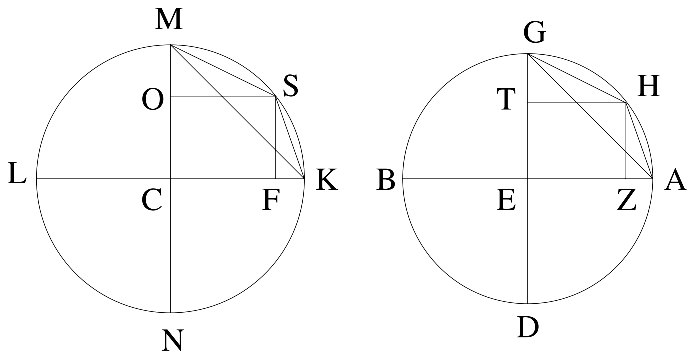

Figure 32

$< 32 >$ (Fig. 32) If the circle $AGBD$ is unknown and the diameter AB intersects diameter GD at right angles, and in quadrant AG the rectangular area $HTEZ$ has been inscribed in such a way that each of the lines AZ, GT is known (in magnitude), and we want to know diameter GD, I say: it is known (in magnitude).

Proof of this: the area BA times AZ is equal to the square of AH, and the area AB times GT is also equal to the square of GH, so AB has been multiplied by the two known magnitudes AZ, GT and (the products) of them were the squares of AH, GH. Thus so the ratio of ZA to GT is known,[84] and it is as the ratio of the square of AH to the square of GH, so the ratio of AH to GH is known.

We describe a circle $KMLN$ with known diameter, and let the two diameters KL, MN in it intersect at right angles. We draw KM, the chord of a quadrant, then it is known. We erect on arc KM two lines KS, SM such that the ratio between them is as the ratio of AH to HG, which is (a) known (ratio). Then the ratio of KS to SM is known and each of them is known (prop. 15). We drop the two perpendiculars SF, SO onto the two diameters KL, MN, then each of them (the perpendiculars) is known, and each of the arrows KF, MO is known, and the ratio of KF to KL is known, and it is as the ratio of AZ, which is known, to AB, so AB is known. That is what we wanted.[85]

[84]The manuscript has: "so the ratio of ZH to HT is known."

[85]Algebraically, if $ZA = p, GT = q, AE = EG = r, p < r, q < r$, we have e.g., $AZ \cdot ZB = HZ^2$, so $p(2r - p) = (r - q)^2$, whence $r = p + q + \sqrt{2pq}$. Thus the solution by Nu^{c}aim is unnecessarily complicated.

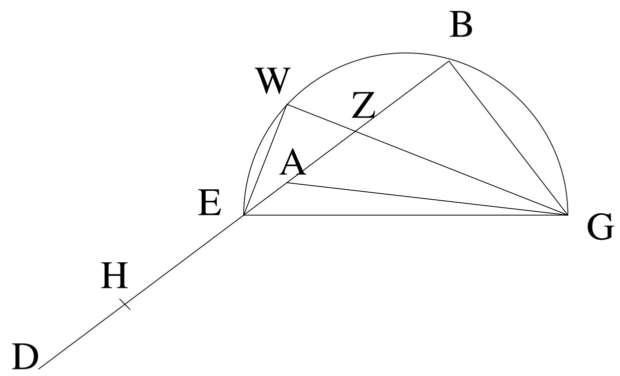

Figure 33

< 33 > (Fig. 33) If triangle ABG is right-angled and we want to draw from point G a line towards AB such that the drawn line together with the part (of AB) between it (the line) and B is equal to line AG together with the other part of line AB, then we extend BA in a straight line to D and we make AD equal to AG [264]. We bisect BD at E. We join EG and we construct on it the semicircle GBE. We draw from point E chord EW equal to half of BG. We draw WG to intersect AB at Z. Then I say: DZ is equal to the two lines GZ, ZB together.[86]

Proof of this: Triangle WZE is similar to triangle BZG and line WE is half of BG, so line EZ is half of ZG. We make DH equal to BZ, then EH is equal to EZ, so line ZH is equal to ZG, and BZ is common, so (the sum of) lines GZ, ZB is equal to HB, and HB is equal to DZ, and DZ is equal to (the sum of) GA, AZ, so (the sum of) lines GA, AZ is equal to (the sum of) lines GZ, ZB. And similarly if the angle is acute or obtuse, the construction is the same.[87]

[86]Note that $DZ = DA + AZ = AG + AZ$.

[87]In this case, the circle through E, B, G is not a semicircle. In the *Book of Assumed Things* (Kitāb al-Mafrūḍāt), Thābit ibn Qurra solves the same problem in the course of three propositions (props. 8,9,10 in Thābit's original, cf. [8, p. 217] and props. 3,5,6 in the revision by Naṣīr al-Dīn al-Ṭūsī, see [30, pp. 3-4]). The only essential difference between the solutions is as follows. Thābit does not use the circle through E, B, G and he does not introduce point W. Instead, he constructs the circle with center B and radius BD. He extends EG to intersect the circle at point K and he draws KB. He constructs point Z on BA by drawing GZ parallel to BK. Then $GZ : ZE = KB : BE = DB : BE = 2 : 1$ so $GZ = 2ZE$ as in the proof by Nu^{c}aim. Abū Sahl al-Kūhī solves the problem for an arbitrary triangle ABG, see [5, pp. 171-173].

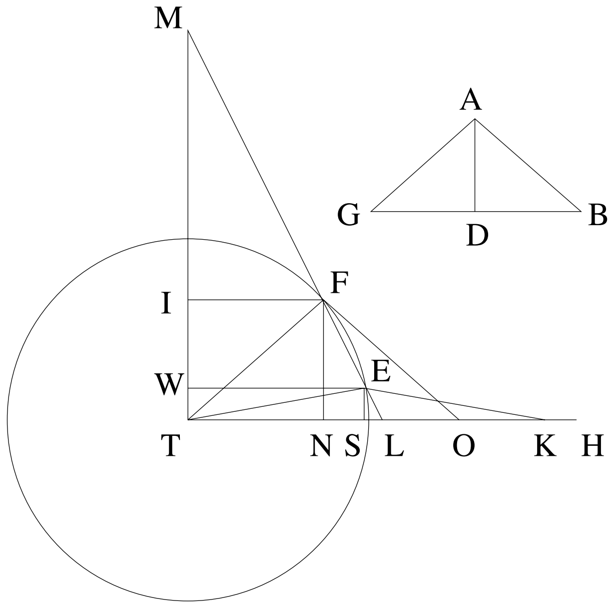

Figure 34

34. (Fig. 34) If triangle ABG is isosceles and the two (equal) arms BA, AG are known, and (the sum of) the perpendicular AD together with the base BG is known, and we want to know each of them, then we say that this problem can always be (solved) in two ways, unless the ratio of AB to half (the sum) of AD, BG is as the ratio of (the sum of) AD, BG to the square root of the (sum of the) squares of AD, BG and half of it together.[88] For if it is like this, the problem can only be (solved) in one way.

Proof: we draw line HT equal to lines AD, BG together, and we bisect it at L. We describe with center T and radius AB circle EF. We erect at point T the perpendicular TM equal to twice TL, and we draw LM. If the ratio of AB to TL is as the ratio of TM to ML, then it touches circle EF,[89] and AD, BG are (to be found) in one way.

[88]The text is mathematically incomplete. The condition $AB : \frac{1}{2}(AD + BG) = (AD + BG) : \sqrt{(AD + BG)^2 + (\frac{1}{2}(AD + BG))^2}$ is equivalent to $AB : (AD + BG) = 1 : \sqrt{5}$. To analyse the problem we put $\angle A = 2\alpha$, so $(AD + BG)/AB = \cos\alpha + 2\sin\alpha = \sqrt{5}\sin(\alpha + \phi)$ with $\phi = \arcsin(1/\sqrt{5}) \approx 26.5^o = \angle M$. A solution of the problem is only produced for $0^o < \alpha < 90^o$. Thus the problem has one solution if $(AD + BG)/AB = \sqrt{5}$ (the condition in the text is correct); two solutions only if $\sqrt{5} > (AD + BG)/AB > 2$, one solution if $2 \geq (AD + BG)/AB > 1$, and no solution if $AD + BG \leq AB$. The text ignores the latter two cases. If $(AD + BG)/AB = 2$, the circle in Fig. 34 passes through L.

[89]The circle touches ML at E if $\angle TEL$ is a right angle, that is to say if $AB : TL = TE : TL = TM : ML$.

If this is not the case, and ML intersects the circle EF, then let it intersect it at two points E, F. We draw from the two points E, F two perpendiculars ES, FN onto HT and two perpendiculars EW, FI onto MT, and we join ET, FT. We cut off from HT (segment) HO equal to twice NL, and (segment) HK equal to twice LS [265] and we join EK, FO.

Then, since TM is twice TL, FN is twice NL, and ES is twice SL. Similarly HO is equal to FN and HK is equal to ES. By subtraction KT is twice ST since MW, which is equal to KT, is twice EW, which is equal to ST. By subtraction also OT is twice NT. So KE is equal to ET and OF is equal to FT. But SE, KT together have been shown to be equal to FN, OT together. It is clear from this that the problem can be solved in two ways as we have said before, and that is what we wanted.

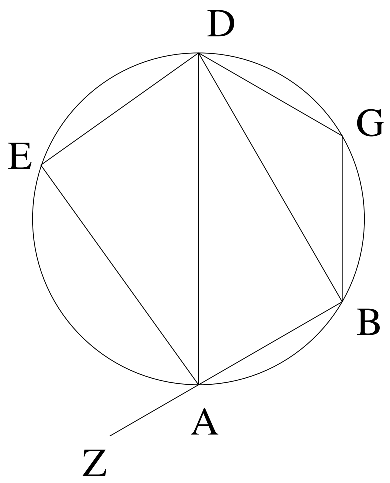

Figure 35

35. (Fig. 35) The sum of the chord of the hexagon and the chord of the decagon in every circle is equal to the chord of three-tenths of it.[90]

Thus let the circle be $ABGDE$ and the diameter AD. We draw lines AB, BG, GD, and each of them is the chord of a sixth. We draw BD, the chord of a third, and DE, the chord of a fifth, and EA, the chord of three-tenths. We extend BA and we cut off AZ, the chord of a tenth. I say that AE is equal to ZB.

Proof of this: ZB is divided at A in extreme and mean ratio [*El.* XIII:9], so BZ times ZA is equal to the square of BA. The (sum of the) squares of ZA, AB is equal to the square of DE [*El.* XIII:10], and the square of BD is three times the square of AB, that is, equal to the (sum of the) square of BA and twice BZ times ZA. But BZ times ZA is equal to (the sum of) BA times AZ and the square of

[90]In a circle with radius r, the chord of an arc α is in modern terms $r\sin(\alpha/2)$. Thus prop. 35 is equivalent to $\sin 30^o + \sin 18^o = \sin 54^o$.

AZ [*El.* II:3]. So the square of DB is equal to the (sum of the) square of BA and twice the product BA times AZ and twice the square of AZ, that is, equal to the (sum of the) squares BZ, ZA [*El.* II:4]. We make the square BA common, then the (sum of the) squares DB, BA, that is the square of the diameter DA, is equal to the (sum of the) square of BZ and the square of BA and the square of AZ, that is, equal to the (sum of the) square of BZ and the square of DE. But it was equal to the (sum of the) squares of DE, EA. [266] So the (sum of the) squares of BZ, DE is equal to the (sum of the) squares of EA, DE. We subtract the common square of DE, then the squares of EA, BZ are equal, so EA, BZ are equal, and that is what we wanted.

I say: there is no use in my drawing the two lines BG, GD in it (the figure). It has become clear that the chord[91] of three-tenths of the circle and the chord of a sixth if they are joined, are (the two parts of a segment) divided in extreme and mean ratio, and the shorter part is the chord of a sixth,[92] and that the (sum of the) squares of the chord of three-tenths and the chord of a tenth is equal to the square of the chord of a third.[93]

It is necessary to know that the ratio of the chord of two-fifths to the chord of a fifth is as the ratio of the chord of a sixth to the chord of a tenth since the chord of two-fifths and the chord of a fifth, if they are joined, are (the two parts of a segment) divided in extreme and mean ratio [*El.* XIII:8], just as the chord of a sixth and the chord of a tenth [*El.* XIII:9].

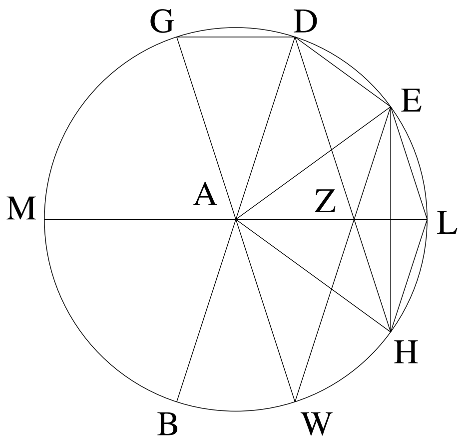

Figure 36

36. (Fig. 36) Another proof of (the theorem) that (the sum of) the chord of a sixth and the chord of a tenth together are equal to the chord of three-tenths of

[91]Instead of "the chord," the manuscript has "the square."

[92]This theorem is a consequence of prop. 35 and *El.* XIII:5, 9.

[93]This theorem is the equivalent of $\sin^2 54^o + \sin^2 18^o = \sin^2 60^o$. It is a consequence of $DB^2 = BZ^2 + ZA^2$ and $BZ = EA$ in Fig. 35.

the circle. Let in circle LWD the diameters BD, GW be drawn which intersect at the angle of a tenth (i.e., 36^o), and let us draw DH parallel to GW. Then HW is also a tenth of the circle, and HD is three tenths of it. We draw WZE parallel to BD. Then ED is also a tenth. Since the area $WADZ$ is a parallelogram, and side DA is equal to AW, the area $WADZ$ is a rhombus, of which each side is equal to half the diameter, so side WZ is half the diameter. Angle ZDA is equal to angle EZD[94] because since EW, DB are parallel. We join ED. Angle ZDA is also equal to angle EDZ because they stand on the fifth (part of the circumference), so angles EZD, EDZ are equal, so EZ is equal to ED. But ED is the chord of a tenth, so EZ is the chord of a tenth, so EW is the chord of a tenth and the chord of a sixth together, but it is the chord of three tenths of the circle, and that is what we wanted.

And if AZ, DG are joined, they are parallel and equal, so AZ is equal to the chord of a tenth. If one extends [267] AZ to L, the arcs HL, LE are equal, and that is (true) because of the equality of angles EAL, HAL and the equality of arcs ED, HW. If we join HL, LE, line HZ is equal to each of the lines HL, LE, and triangle HLZ is similar to triangle WZA, so the ratio of HZ to ZL is as the ratio of WA to AZ, and the area ZL times WA, that is ZL times LA, is equal to the area HZ times AZ, that is the square of AZ. So LA is divided in extreme and mean ratio at Z, and its greater part is AZ. If LA is extended towards M it is clear that ZM is divided at A in extreme and mean ratio, and the longer (part) AM is the chord of a sixth.[95]

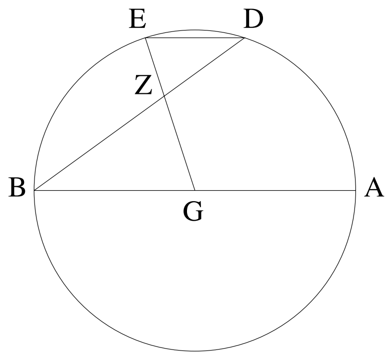

Figure 36a

I say: In another way, from the *Fawā'id* ("Useful things") of Mu'ayyad al-Dīn al-^{c}Urḍī:[96]

[94]The manuscript has "Angle ZWA is equal to angle EAD." Although this statement is mathematically correct, it does not follow from "EW, DB are parallel," so I have emended the text.

[95]The same theorem is proved in a different way in *El.* XIII:9.

[96]On Mu'ayyad al-Dīn al-^{c}Urḍī (died 1266) see [16, II:414]. He was a colleague of Naṣīr al-Dīn al-Ṭūsī. I have no further information on the *Fawā'id*.

(Fig. 36a) Let there be the circle ADB with diameter AB and center G, and let each of the arcs AD, BE be a fifth of the circle. Then, by subtraction, DE is a tenth. We join DE, DB, EG which intersect at Z. Then DE is parallel to AB and triangles DZE, GZB are similar. Angle AGZ, which is at the center on three-tenths of the circle, is equal to (the sum of) the angles GZB, GBZ. And, angle GBZ which is at the circumference on two tenths, is of the same magnitude as an angle at one tenth at the center. By subtraction, angle GZB is equal to an angle at the center on two tenths, but angle ZGB is like this, so they are equal, so BG is equal to BZ, so it is the chord of a sixth. Similarly, DZ is equal to DE, the chord of a tenth. So DB, the chord of three tenths of the circle, is equal to (the sum of) the chords of a sixth and a tenth together. That is what we wanted.

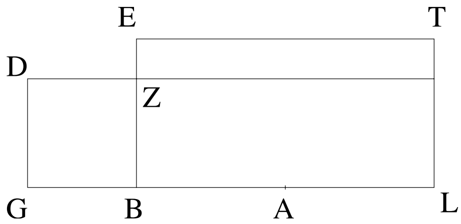

Figure 37

< 37 > (Fig. 37). We want to divide a known line into two parts in such a way that the product of the line and two times or many times one of its parts is equal to the square of the other part.

Thus we make the line AB. We want to divide it first into two parts in such a way that the product of the line and two times one of its parts is equal to the square of the other part.

Then we extend AB to L and we make AL equal to AB. We erect at point L of line [268] AL the perpendicular LT equal to AB, and we complete the parallelogrammic area LE. We apply to line LB a parallelogrammic area equal to the area LE and exceeding its completion by a square, namely (area) LD [*El.* VI:29]. Then LD is equal to LE. We subtract LZ, which is common, then the area TZ is equal to the square ZG. But the area TZ is two times the product BE times EZ. Thus we have divided BE at point Z and the product of the line and two times one of its parts is equal to the square of the other part. And by this reasoning, if we want the product of the line and three times, or more than this times, one of its parts to be equal to the square of the other part, then we act in this (case) by multiplying AB by the amount of the number of times.

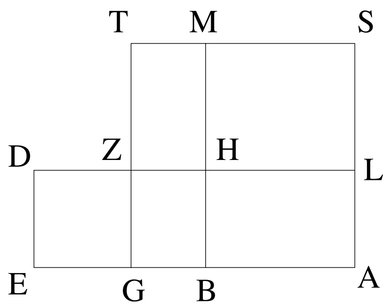

Figure 37a

(Fig. 37a) Then, if we want to divide a known line into two parts in such a way that a number of times the square of one of its parts is equal to the product of the line and the other part: let the line be AG. We want to divide it in such a way that three times the square of one of its parts is equal to the product of the line and the other part. Then we construct on AG the square AT, and we cut off from AG a third of it, namely GB. We draw BM parallel to GT. Then the area BT is a third of the square AT. We apply to BG a parallelogrammic area BD < equal to the area BT, > which exceeds its completion by the square GD [*El.* VI:29]. If we subtract BZ, which is common, the square GD is equal to ZM, that is a third of LT. So the area LT is three times the square GD. But the area LT is the product of GT, that is AG, times TZ, so the product GT times TZ is equal to three times the square of GZ. That is what we wanted.

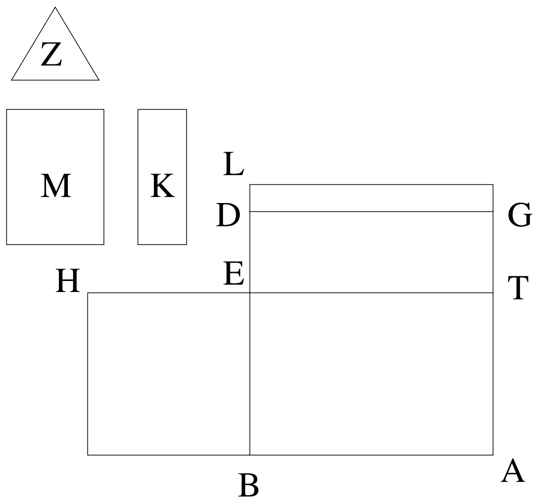

Figure 38

< 38 > (Fig. 38).[97] If line AB is assumed, and we want to divide it in such a way that (the area) which exceeds the product of AB and one of its parts, a number

[97] To the right of triangle Z the manuscript has a note which I tentatively read as follows: *lḥ shakl z muthallath*, "38, the figure of Z is a triangle."

of times, by the area M is equal to (the area) which exceeds a number of times the square of the other part by the area Z:

If M is equal to Z, then if we divide AB in extreme and mean ratio [*El.* VI:30],[98] it is what we want. If [269] M is not equal to Z, let M be greater than Z, and let the excess of M over Z be the magnitude K. We construct on AB the quadrilateral $AGDB$ with equal sides and angles (i.e., a square). We apply to line GD the area GL equal to the area K [*El.* I:44]. Then we apply to AB the parallelogrammic area AH which exceeds its completion by the square BH and such that the applied area is equal to area AL [*El.* VI:29]. If we subtract the common area AE, the square BH is equal to the area TL. But the area TL is equal to BD times DE together with the area K, which is the excess of the area M over the area Z. So the product BD times DE together with the area M is equal to the square of BE together with the area Z.

And if the area Z is greater than the area M, we take the excess of it over it, and we apply to line GD on the other side which is between the two points[99] GD, AB a parallelogrammic area equal to the excess (of Z over M), and then we argue as before, and that is what we wanted.[100]

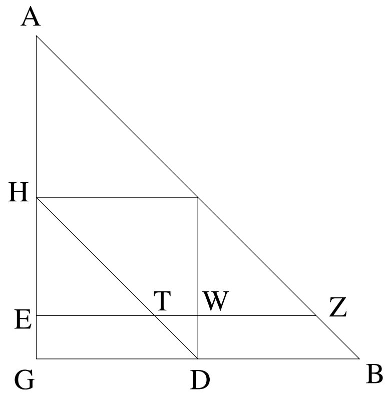

Figure 39

< 39 > (Fig. 39). We want to divide a known line into two parts in such a way that the product of the line and two or more times one of its parts is equal to the

[98]If $M = Z$ and the "number of times" is not one, the problem can be solved as in prop 37.

[99]One would expect: "the two lines."

[100]The problem has a solution only if the excess of Z over M is less than the square of AB. In the beginning of prop. 38, Nu^{c}aim mentions the more general problem of dividing a given segment BD in point E in such a way that $c_1 BD \cdot DE + M = c_2 BE^2 + Z$, where c_1, c_2 are given numbers, not necessarily equal to one. This problem can be solved as follows: For $Z = M$, see prop. 37. Suppose $Z \neq M$. Without loss of generality, we can assume $c_2 = 1$. Construct the rectangle $ABDG$ in such a way that $AB = c_1 BD$. The rest of the construction is as in the case $c_1 = 1$.

square of the other part.[101] Thus let the line be GD. We construct on it the square DH, and we draw the diagonal DH. We extend GD and we construct DB equal to DG. We draw from B a line parallel to the diameter DH, and we extend GH to meet it at the point A. We construct the triangle AZE similar to the triangle AGB and equal to the area $BDHA$ [*El.* VI:25]. We subtract the common area $AZTH$, then the area $BDTZ$ is equal to the triangle TEH. But the area $BZTD$ is equal to the area $DWEG$ which is equal to the area EW times WT, and the triangle HTE is half the square of TE. So twice the area EW times WT is equal to the square of ET and WE is equal to DG [270]. So we have constructed what we wanted.

And if we want that three times the area GD times one of the two parts is equal to the square of the other part, we add to GD one and a half times it, and we construct as we have constructed before.[102]

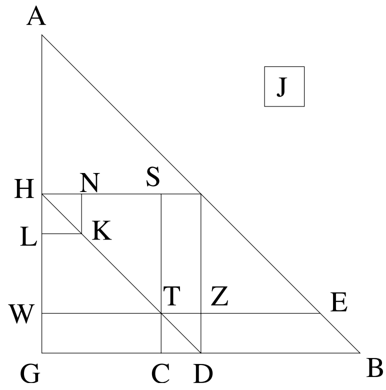

Figure 40

40 (Fig. 40). We want to divide a known line into two parts in such a way that twice the product of the line and one of the parts together with the square of another line is equal to the square of the other part.

Thus let the known line be DG, and (let) the square of the other line (be) the area J. We construct on DG the square DH and we draw the diagonal DH. We construct on the diagonal the area NL equal to the area J. We construct the triangle ABG[103] and we construct the triangle AEW similar to it and equal to (the sum of) the two areas $ABDH, KHL$ [*El.* VI:25]. We subtract the common areas

[101] This is exactly the same problem as in prop. 37.

[102] To find T such that $\gamma \cdot EW \cdot WT = ET^2$, choose the point B on GD extended such that $DB = (\gamma/2) \cdot GD$, draw $BA//DH$, and define $WE//GB$ in such a way that $\Delta AZE = [BDHA]$. For an algebraic transcription see the footnote to prop. 40.

[103] The text implicitly assumes that AB is parallel to HD.

$AETH, KHL$, then area $BEDT$ is equal to the trapezium $TKLW$. But[104] twice the area $DZWG$, which is the result of the product of WZ times ZT two times, is equal to twice the trapezium $TKLW$, which is the gnomon $KNSTLW$. We make the square NL common, then the product WZ times ZT two times together with the square NL, that is area J, is equal to the square TH, which is the square of line TW. That is what we wanted.[105]

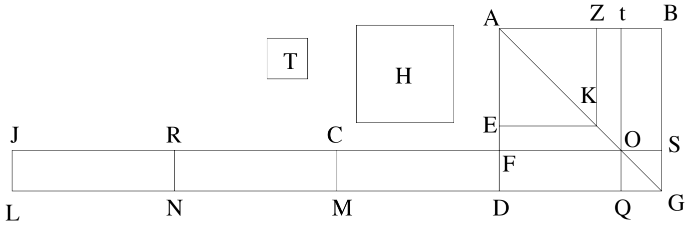

Figure 41

< 41 > (Fig. 41).[106] We want to divide a known line into two parts in such a way that the product of the line and < twice > one of the parts together with the square of another (known) line is equal to the square of the other part together with the square of another (known) line.

Thus we make GD the known line and area H the square of the (known) line which is together with the product of the line and two times one of the parts, and area T the square of the (known) line which is together with < the square of > the other part. We construct on GD the square GA, and we draw the diagonal GA. We construct on it the square EZ equal to the area H. [271] We extend GD towards L and we make GL four times GD. We apply to GL a parallelogram JQ which is deficient from its completion by the square SQ and which is equal (in area) to (the sum of) the gnomon $BZKEDG$ and the area T. We say: the line SF, which is equal to the line GD, has been divided at O as we wanted.

Proof of this: we draw tOQ parallel to BG. Then the area BQ, which is equal

[104]The word "but" (*walākin*) is strange here, but cannot easily be interpreted as a mechanical scribal error for the word "so" (*fa-*), which one would expect here. Perhaps Naṣīr al-Dīn had not really worked through the argument.

[105]The solution can be generalized: if $\gamma \neq 2$ is a given number, define $DB = (\gamma/2)CD$ and let the other definitions be the same, then $\gamma \cdot WZ \cdot ZT + J = TW^2$. In this case the line AB does not pass through the fourth angular point of the square DHG. Algebraically, the problem of dividing a into segments x and y with $a = x + y$, $\gamma ax + c^2 = y^2$ can be reduced to $\frac{1}{2}(y + \frac{1}{2}\gamma a)^2 = \frac{1}{2}c^2 + \frac{1}{2}((a + \frac{1}{2}\gamma a)^2 - a^2)$. For $c = 0$ we obtain the solution of prop. 39. In prop. 40, the text only deals with $\gamma = 2$.

[106]Apparatus to the figure: the labels Q, R in Fig. 41 are my emendations of the labels h, Q in the manuscript.

to the area SD, is equal to the area FM.[107] We make the area OD common. Then the gnomon BOD is equal to the area OM, and by subtraction, the gnomon tKF together with the area T is equal to the area CL, that is the area SM. We make the area ZE, that is the area H, common, then the area SM, which is equal to the product of twice SF times SO together with area H, is equal to the square tF, which is equal to the square of line OF together with the area T. Thus we have divided the line GD at Q in such a way that two times the product QG times GD together with the area H is equal to the square of QD together with the area T, and that is what we wanted.

And if we want the product of the line and one time one of its parts together with the above-mentioned area to be equal to the square of the other part together with the (other) above-mentioned area, we restrict ourselves to GN [272], which is three times GD (Fig. 41), and we explain as we have explained previously, and that is what we wanted to explain.[108]

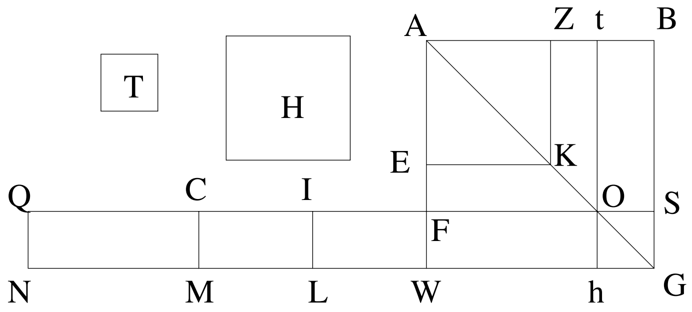

Figure 42a

< 42 > (Fig. 42a) And in the way we have described, we (can) divide AB into two parts in such a way that the product of one of the parts times the other (part), once or twice or more (times), together with the square of another (known) line, is equal to the square of the other part together with the square of another (known) line.

I say:[109] we repeat the figure,[110] and let it first be the product of one of the parts once (times the other part). We make WL equal to half GW, and we subtract from the square TF the square ZE equal to the area H. We construct on the line GL

[107]Point M is supposed to be on GD extended such that $DM = GD$, and MC is parallel to AD.

[108]If γ is a given number, and if we define $GL = (\gamma + 2) \cdot GD$ and let all other definitions be the same, we have $\gamma QG \cdot GD + H = QD^2 + T$. Algebraically, the problem of dividing a into segments x and y with $a = x + y$, $\gamma ax + H = y^2 + T$ is reduced to $x \cdot ((\gamma + 2)a - x) = a^2 - H + T$. In prop. 41 $a = GD, x = GQ, y = QD$.

[109]The words "I say" indicate that the first part of the construction and fig. 42a are the work of Naṣīr al-Dīn al-Ṭūsī.

[110]Point D in Fig. 41 appears as W in Figure 42a and in the first part of prop. 42.

a parallelogrammic area deficient from its completion by a square, and equal to the area $GKEW$ together with half the area T [*El.* VI:28]. That area is hI, and its double hQ[111] is equal to the excess of the square GA over the area H together with the area T.

If we subtract from the excess of the square GA over the area H the complement $BO\ OW$, and from the area hQ the two equal areas hF, MQ, the complement tFK together with the square Sh and the area T is equal to area WC, that is the area GF. We subtract from it the common square Sh, so OW is equal to the (sum of the) complement tKF and the area T. < And if > we add the area H to them, the area hF together with the area H is equal to the square OA together with the area T. And by analogy to this, (treat the case of) more than one time.[112]

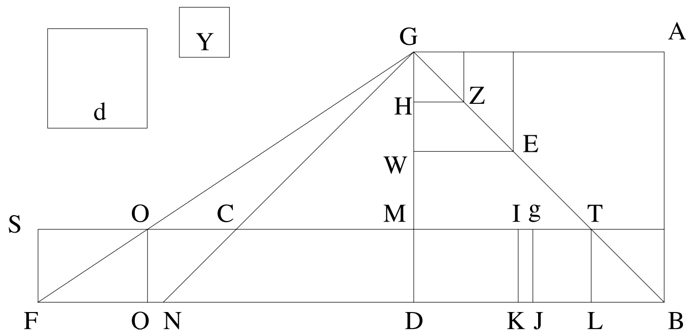

Figure 42b

(Fig. 42b) He said:[113] We want to divide a known line into two parts in such a way that one of the parts times the other part, two times or three times or as many (times) we want, together with the square of another (known) line < is equal to the square of the other part together with the square of another (known) line >.

Then we make it (for the case of) three times: we make the line BD, and we make line Y the (known) line which is together with one of the parts times three

[111] Points M and N are assumed to be on GW extended in such a way that $LM = LW, NM = WH$, and MC, NQ are perpendiculars drawn onto SF extended.

[112] Algebraically, the problem is to divide segment a into two parts x and y such that $a = x+y$, $\gamma xy + H = y^2 + T$. This problem can be reduced to the equation $x((\gamma+2)a - (\gamma+1)x) = a^2 - H + T$. Al-Ṭūsī treats $\gamma = 1$ but the generalization to arbitrary γ is not obvious from what he says.

The manuscript has on f. 272 a mysterious figure which I have rendered as Fig. 42c below, and which is not discussed in the extant text.

[113] The words "he said" indicate that the following argument and Fig. 42b are the work of Nu^{c}aim. The scribal errors in this part of the manuscript suggest to me that Naṣīr al-Dīn al-Ṭūsī did not understand the construction and reproduced the text in the corrupted form. I have reconstructed the original text by Nu^{c}aim, and my emendations are listed in the critical apparatus to the Arabic edition.

times the other part, and line d the (known) line which is together with the other part. We want to divide BD into two parts in such a way that the product of one of the parts times three times the other part, together with the square of Y, is equal to the square of the other part together with the square of d.

Then we add to line DB (segment) DF and we make DF one and a half times BD. We take from it (a segment) equal to BD, namely [273] DN. We draw NG and we draw FG and we construct the square EG equal to the square of d,[114] and the square ZG equal to the square of Y.[115] We apply to line FB an area $KISF$ equal to area $ZHBD$ and triangle GEW together, and deficient from its completion by an area whose length is twice its width. Then line KL is equal to KI and (area) BT (i.e., square BLT) is equal to the square of TL.[116]

The oblong area $OQFS$ is one and a half times the square of TL since FQ is one and a half times QO. We take line LJ from line LK and we make it equal to three quarters of LK. We draw Jg at right angles. Then the area (i.e., rectangle) TLJ is equal to the triangle OSF, and the area JI is the excess of the area $CONF$ over half the area $TLDM$, since the area $CONF$ is half of the area $TBDM$, since BD is twice NF, and since JK is a quarter of BL.

Then we subtract the two equal areas $BTDM, DMNC$, so the area $TLDM$ and half the area $TLDM$ together are equal to (the sum of) the area $ZTHM$ and the triangle GEW, since half of the area $TLDM$ together with the area KJg is equal to the area $CONF$. We make the triangle ZGH common, then the area $TLDM$ and half of the area $TLDM$ together, together with the triangle GZH, are equal to the triangle GTM together with the triangle GEW. Thus it has become clear that we have divided BD into two parts at L in such a way that the product DL times BL three times, together with the square ZG is equal to the product TM times itself together with the square EG, and that is what we wanted to explain.[117]

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[114] The manuscript has incorrectly Y.

[115] The manuscript has incorrectly d.

[116] Point T is the intersection of SI extended and the diagonal BG, and TL is perpendicular to BD.

[117] Algebraically, the problem to divide a given segment a into two parts x and y such that $a = x + y$, $\gamma xy + c = y^2 + d$ for given c, d can be reduced to the equation $x((1+\frac{1}{2}\gamma)a - \frac{1}{2}(\gamma+1)x) = \frac{1}{2}(a^2+d-c)$. In the text, $\gamma = 3, a = DB, x = LB, y = DL$, the left side of this equation is the area $[KISF]$, and the right side is the area $[ZHBD] + \Delta GEW$. For general γ, the definitions of the text should be modified as follows: $ABGD$ is a square with side a, $DN = a, DF = \frac{1}{2}\gamma a$. The rectangle $KISF$ should be applied to segment BF in such a way that the length BK of the "deficient" rectangle is $\frac{1}{2}(\gamma + 1)$ times its width KI. Since $[KISF] = [ZHBD] + \Delta GEW$ and $[DMNC] = [DLTM]$, by subtraction $[CNFS] + [KIMD] = [ZHMT] + \Delta GEW$ (1).

We have $KB = \gamma LB$; if we define J on KL such that $LJ = \frac{1}{4}\gamma LB$, then $KJ = \frac{1}{2}(\frac{1}{2}\gamma - 1)LB$. Then $OS = 2LJ$ so $\Delta FOS = [TLJg]$. We now study the left side of equation (1). We have

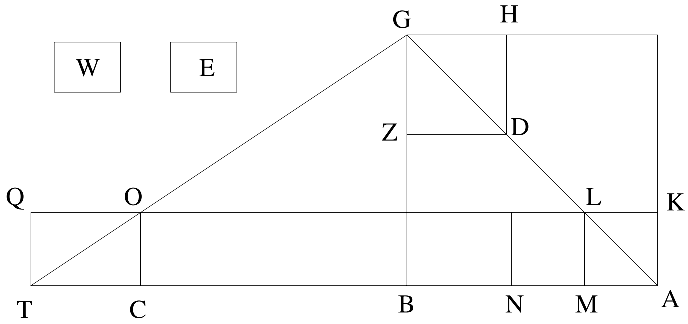

Figure 42c

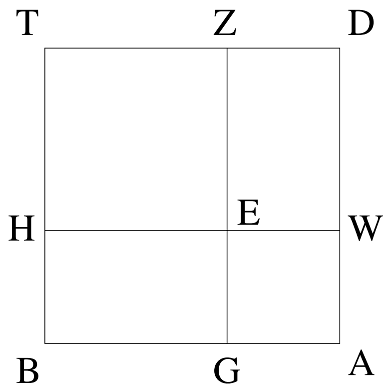

Figure 43

I have found the end of the propositions of a book by the Banū Mūsā to be propositions that add to (prop.) 42.

(Fig. 43)[118] (For) every line divided in extreme and mean ratio, the product of the line and the longer part together with the square of the shorter part is equal to twice the square of the longer part. This has been explained, and it is clear from this figure by means of the gnomon. And together with this it is shown from this figure that the square of the line, namely AB, together with the square of the shorter part, namely AG, is three times [274] the square of the longer part, namely BG [*El.*

$[CNFS] = [CNFO] + \Delta FOS$. But $[CNFO] = (\frac{1}{2}\gamma - 1)[TBDM] = (\frac{1}{2}\gamma - 1)[TLDM] + (\frac{1}{2}\gamma - 1)\Delta TBL$, and $(\frac{1}{2}\gamma - 1)\Delta TBL = [JKIg]$, so $[CNFS] = (\frac{1}{2}\gamma - 1)[TBDM] + [JKIg] + [TLJg]$. Hence $[CNFS] + [KIMD] = (\frac{1}{2}\gamma - 1)[TBDM] + [JKIg] + [TLJg] + [KIMD] = \frac{1}{2}\gamma[TBDM]$. In conclusion $\frac{1}{2}\gamma[TBDM] = [ZHMT] + \Delta GEW$, so $\gamma[TBDM] = MT^2 - HZ^2 + EW^2$ as required.

[118] The figure is not completely defined in the text. Segment AB has been divided in extreme and mean ratio at point G and BG is the longer part. Apparently $ABTD$ and $AWEG$ are squares, and the sides GE and WE have been extended to meet lines TD and TB at points Z and H.

XIII:4].[119]

This is the end of the book of Nu^{c}aim ibn Muḥammad ibn Mūsā on the geometrical propositions.

[119] The proposition may be proved as follows: Since AB has been divided in extreme and mean ratio at G and BG is the greater segment, $BG^2 = AB \cdot AG$, that is to say that $[EHTZ] = [ABHW]$. Thus $AB \cdot BG + AG^2 = [BGZT] + [AWEG] = [EHTZ] + [AWHB] = 2[EHTZ]$, and $AB^2 + AG^2 = [EHZT] + \text{gnomon}[ZDABHE] + [AGEW] = [EHTZ] + 2[AGZD] = 3[EHTZ] = 3BG^2$.

Arabic Text

منقول من خط المخدوم السعيد خواجة نصير الدين قدّس الله روحه.
هذه مسائل هندسية من كتاب نعيم بن محمد بن موسى المنجّم نقلتها من نسخة في غاية الفساد أصلحت ما فهمت منها ونقلت ما لم أفهم على الوجه الفاسد كما كان في النسخة والله المستعان.

١ إذا كان مربع ابجد معلوم الأضلاع وزاويتا ابج دجب متساويتان ونخرج عمود از [وليكن معلومًا] وأردنا أن نعلم قطري المربع فلنخرج اه[1] موازيًا لـ دج ونخرج اد هج إلى أن يلتقيا على ح ونخرج دم موازيًا لخط بجح فيكون مثل هج ولكون اب مثل اه و مه مثل دج يكون ام مه معلومين (٢٤٧) ونسبة ام إلى مه كنسبة اد إلى دح و اد معلوم فـ دح معلوم. ومربع اح المعلوم مثل مربعي از زح بل مثل مربعي اه هح وسطح به في هح أعني مربع اه وسطح بح في حه ومربع اه معلوم يبقى سطح بح في حه معلومًا وكان بج معلومًا.
وليكن نسبة هج إلى جح المعلومة لكونها مساوية لنسبة اد إلى دح نسبة الواحد إلى الاثنين ونعمل سطح لكس قائم الزوايا على أن لك < مثل و > نصف كس فإذا أضفنا إلى خط بج المعلوم سطح بح في حه المعلوم زائد على تمامه سطحًا شبيهًا لسطح لكس على أن جح الحادث يكون نظير كس يكون جح معلومًا وجميع بح معلومًا ونعلم بعد ذلك قطري المربع وذلك ما أردناه.

٢ نريد أن نعمل مثلثًا قائم الزاوية تكون نسبة أقصر أضلاعه إلى [أضلاعه إلى] أوسطها كنسبة أوسطها إلى أطولها. فلنفرض خطًا مستقيمًا كخط بد ونرسم عليه نصف دائرة باد ونقسم الخط على ج على ذات وسط وطرفين ونخرج عمود جا ونصل اب فسطح دب في بج مثل مربع جد بل مربع اب ونسبة بج إلى جا كنسبة جا إلى جد أعني اب وذلك ما أردناه.

٣ إذا كان مثلث ابج فيه خط اد كيف وقع وعلى اد نقطة مثل ه ونريد أن نجيز عليها خطًا ينتهي على اب اج ويكون ما يقع بين اب اد مثلي ما يقع بين اد اج أو ثلاثة أمثاله أو أي نسبة شئنا ولتكن مثلًا مثلاه. فنتعلم[2] على اب نقطة ط كيف وقعت ونخرج طو موازيًا لـ

[1] ولنخرج له : فلنخرج اه .
[2] ونتعلم : فنتعلم .

اد ونجعل ود مثلي دل ونصل طسل فيكون طس مثلي سل ونخرج لك موازيًا ﻟ اد ونصل طك فيكون طع مثلي[3] عك ونخرج من ه خطًا موازيًا (٢٤٨) لخط طك إلى أن يقع على الضلعين على نقطتي ز ح فيكون زه مثلي هح وذلك ما أردناه.

٤ إذا كان مال وعدة أجذار يعدل عددًا معلومًا جعلنا المال مربع ابجد ونضيف إلى ضلع جد منه سطحًا يساوي عدة أجذار[4] وهو سطح جده ﻓ جه عدة الأجذار وهي معلومة. وجميع سطح اه معلوم وهو مجموع المال والأجذار المساوي للعدد المعلوم. وإذا أضفنا السطح المعلوم إلى جه المعلوم بحيث يزيد على تمامه مربعًا يكون المربع الزائد هو المال.

وله برهان آخر ننصف جه على ح ﻓ جه المعلوم نُصِف على ح وزيد فيه جب ﻓ هب في بج المعلوم مع مربع جح المعلوم يساوي مربع بح فهو معلوم و بح معلوم و جح معلوم ﻓ بج معلوم وهو الجذر.

وله برهان ثالث ليكن اب نصف عدة الأجذار وهو معلوم ونرسم عليه مربع ابجد ثم نعمل سطحًا يشبه[5] المربع المذكور ويساوي < المربع المذكور مع > العدد المعلوم وهو سد فجذره حد معلوم و جد معلوم ﻓ جح الباقي معلوم وهو الجذر لأن العَلَم يساوي المال مع الأجذار والمتمّمان هما[6] الأجذار لأنّهما سطح عدة الأجذار في الجذر.

وإذا قيل مال وعدد معلوم عندي ونحوي جذرٌ[7] من أجذار المال نجعل المال سطح ملجه ونضيف إلى أحد أضلاعه وهو جه[8] سطحًا مساويًا للعدد المذكور مع المال وليكن هجاب فخط ال معلوم[9] لتساوي عدة ما فيه عدة الأجذار المعادلة للمال < والعدد > فنضيف إلى خط ال سطحًا ينقص عن تمامه مربعًا ويكون المضاف مساويًا للعدد المذكور مع المال فيكون المربع الناقص هو المال.

وله برهان آخر ننصف ال (٢٤٩) على ز وقد انقسم بمختلفين على ج فضرب اج[10] في

---

[3] مثلي : مثل .

[4] أجذار : الأجذار .

[5] يشبه : نسبه .

[6] هما: هو.

[7] جذرٌ : جذرًا .

[8] جه : جد .

[9] معلوم : لوم ه .

[10] اج : اب .

جل هو سطح اه المعلوم[11] ألقيناه من مربع < نصف > الخط المعلوم بقي مربع جز < فمربع جز > معلوم وجذره وهو جز معلوم فلج الباقي جذر المال.

وله برهان ثالث ننصف عدة الأجذار ونجعل جد ما فيه من العدد مثل عدة نصف الأجذار ونعمل مربع جز المعلوم وننقص من مربع جز المعلوم العدد المعلوم المذكور مع المال ونعمل ممّا يبقى مربعًا شبيهًا بمربع جز وهو معلوم وخط از معلوم فخط اه معلوم وهو جذر المال.

< ٥ > مثلث بهد قائم زاوية ه وأخرج منها عمود هج على بد وأخرج من نقطتي ب د خطا بح دح يساوي كل واحد منهما هد وأخرج حج. أقول إنّه مثل جد. برهانه نخرج عمود حك < و > نفصل طك مثل كج ويبقى بط مثل جد ومربع ده مثل مسطح بد في دج فمربع دح مثل بد في دج بل [بل] مربع جد وسطح بج في جد. ولأن زاوية دجح منفرجة يكون مربع دح مثل مربع جد ومربع حج وسطح طج في جد أعني مثل مربع حج وسطح بج في جد ونلقي سطح بج في جد المشترك يبقى مربع جد مثل مربع حج فحج مثل جد وذلك ما أردناه.

٦ مثلث ابج أخرج ضلع بج في جهة ك و أعلم على ضلع اب نقطة د كيف وقعت ونريد أن نخرج منها خطًا ينتهي إلى بجك حتى يكون المثلث الحادث منه على خط بجك مثل المثلث المفصول الباقي من مثلث ابج أو مثليه أو أي قدر معلوم أردنا منه.

فليكن أولًا مثليه ولنخرج في مثلث ابج من نقطة د خط دط كيف (٢٥٠) وقع ونخرجه إلى ع ونجعل دع مثلي دط ونخرج عدط في جهة ز ونجعل نسبة عد إلى دز كنسبة بد إلى دا ونجيز على نقطتي ع ز خطي عس حز موازيين لضلع اج ونخرج حز إلى أن يلقى بج على ه ونخرج هد إلى أن يلقى عس على س ونصل سا.

فبين[12] أن سد دم على نسبة عد دط فيكون سد مثلي[13] دم ومثلث ساد مثلي مثلث ادم لكن مثلث سدا مثل مثلث بده لأن نسبة سد إلى ده الذي هو كنسبة عد إلى دز كنسبة بد إلى دا بالتكافؤ وزاوية[14] سدا مثل زاوية بده فإن مثلث بده مثلا مثلث ادم وذلك ما أردناه.

[11] المعلوم : معلوم.
[12] فبين : فنبى.
[13] مثلي : مثل.
[14] وزاوية : وزاويتا.

٧ مثلث ابد فيه ضلعا اد دب مجموعين معلومين وقد أخرج منه عمود اجـ وهو معلوم ويصير[15] نسبة بجـ إلى جد معلومة ونريد أن نعلم كل واحد من خطي اد دب ·

فنخط لذلك خط [ز من] زط ونجعله مثل خطي اد دب مجموعين ونعمل عليه مربع زه ونخرج قطر زه ونعمل مربع زح مثل مربع اجـ وليكن في هذه المسألة جد مثلي جب فمربع بد مثلا مربع دجـ ومثل ربع مربع دجـ جميعًا وخط زط مثل خطي اددب[16] مجموعين· فإذا قسمنا زط بقسمين حتى يكون مربع أحد القسمين مثل مربع زح ومثل مربع[17] ثلثي القسم الآخر فقد عملنا ما أردنا·

برهانه نخرج لح إلى ق ونجعل حق مثل لح حس ونجعل قن مثل < ربع > لح ونخرج نت موازي قطر هز ونجعل نش مثل حق ونخرج شث[18] على زاوية قائمة· ونجعل سطح ف طوله مثل وثمن[19] عرضه ونضيف إلى حش سطح[20] مصحقع[21] يزيد على تمامه سطح شم ويكون شم شبيهًا بسطح ف ويكون المضاف مساويًا لمثلث هحق · فإذا جعلنا ذلك وألقينا سطح شنتث الذي هو مثل حصقك يبقى سطح كع مثل مثلث هصك وضعف كع معدل مربع هك وخط قش مثل لح حس ومثل قن[22] < ربع > لح وسطح شم مثل مربع ثش وثمنه[23] وذلك ما أردناه·

ما رأيت هاهنا غير هذه الدعوى وهذا الشكل ولم أفهم منه شيئًا فنقلته هكذا·

٨ مثلث ابجـ معلوم وعلى اجـ نقطة د وقد أخرج بجـ إلى ح ونريد أن نخرج من نقطة د خطًا ينتهي إلى خطي اب بجـ حتى يكون المثلث الذي فيما بين اجـ جح مثلي المثلث الذي قطعه من مثلث ابجـ مثلًا·

فنجيز على نقطة د خط دز كيف وقع ونخرجه إلى م ك ونجعل دز مثل مز ونجيز على م خط مع يوازي اب ونجعل نسبة مد إلى دك كنسبة جد إلى دا ونجيز على ك خط كه

[15] ويصير : وصير .

[16] دب : جب .

[17] مربع : مربعي .

[18] شث : ست .

[19] وثمن : وتر .

[20] سطح : سطحًا.

[21] مصحقع : فصفح تع .

[22] قن : تن .

[23] ثشوثمنه : وشد ت ن ق .

يوازي اب و نخرج من نقطة ه خط هدوع فخط وع مثل ود ونسبة جد إلى دا التي هي كنسبة مد إلى دك[24] (٢٥١) تكون كنسبة عد إلى ده على التكافؤ ونصل عج اه فيكونان متوازيين ونصل اع فمثلث ادع مثل مثلث جده ومثلث اعد مثلا مثلث اود فمثلث جده مثلا مثلث اود وذلك ما أردناه.

٩ مثلث ابج نُصِف بج منه على د وأخرج اد وأعلِم عليه ح وأخرج بج في الجهتين وفصل منه به جز متساويين وأخرج حله حطز[25] أقول فمثلثا الح اطح متساويان.

برهانه نجيز على ح خط كحم موازيًا لـ بج فـ كح يساوي حم لأن بد مساو لـ دج .[26] ومثلثا الكح امح متساويان ونخرج كم في الجهتين ونفصل كش مس مساويين لـ هد دز المتساويين ونخرج من نقطتي ش س خطين يوازيان خطي اب اج وهما خطا شع سص ونخرج خطي حه حز[27] إلى أن تلاقياهما على ق ن

ونصل قن فهو يوازي عص وذلك لأن عه زص المساويين لـ بد دج متساويان و شح سح متساويان فمثلث شقح يشبه[28] مثلث عقه ونسبة عه إلى شح كنسبة قه إلى قح وكذلك مثلث سنح يشبه[29] مثلث صنز ونسبة صز إلى سح كنسبة نز إلى نح فنسبة حق إلى قه كنسبة حن إلى نز و قن يوازي هز .

وأيضًا مثلث حشق يشبه[30] مثلث حكل فنسبة شك إلى كح كنسبة قل إلى لح ومثلث حسن يشبه[31] مثلث حمط ونسبة سم إلى مح كنسبة نط إلى طح فنسبة قل إلى لح كنسبة نط إلى طح وإذا وصلنا لط كان موازيًا لـ هز فمثلثا لكح طمح على قاعدتين متساويتين وبين خطين متوازيين فإذن جميع مثلث الح مساو لجميع مثلث اطح وذلك ما أردناه.

(٢٥٢) ١٠ نريد أن نعمل على مثلث مختلف الأضلاع كمثلث جدط مربعًا متساوي الأضلاع والزوايا يحيط بالمثلث إن أمكن . فنخرج عمود المثلث وهو جم ونخرجه ونفصل جح

[24] إلى دا التي هي كنسبة مد إلى دك : in the margin only.
[25] حطز : حرطر.
[26] لـ دج : لـ رد .
[27] حهحز (i.e., ح ه ح ز) : م ه م ز .
[28] يشبه : نسبه .
[29] يشبه : نسبه .
[30] يشبه : نسبه .
[31] يشبه : نسبه .

مثل دط ونصل دح ونخرج من[32] ط عليه عمود طه ونخرج هط ومن ج عليه عمود جب ونخرج من د خطًا يوازي خط هط وهو دا ونخرج بج إلى أن يلقاه على ا . أقول فه اه مربع.

برهانه إنا نخرج عمود جز على ده ففي جزح دمح زاويتا م ز قائمتان وزاوية ح مشتركة فمثلثا جزح دمح متشابهان وبمثله نبين أن مثلثي دمح دهط متشابهان فمثلث جزح يشبه[33] مثلث دهط فه جز مثل ده و جز مثل به فه ده مثل به فسطح اه مربع وذلك ما أردناه.

واعلم أن المثلثات ما لا يحيط به مربع وهو أن يكون عموده مثل نصف القاعدة أو أقلّ من ذلك لأنه إذا كان جم مثل مح و دم مشترك والزاويتان اللّتان عن جنبتي م قائمتين فه دج إذًا مثل دح وهذا خلف لأن دح ينبغي أن يكون أطول من ده في المربع وذلك ما أردناه.

١١ نريد أن نعمل في مثلث مختلف الأضلاع مربعًا يحيط به المثلث وليكن المثلث حبز . فنخرج من نقطة ح عمود حد ومن ب خطًا يوازي دح ونفصل منه مثل بز ونصل اد ونخرج من نقطة < م > عمود < مج > وخط مو موازيًا لـ بز ومن و خط وه موازيًا لـ مج فأقول إن مه مربع.

برهانه فلأن اب مواز لـ مج تكون نسبة اب إلى مج كنسبة اد إلى دم ونسبة اد إلى دم كنسبة بح إلى حم وهي كنسبة بز إلى مو فبالإبدال[34] نسبة اب إلى بز كنسبة مج إلى مو و اب مثل بز فه مج مثل مو فه مه مربع (٢٥٣)

١٢ نريد أن نبين أن كل مثلث فإن أعمدته الثلاثة إذا أخرجت التقت على نقطة واحدة. فليكن مثلث ابج أولًا حاد الزوايا ونخرج فيه عمودي به جد يلتقيان على ز ونخرج ازح فأقول[35] إنه عمود على بج .

برهانه إنا نصل ده فلأن زاويتي بهج جدب قائمتان تحيط بمنحرف بدهج دائرة وتكون زاوية جده مثل زاوية جبه وكذلك أيضًا لأن زاويتي ادز اهز قائمتان يكون منحرف ادزه تحيط به دائرة وتكون زاوية حاه مثل زاوية جده أعني زاوية جبه فزاويتا جاح جبه متساويتان وزاوية ج منهما مشتركة وتبقى احج مثل زاوية بهج[36] القائمة فهي قائمة.

وإن كان المثلث منفرج الزاوية مثل مثلث بزج وزاوية ز منفرجة فإذا أخرجنا عمودين من

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[32] و : من .
[33] نسبه : يشبه .
[34] وبالابدال : فبالإبدال .
[35] و اقول : فأقول .
[36] ده ج : بهج .

نقطتي ب ج والتقيا عند ا كان مثلث ابج الحادث حاد الزوايا وبيّن أن از إذا أخرج لقى قاعدة بج على قائمة فكل[37] مثلث فإن أعمدته تلتقى على نقطة وذلك ما أردناه.

١٣ إذا كان مثلث بدو معلوم الأضلاع وأردنا أن نخرج فيه خطًا يوازي دو ويكون الخط المخرج مثل الخطين اللّذين فصلهما[38] ذلك الخط من الضلعين مما يلي د وفإنا نقسم خط دو بقسمين على ه حتى تكون نسبة ده إلى هو كنسبة دب إلى بو ونصل هب ونخرج من ب خط با يوازي دو وليكن با مثل بد ونخرج ود على استقامة ونخرج من ا خط اج يوازي بد وبين أن اب يكون مساويًا لـ اج ونخرج اه يقطع بد على ح ومن ح خط حز موازيًا لـ دو[39] فأقول إن خط حز مثل خطي[40] حد زو.

برهانه إن نسبة اب إلى لح كنسبة اه إلى هح لتوازي اب حل (٢٥٤) ونسبة اه إلى هح كنسبة اج إلى حد فنسبة اج إلى حد كنسبة اب إلى حل وبالإبدال نسبة اج إلى اب كنسبة حد إلى حل و اج مثل اب فـ حد مثل حل ونسبة حد إلى زو كنسبة حل إلى لز وبالإبدال نسبة حد إلى حل كنسبة زو إلى لز و حد مثل حل فـ لز مثل زو وخط حلز مثل خطي حد زو جميعًا وذلك ما أردناه.

وإن أردنا أن يكون الخط الموازي المخرج مثلي الخطين اللّذين فصلهما من المثلث فجعلنا اب ضعف بد وكذلك أي مقدار أردناه.

١٤ وإن أردنا أن يكون مربع الخط الموازي المخرج مثل مربعي الخطين المفصولين[41] من المثلث فإنا[42] نجعل في مثلث ابج بعد إخراج جا مربع جز مثل مربعي جا اب ونخرج كهح موازيًا لـ جز ومساويًا لـ كج فمربع كح أعني مربع هد يساوي مربعي كه هب أعني مربي جد هب وبمثل هذا التدبير لو أردنا أن يكون مربع ده مثلي[43] مربعي به دج أو أي قدر شئنا وذلك ما أردناه.

١٥ إذا كان في دائرة ابج وتر بج معلومًا وقطرها معلوم ورُسم على وتر بج خطا با اج

[37] فكل : وكل .
[38] فصلهما : فضلهما .
[39] لـ دو : لـ در .
[40] خطي : خط .
[41] المفصولين : المصولين .
[42] فإنا : فان .
[43] مثلي : مثل .

وكانت نسبة أحدهما إلى الآخر معلومة ونريد أن نعلم كل واحد منهما، فنقسم زاوية باج بنصفين بخط اد ونخرج بد دج فهما متساويان وكل واحد منهما معلوم. ونسبة با إلى اج كنسبة به إلى هج فكل[44] واحد من به هج معلوم ويكون ده معلومًا (٢٥٥) ويبقى اه معلوم وسطح هد[45] المعلوم في با اج معلوم فخطا با اج مجموعين معلومان ونسبة أحدهما إلى الآخر معلوم فكل[46] واحد منهما معلوم.

وفي مثل هذه الصورة إذا كان قطر الدائرة معلومًا وخط بج معلوم وخطا با اج مجموعين معلوم ونريد أن نعلم كل واحد منهما، فيكون كل من بد دج معلومًا ومضروب بد في با اج مجموعين معلوم وهو مثل اد في بج فـ اد معلوم و بد معلوم فـ اب معلوم.

١٦ إذا كان مثلث ابج معلومًا وقد أخرج فيه بج إلى د وهو معلوم ونريد أن نخرج من نقطة د خطًا ينتهي إلى اب حتى يكون المثلث الذي يحدث فيما بين الخط المخرج وخطي اج جد مثل المثلث الذي حدث من الخط وخطي با اج ، فنصل اد ونخرج من ج جه موازيًا لـ اد ونصل هد فمثلثا اهد اجد متساويان لكونهما على قاعدة اد وفيما بين متوازيي هج اد ونلقي مثلث ازد المشترك يبقى مثلثا دزج ازه متساويين وذلك ما أردناه.

١٧ فإن أردنا أن يكون مثلث دحج نصف مثلث اهح[47] فإنّا نصل دا ونخرج من نقطة ج خط جل يوازيه فمتى أخرجنا من د خطًا يقطع اج لج وينتهي إلى اب حتى يصير الخط الذي فيما بين اج لج مثل الخط الذي فيما بين لج اب عملنا ما أردنا.

وإخراج هذا الخط يكون كما نبين. لنضع ذلك الخط خط دحزه يقطع اج لج على نقطتي ح ز فـ حز مثل زه ونسبة ده إلى هز كنسبة دا إلى زل ونسبة ده إلى هز كنسبة دز مع زح إلى زح ونسبة دز مع زح إلى زح كنسبة اد مع ضعف زج إلى زج فنسبة اد مع ضعف زج إلى زج (٢٥٦) كنسبة اد إلى لز ونخرج لج إلى م ونجعل جم مثل نصف اد فنسبة مز إلى زج كنسبة جم إلى لز فسطح جم في جز مثل سطح زم في زل .

فإذا أردنا أن نقسم لم هذه القسمة حتى يكون مضروب أحد قسميه في الآخر مثل مضروب جم في فضل لج على أحد ذينك القسمين فإنّا نعمل على لم مربع مك ونخرج قطر كم ونخرج من ج خط جس إلى كم مثل جم ونخرج طس يوازي لم ونخرج لف مثل لط ثم

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[44] فكل : وكل .

[45] هد : بد .

[46] فكل : وكل.

[47] اهح : اه ر .

نضيف إلى كف سطحًا متوازي الأضلاع ينقص عن تمامه مربعًا ويكون مساويًا لسطح طسجل المستطيل. فإذا فعلنا ذلك يبقى سطح لو مثل سطح عج الذي هو سطح سج في جز وسطح ول هو سطح مز في زل وكذلك إن أردنا أن يكون قدر مثلث اهح أي[48] قدر شئنا دبّرنا فيه هذا التدبير.

١٨ ليكن سطح ابجد المتوازي الأضلاع معلومًا وقد أخرج فيه قطر بج وأخرج بد إلى لا[49] نهاية ونريد أن نخرج من ا خطًا يقطع بج وينتهي إلى ه من خط بد خارج السطح ويكون المثلث الذي يقع فيما بين خطي بج دج مثل المثلث الحادث خارج السطح أعني مثلثي جحز زده . فنجعل السطح الباقي من مثلث بجد مشتركًا فيصير مثلث بجد مثل مثلث بحه. ولكن مثلث بجد مثل مثلث ابج ونسبة مثلث بحه إلى مثلث ابح كنسبة هح إلى حا أعني نسبة بح إلى حج . وأيضًا مثلث بحه مثل مثلث ابج ونسبة مثلث ابج إلى مثلث ابح كنسبة جب إلى بح فنسبة جب إلى بح كنسبة بح إلى حج وسطح بج في جح مثل مربع بح (٢٥٧) فإذن إذا قسمنا بج على ح على نسبة ذات وسط وطرفين وأخرجنا من ا خط احه صار مثلث جحز مثل مثلث دزه وذلك ما أردناه.

١٩ ليكن ازحب متوازي الأضلاع وقد أخرج زح إلى ط ونريد أن نخرج من ط خطًا إلى از حتى يكون المثلث الذي حدث مساويًا للسطح الذي فوقه وهو الذي قطعه من سطح ازحب .

فنخرج من ط خط طه كيف اتفق وليقطع ضلعي از بح على نقطتي و س ونعمل مثلث طهد شبيهًا بمثلث طوز ومساويًا لمثلث طوز ولسطح وزحس[50] ونخرج ده ونعمل على دط مثلث دجط مساويًا لسطح ازحب نقول فخط طم هو الذي أردناه.

فلأن في مثلث < دجط > خطي زم حل موازيان لخط جد وقد أخرج فيه طوه وصيّر سطح هدزو مثل سطح وزحس[51] يكون سطح جدزم مثل سطح مزحل وإذا ألقينا سطح مزحل المشترك بين سطح[52] ازحب ومثلث طجد المتساويين يبقى مجموع سطح جمزد ومثلث

48 أي : الى .
49 إلى لا : لا الى .
50 وزحس : وع س .
51 وزحس : ٥ رح س .
52 سطح : سطحي .

طلح أعني مثلث طمز مساويًا لسطح املب وذلك ما أردناه.

وإن أردنا أن يكون المثلث نصف السطح أو مثليه أو أي قدر شئنا منه فعلى ما وصفنا نعمل ما نريد.

٢٠ مربع ابدج فيه خط هز مواز لـ اج وذراع من سطح از بخمسة أذرع من سطح زب قيمة[53] فنريد أن نخرج من نقطة د خطًا إلى اج حتى يقطع سطحي جه زببقسمين وتكون قيمة[54] أحد قسمي مربع ابدج مثل قيمة[55] القسم الآخر منه.

فنخرج خط دمل إلى قدر (٢٥٨) معلوم من اج وهو جل فسطح لمزج وقيمته معلومان وكذلك مثلث مزد وقيمته وقيمة نصف سطحي از زب معلومة ونسبة قيمة مثلث لجد إلى قيمة مثلث مزد المعلومتين كنسبة نصف قيمة سطحي از زب المعلومتين إلى قيمة نظير مثلث مزد وهو مثلث زدط فقيمته وسطحه معلومان ونخرج دط إل ح فتصير قيمة مثلث دجح مثل قيمة نصف سطحي از زب وذلك ما أردناه.

٢١ إذا كان مربع اطجل مختلف الأضلاع معلومًا وفيه زاوية جطا قائمة وقد أخرج طج إلى غير نهاية ونريد أن نخرج من نقطة ا خطًا يقطع لج وينتهي إلى خط طج حتى يصير المثلث الذي حدث من الخط المخرج وخطي لج جح مثل مثلث ب المعلوم فنخرج من نقطة ا خط اح يوازي خط لج فيكون جح معلومًا وذلك لكون طج وزاوية احط أعني زاوية لجط معلومتين. وليكن الخط المخرج خط اس يقطع لج على ز ونخرج زو موازيًا لخط جح فيكون مساويًا له ومعلومًا. ونضيف إلى وز سطحًا متوازي الأضلاع مثل ضعف مثلث ب وهو سطح دوزه وهو مثلا ب ونصل اه هس اج فمثلث ازه يكون مثل مثلث ب بل مثل مثلث زجس فخط هس يوازي خط اج وكان هج يوازي اح فمثلث هجس يشبه[56] مثلث احج ومثلث احج معلوم الأضلاع فنسبة هج إلى جس معلومة وخط زه معلوم ومثلث زجس معلوم المساحة فخطا زس جس معلوم. وبمثل هذا التدبير نخرج من نقطة ا خطًا يصير مثلث زجس مثل مثلث ب وإن لم يكن زاوية (٢٥٩) ط قائمة .

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[53] قيمة : ثلثه .

[54] قيمة : قسمه .

[55] قيمة : قسمه .

[56] يشبه : نسبه .

٢٢ سطح ابجد متوازي الأضلاع أخرج[57] ضلع بج إلى غير نهاية وقد أخرج فيه خط هز يوازي ضلعي اب جد ونريد أن نخرج من نقطة ا خطًا ينتهي إلى خط بجل ويقطع خطي جد زه ويصير المثلث الذي حدث خارجًا عن سطح ابجد مثل السطح الذي قطعه الخط المخرج من سطح هدجز · فنخرج خط اطح كيف ما وقع ونقسم جد بقسمين على م حتى تكون نسبة دم إلى مج مثناة كنسبة مثلث احد إلى سطح هطحد ونخرج امل فأقول إن سطح هكمد مثل مثلث مجل ·

برهانه إن نسبة مثلث احد إلى سطح هطحد التي هي كنسبة مثلث امد إلى سطح هكمد كنسبة دم إلى مج مثناة أعني كنسبة مثلث امد إلى مثلث مجل فنسبة مثلث امد إلى سطح هكمد ومثلث مجل واحدة فإذن هما متساويان وذلك ما أردناه.[58]

٢٣ إذا كان مربع ابجد وقطره بج وأخرج في مثلث ابج خط لص يوازي بج ونريد أن نخرج من نقطة ا خطًا ينتهي إلى جد ويقطع خطي لص بج ويكون الخط الذي وقع منه فيما بين نقطة ا وخط لص مثل الخط[59] الذي يقع فيما بين بج وبين دج أو مثليه أو أي قدر شئنا. فنعمل أولًا على أنه مثله ونخرج خط ازح كيف ما وقع ونجيز على ا خط اع يوازي خطي لص بج ونخرج حزا[60] إلى ه ونجعل اه مثل زح ونخرج ازح إلى ط ونجعل حط مثل از وكان ها مثل زح فـ هز مثل زط ونجيز على نقطة ط خطًا موازيًا لخط لص والخط الذي كان على نقطة ا وهو اع وليقع على دج على نقطة س ونخرج سا وننفذه إلى ك · فلأن ها مثل زح و زا مثل حط فـ ام مثل وس وذلك ما أردناه. وهكذا نفعل أي قدر شئنا. كان في هذا الشكل زيادات. (٢٦٠)

٢٤ إذا كان مربع ابجد وقطره بد وأخرج لط يوازي بد وأخرج از يقع من جد على ز ويقطع بد على ح وأخرج طه موازيًا لـ از ونخرج حه فأقول إن نسبة اط إلى طد كنسبة مثلث حزه إلى مثلث هحد ونسبة اد إلى دط كنسبة مثلث حزد إلى مثلث حهد · برهانه إن طه يوازي از فنسبة اط إلى طد كنسبة زه إلى هد أعني نسبة مثلث حزه إلى مثلث حهد ونسبة اد إلى طد كنسبة مثلث حزد إلى مثلث حهد وذلك ما أردناه.

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[57] أخرج : اجـ ح .

[58] ما أردناه in the margin only.

[59] الخط : in the margin only.

[60] حزا : حب ا .

٢٥ مربع ابجد قطره بد وفيه خط زه يوازي بد وهو يقطع اد على نسبة معلومة ولتكن الثلث[61] ونريد أن نخرج من آ خطًا ينتهي إلى جد ويكون المثلث الذي فيما بين بد[62] جد مثل المثلث الذي قطعه الخط المخرج من مثلث ازه أو مثليه أو أي قدر أردنا.

فليكن أولًا مثله ونصل اط كما أردنا [فرضنا] و هح موازيًا له و اح لح . فلأن نسبة اد[63] إلى ها معلومة وهي كنسبة مثلث لطد إلى مثلث لطح و دا ثلاثة أمثال ها فمثلث لطد ثلاثة أمثال مثلث لحط ومثلث اسه فرضناه مثل مثلث دلط فمثلث اسه ثلاثة أمثال مثلث لحط وهما بين متوازيين فـ اس ثلاثة أمثال لط فمتى كان اه ثلث اد وأردنا أن نخرج من نقطة آ خطًا ينتهي إلى جد ويكون المثلث الذي قطعه الخط المخرج من مثلث ازه وهو مثلث اسه مثل المثلث الذي حدث فيما بين بد دجـ فإنّا نخرج من آ خطًا ينتهي إلى جد بحيث يكون الذي يقع منه فيما بين آ وخط زه ثلاثة أمثال الخط الذي يقع فيما بين بد جد وعلى هذا المثال (٢٦١) الذي وصفنا إن أردنا مثليها أو أي قدر شئنا فعلنا كذلك وذلك ما أردناه.

٢٦ إذا كان مثلث ابجـ قائم الزاوية وتكسيره ومجموع أضلاعه معلومان وأردنا أن نعلم كل واحد من أضلاعه فندير فيه دائرة وهد على مركز ز ونخرج أعمدة زد زه زو وقد علمنا أنّا متى قسمنا التكسير على < نصف > المحيط خرج نصف القطر فخط دز معلوم وهو مثل به ومثل بد فخطا دب به مجموعين معلوم وخط دا مثل او وخط وجـ مثل هجـ فخطا دا هجـ مثل اجـ فـ اجـ نصف الباقي معلوم وبعد ذلك نعلم كل واحد من اد هجـ .

٢٧ وإذا أردنا أن نعرف في مثل المثلث كل واحد من الأضلاع برهانه فإنّا نخط ده بقدر اب بجـ اجـ هي ولم يكتب بعد هذا في النسخة شيء وكانت صورته هكذا فما فهمت منه مراده.

< ٢٨ > إذا كان مثلث اجب مختلف الأضلاع وعمود اد معلوم وهو يقسم بجـ على نسبة معلومة وضلعا اجـ جب معلومين ونريد أن نعلم كل واحد من أضلاع المثلث فنخط هز ونجعله[64] مثل اجـ جب مجموعين ومربع اد معلوم ومربع اجـ مثل مربعي اد دجـ فمتى قسمنا هز قسمين حتى يصير مربع أحد القسمين مثل مربع اد مع تسع مربع القسم الآخر إن كان

[61] الثلث : المثلث.
[62] بد : د.
[63] اد : ده.
[64] ونجعله : ونجعل.

ثلث بج فقد علمنا ما أردنا وهذه المسألة[65] تقسم وتعلم بالصورة التي قبل هذه.

< ٢٩ > مثلث ابج قد أخرج فيه من نقطة ا عمود اد والعمود معلوم و بج معلوم ونسبة اب إلى اج معلومة ونريد أن نعرف كل واحد من اب اج فلأن نسبة اب إلى اج معلومة تكون نسبة مربع (٢٦٢) اب إلى مربع اج معلومة ونجعل خط حط مثل بج ونريد أن نقسمه بقسمين حتى يكون مربع اد مع مربع أحد قسمي حط مثل ثلث مربع[66] القسم الآخر مع مربع اد إن كان مربع اب ثلث مربع اج وربعه إن كان مربع اب ربع مربع اج وبالجملة فإنّا نقسم حط بقسمين حتى تكون نسبة مربع أحدهما مع مربع اد إلى مربع الآخر مع مربع اد كنسبة مربع اب إلى مربع اج فإذا فعلنا ذلك فقد علمنا كل واحد من اب اج[67] ونعلم ذلك من الصورة التي قبل وذلك ما أردناه.

< ٣٠ > إذا كان مثلث اجب مختلف الأضلاع وعمود اد[68] معلوم وهو يقسم بج على نسبة معلومة وجميع اج جب معلومين ونريد أن نعلم كل واحد من أضلاع المثلث فنخط هز ونجعله مثل اج جب مجموعين ومربع اد معلوم ومربع اج مثل مربعي اد دج فمتى ما قسمنا هز قسمين حتى يصير مربع أحد القسمين مثل مربع اد وتسع[69] مربع القسم الآخر إن كان دجثلث جب فقد علمنا ما أردنا. وهذه المسألة تقسم وتعلم بالصورة التي قبل هذا.

< ٣١ > مثلث ابج قد خرج فيه من نقطة ا عمود اد والعمود معلوم و بج معلوم ونسبة اب إلى اج معلومة ونريد أن نعرف كل واحد من اب اج وقد قلنا أن نسبة اب إلى اج معلومة فنسبة مربع اب إلى مربع اج معلومة ونجعل حط مثل بج ونقسمه قسمين حتى يكون مربع اد مع مربع أحد قسمي حط مثل ثلث مربع[70] القسم الآخر مع مربع اد إن كان مربع اب ثُلث مربع اج وربعه إن كان مربع اب ربع مربع اج وبالجملة فإنّا نقسم حط قسمين حتى يكون < نسبة > مربع أحدهما مع مربع اد إلى مربع الآخر مع مربع اد كنسبة مربع اب إلى مربع اج فإذا فعلنا ذلك فقد علمنا كل واحد من اب اج ونعلم ذلك من الصورة

[65] المسألة : (المسئلة) المثلث .
[66] مربع : ثلث مربع .
[67] اج : بج .
[68] اد : ده .
[69] وتسع : وربع .
[70] ثلث مربع : مربع ثلث .

(٢٦٣) التي قبل وذلك ما أردناه.

< ٣٢ > إذا كانت دائرة اجب مجهولة وقطر اب يقطع قطر جد على زوايا قائمة وقد وقع في ربع اج منها سطح حطهز قائم الزوايا فصيّر خطا از جط كل واحد منهما معلومًا ونريد أن نعلم قطر جد أقول[71] فهو معلوم.

برهانه إن سطح با في از مثل مربع اح وسطح اب في جط أيضًا مثل مربع جح فه اب ضُرب في مقداري از جط المعلومين وكان منهما مربعا اح جح فنسبة زا إلى جط[72] معلومة وهي كنسبة مربع اح إلى مربع جح فنسبة اح إلى جح معلومة.

ونخط دائرة عليها كملن وقطرها معلوم وليتقاطعا فيها قطرا كل من على زوايا قائمة ونخرج كم وتر الربع فهو معلوم ونقيم على قوس كم خطي كس سم نسبة أحدهما إلى الآخر كنسبة اح إلى حج التي هي معلومة فتكون نسبة كس إلى سم معلومة وكل واحد منهما معلوم ونخرج عمودي سف[73] سع على قطري كل من فكل واحد منهما معلوم وكل واحد من سهمي كف مع معلوم ونسبة كف إلى كل معلومة وهي كنسبة از المعلوم إلى اب فه اب معلوم وذلك ما أردناه.

< ٣٣ > إذا كان مثلث ابج قائم الزاوية وأردنا أن نخرج من نقطة ج خطًا إلى اب يكون الخط المخرج مع ما بينه وبين ب مجموعين مثل خط اج مع القسم الآخر من خط اب فنخرج با على استقامة إلى د ونجعل اد مثل اج (٢٦٤) وننصف بد على ه ونصل هج ونعمل عليه نصف دائرة جبه ونخرج من نقطة ه وتر هو مثل نصف بج ونخرج وج يقطع اب على ز فأقول إن دز مثل خطي جز زب مجموعين.

برهانه إن مثلث وزه شبيه بمثلث بزج وخط وه نصف بج فخط هز نصف زج ونجعل دح مثل بز فيصير هح مثل هز فخط زح مثل زج و بز مشترك فخطا جز زب مثل حب و حب مثل دز و دز مثل جا از فخطا جا از[74] مثل خطي جز زب وكذلك لو كانت الزاوية حادة أو منفرجة لكان العمل واحدًا.

[71] فأقول : وأقول .

[72] زا إلى جط : زح إلى حط .

[73] سف : س ن .

[74] خطا جا از only in the margin.

٣٤ إذا كان مثلث ابج متساوي الساقين وساقا با اج معلومان وعمود اد مع قاعدة بج مجموعين معلومين ونريد أن نعلم كل واحد منهما فنقول إن[75] هذه المسألة تكون على وجهين أبدًا إلّا أن تكون نسبة اب إلى نصف اد بج كنسبة اد بج إلى جذر مربعي اد بج ونصفه مجموعين فإنّها إذا كانت هكذا فليس تكون إلّا على وجه واحد.

برهانه إنّا نخرج خط حط مثل اد بج مجموعين وننصفه على ل وندير على مركز ط وببعد اب دائرة هف ونقيم على نقطة ط عمود طم مثلي[76] طل ونخرج لم فإن كانت نسبة اب إلى طل كنسبة طم إلى مل فإن مل حينئذ يماس دائرة هف ويكون اد بج على وجه واحد فإن لم يكن كذلك وكان مل يقطع دائرة هف فليقطعه على نقطتي ه ف ونخرج من نقطتي ه ف عمودي هس فن إلى حط وعمودي هو في إلى مط ونصل هط فط ونفصل من حط حع مثلي نل و حك مثلي لس (٢٦٥) ونصل هك فع.

فلأن طم مثلا طل يكون فن مثلي نل و هس مثلي سل وكذلك يكون حع مثل فن و حك مثل هس ويبقى كط مثلي سط لكون مو المساوي لـ كط مثلي هو[77] المساوي لـ سط ويبقى أيضًا عط مثلي نط فيكون كه مثل هط و عف مثل فط وقد تبين أن سه كط مجموعين مثل فن عط مجموعين وتبين من ذلك أن المسألة تخرج على وجهين كما قدّمنا وذلك ما أردناه.

٣٥ مجموع وتر المسدس ووتر المعشر في كل دائرة يساوي وتر ثلاثة أعشارها.

فلتكن الدائرة ابجده والقطر اد ونرسم خطوط اب بج جد كل واحد وتر السدس ونخرج بد وتر الثلث و ده وتر الخمس و ها وتر ثلاثة أعشار ونخرج با ونفصل از وتر العشر وأقول إن اه مثل زب[78].

برهانه زب مقسوم على ا على نسبة ذات وسط وطرفين فـ بز في زا مثل مربع با ومربعا زا اب مثل مربع ده ومربع بد ثلاثة أمثال مربع اب أعني مثل مربع با ومثلي بز في زا و بز في زا مثل با في از ومربع از فمربع دب يكون مثل مربع با وضعف ضرب با في از وضعف مربع از أعني مثل مربعي بز زا ونجعل مربع با مشتركًا فيكون مربعي دب با أعني مربع دا القطر [يكون] مثل مربع بز ومربع با ومربع از أعني مثل مربع بز ومربع ده وكان مثل مربعي ده ها (٢٦٦) فمربعي بز ده مثل مربعي ها ده ونلقي مربع ده المشترك يبقى

[75] اي : إن.
[76] مثل : مثلي.
[77] م و : هو
[78] ره : زب.

مربعي ها بز متساويان ف ها بز متساويان وذلك ما أردناه.

أقول ليس في إخراجي خطي بج جد فائدة فيه وقد بان أن وتر[79] ثلاثة أعشار الدائرة ووتر السدس إذا اتّصلا[80] انقسما على نسبة ذات وسط وطرفين وأقصرهما وتر السدس وأن مربعي وتر ثلاثة أعشار ووتر العشر مثل مربع وتر الثلث.

وينبغي أن نعلم أن نسبة وتر الخمسين إلى وتر الخمس[81] كنسبة وتر السدس إلى وتر العشر لأن وتر الخمسين ووتر الخمس إذا اتّصلا انقسما على نسبة ذات وسط وطرفين مثل وتر السدس ووتر العشر.

٣٦ برهان آخر على أن وتر السدس ووتر العشر مجموعين مثل وتر ثلاثة أعشار الدائرة فليكن دائرة لود قد أخرج فيه قطرا بد جو يتقاطعان على زاوية العشر ولنخرج دح يوازي جو فيكون حو أيضًا عشر الدائرة و حد ثلاثة أعشارها. ونخرج وزه يوازي بد فيكون هد أيضًا عشرًا ولأن سطح وادز متوازي الأضلاع وضلع دا يساوي او[82] فسطح وادز معيّن كل ضلع منه مثل نصف القطر فضلع وز نصف القطر وزاوية زدا[83] مثل زاوية هزد[84] لتوازى هو دب ونصل هد ويكون زاوية زدا[85] أيضًا مثل زاوية هدز لأنّهما على الخمس فزاويتا هزد هدز متساويتان ف هز مثل هد[86] و هد وتر العشر ف هز وتر العشر [ف هز وتر العشر] ف هو وتر العشر ووتر السدس مجموعين وهو وتر ثلثة أعشار الدائرة وذلك ما أردناه.

وإذا وصل از دج كانا متوازيين ومتساويين ف از[87] يساوي وتر العشر وإذا أخرج (٢٦٧) از إلى ل صار قوسا حل له متساويين وذلك لتساوي زاويتي هال حال[88] وتساوي قوسي هد حو وإذا وصلنا حل له كان خط حز مثل كل واحد من خطي حل له ومثلث حلز شبيه بمثلث وزا فنسبة حز إلى زل كنسبة وا إلى از وسطح زل في وا أعني زل في لا مثل سطح حز في از أعني مربع از ف لا[89] مقسوم على نسبة ذات وسط وطرفين على ز وقسمه

[79] وتر : مربع .

[80] اتّصلا : فصلا .

[81] الخمس : الخمسين .

[82] يساوي او : in the margin only.

[83] زدا : روا .

[84] هزد : ه اد .

[85] زدا : روا .

[86] هد : ه ر .

[87] ف از : فان.

[88] هال حال : دال جـ ال.

[89] ف لا : ف ل ر.

الأطول از وإذا أخرج لا إلى م تبين أن زم ينقسم على ا على نسبة ذات وسط وطرفين والأطول ام وتر السدس.

أقول بوجه آخر من فوائد مؤيّد الدين العرضي لتكن دائرة ادب قطرها اب ومركزها جـ وليكن كل واحد من قوسي اد به خمسًا من الدائرة فيبقى ده عشرًا ونصل ده دب هجـ متقاطعين على ز ف ده يوازي اب ومثلثا[90] دزه جزب متشابهان وزاوية اجز التي على ثلاثة أعشار الدائرة عند المركز يساوي زاويتي جزب جبز وزاوية جبز التي عند المحيط على عُشرَين تكون بقدر زاوية على عشر عند المركز وتبقى زاوية جزب يساوي زاوية مركزية تكون على عُشرَين وزاوية زجب كذلك فهما متساويان ف بجـ يساوي بز فهو وتر السدس وكذلك يكون دز مساويًا لـ ده وتر العشر ف دب وتر ثلاثة أعشار الدائرة مساو لوتري السدس والعشر مجموعين وذلك ما أردناه.

< ٣٧ > نريد أن نقسم خطًا معلومًا بقسمين حتى يكون ضرب الخط في أحد قسميه مرتين أو مرارًا كثيرة مثل مربع القسم الآخر.

فنجعل الخط اب ونريد أن نقسمه أولًا بقسمين حتى يكون ضرب الخط في أحد قسميه مرتين مثل مربع القسم الآخر.

فنخرج اب إلى ل ونجعل ال مثل اب ونقيم على نقطة ل من خط (٢٦٨) ال عمود لط مثل اب ونتمم سطح له المتوازي الأضلاع ونضيف إلى خط لب سطحًا متوازي الأضلاع مساويًا لسطح له ويزيد على تمامه مربعًا وهو لد ف لد مثل له[91] ونلقي لز المشترك فيبقى سطح طز مثل مربع زجـ ولكن خط طز هو مضروب به في هز مرتين فقد قسمنا به على نقطة ز وصار مضروب الخط في أحد قسميه مرتين مثل مربع القسم الآخر. وبهذا التدبير لو أردنا ضرب الخط في أحد القسمين ثلاثة مرارًا أو[92] أكثر من ذلك مثل مربع القسم الآخر لفعلنا في ذلك بأن نضاعف اب بقدر عدد المرار[93] .

فإذا أردنا أن نقسم خطًا معلومًا بقسمين حتى يكون مربع أحد قسميه مرارًا عدة مثل

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[90] ومثلثا : in the margin only.
[91] له : اه .
[92] مرارًا أو : مراراو .
[93] المرار : المراتب .

الخط في القسم الآخر وليكن الخط اج ونريد أن نقسمه بحيث يكون مربع أحد قسميه ثلاث مرات مثل ضرب الخط في القسم الآخر فنعمل على اج مربع اط ونفصل من اج ثلثه وهو جب ونخرج بم يوازي جط فسطح بط ثلث مربع اط ونضيف إلى بج سطح بد المتوازي الأضلاع < مساويًا لسطح بط > يزيد على تمامه مربع جد فإذا ألقينا بز المشترك يبقى مربع جد مثل زم أعني ثلث لط فسطح لط ثلاثة أمثال مربع جد وسطح لط هو مضروب جط أعني اج في طز فمضروب جط في طز مثل ثلاثة أمثال مربع جز وذلك ما أردناه.

< ٣٨ > إذا كان خط اب موضوعًا ونريد أن نقسمه حتى يكون ضرب اب في أحد قسميه مرارًا يزيد على ذلك سطح م مثل مربع القسم الآخر مرارًا يزيد على ذلك سطح ز :

فإن كان م مثل ز فإنّا إذا قسمنا اب قسمة ذات وسط وطرفين فقد كان ما أردنا.

وإن (٢٦٩) لم يكن م مثل ز فليكن م أعظم من ز وليكن فضل م على ز مقدار ك . ونعمل على اب مربع اجدب متساوي الأضلاع والزوايا ونضيف إلى خط جد سطح جل مثل سطح ك ثم نضيف إلى اب سطح اح متوازي الأضلاع يزيد على تمامه مربع بح ويكون السطح المضاف مثل سطح ال وإذا ألقينا سطح اه المشترك بقي مربع بح مثل سطح طل وسطح طل مثل بد في ده مع سطح ك الذي هو فضل سطح م على سطح ز فمضروب بد في ده مع سطح م جميعًا مثل مربع به مع سطح ز .

ولو كان سطح ز أعظم من سطح م أخذنا فضله عليه وأضفنا إلى خط جد في الجهة الأخرى التي ما بين علامتي جد اب سطحًا متوازي الأضلاع مساويًا للفضل ثم دبرنا كما مرّ وذلك ما أردناه.

< ٣٩ > نريد أن نقسم خطًا معلومًا بقسمين حتى يكون ضرب الخط في أحد قسميه مرتين أو أكثر مثل مربع القسم الآخر فليكن الخط جد ونعمل عليه مربع دح ونخرج قطر دح ونخرج جد ونعمل دب مثل دج ونخرج من ب خطًا موازيًا لقطر دح ونخرج جح حتى يلقاه على نقطة ا ونعمل مثلث ازه شبيهًا بمثلث اجب ومساويًا لسطح بدحا ونلقي سطح ازطح المشترك يبقى سطح بدطز مساويًا لمثلث طهح ولكن سطح بزطد مثل سطح دوهج الذي هو مثل سطح هو في وط ومثلث حطه نصف مربع طه فضعف سطح هو في وط مثل مربع هط و وه مثل دج(٢٧٠) فقد علمنا ما أردنا.

وإن أردنا أن يكون سطح جد في أحد القسمين ثلاث مرات مثل مربع القسم الآخر زدنا في دج مرة ونصف مثله وعملنا[94] كما عملنا قبل.

٤٠ نريد أن نقسم خطًا معلومًا بقسمين حتى يكون ضرب الخط في أحد قسميه مرتين مع مربع خط آخر مثل مربع القسم الآخر.

فليكن الخط المعلوم دج ومربع الخط الآخر سطح ش ونعمل على دج مربع دح ونخرج قطر دح ونعمل على القطر سطح نل مساويًا لسطح ش ونعمل مثلث ابج ونعمل مثلث اهو شبيهًا به ومساويًا[95] لسطحي ابدح كحل ونلقي سطحي اهطح كحل المشتركين[96] يبقى سطح بهدط مثل منحرف طكلو ولكن ضعف سطح دزوج الذي هو من ضرب وز في زط مرتين مثل ضعف منحرف طكلو الذي هو عَلَم كنسطلو ونجعل مربع نل مشتركًا فيصير ضرب وز[97] في زط مرتين مع مربع نل أعني سطح ش مثل مربع طح الذي هو مربع خط طو فقد عملنا ما أردناه.

< ٤١ > نريد أن نقسم خطًا معلومًا بقسمين حتى يكون ضرب الخط في أحد القسمين < مرتين > مع مربع خط آخر مثل مربع القسم الآخر مع مربع خط آخر. فنجعل الخط المعلوم جد ومربع الخط الذي هو مع ضرب الخط في أحد القسمين مرتين سطح ح ومربع الخط الذي هو مع < مربع > القسم الآخر سطح ط . ونعمل على جد مربع جا ونخرج قطر جا ونعمل عليه مربع هز مثل سطح ح .[98] ونخرج جد إلى ل ونجعل جل أربع مرات

[94] وعملنا in the margin only.

[95] به ومساويًا : ب ه ومساويًا .

[96] كحل المشتركين in the margin only.

[97] وز : در .

[98] The manuscript repeats the passage with some slight differences, which have been indicated by exclamation marks:

نريد أن نقسم خطًا معلومًا بقسمين حتى يكون ضرب الخط في أحد القسمين مرتين مع مربع خط آخر مثل مربع القسم الآخر مع مربع خط آخر. فليكن (!) الخط جد (٢٧١) والمربع الذي (!) مع ضرب الخط في أحد القسمين سطح ح والمربع الذي (!) مع مربع القسم الآخر سطح ط . ونعمل على جد مربع اجَ (!) ونخرج قطر اجَ(!) وعليه مربع زه (!) مثل سطح ح .

مثل جد ونضيف إلى جل متوازي أضلاع شق[99] ينقص عن تمامه مربع سق[100] ويكون مساويًا لعَلَم بزكهدج ولسطح ط . نقول فقد انقسم خط سف المساوي لخط جد على ع كما أردنا.

برهانه نخرج تعق[101] موازيًا لـ بج فيكون سطح بق المساوي لسطح سد مساويًا لسطح فم ونجعل سطح عد مشتركًا فيكون عَلَم بعد مثل سطح عم ويبقى عَلَم تكف مع سطح ط مساويًا لسطح صل أعني سطح سم ونجعل سطح زه أعني سطح ح مشتركًا فيكون سطح سم[102] الذي هو مثل ضرب ضعف سف في سع مع سطح ح مساويًا لمربع تف الذي هو مربع خط عف مع سطح ط . فإذن قسمنا خط جد[103] على ق بحيث كان ضرب قج[104] في جد[105] مرتين مع سطح ح مساويًا لمربع قد[106] مع سطح ط وذلك ما أردناه.

وإن أردنا أن يكون ضرب الخط في أحد القسمين مرة واحدة مع السطح المذكور مثل مربع القسم الآخر مع السطح المذكور اقتصرنا على جن (٢٧٢) الذي هو ثلاثة أمثال جد وبيّنّا بمثل ما بيّنّاه آنفًا وذلك ما أردنا أن نبيّن.

< ٤٢ > وبمثل ما وصفنا نقسم اب قسمين يكون ضرب أحد القسمين في الآخر مرة أو مرتين أو أكثر مع مربع خط آخر مثل مربع القسم الآخر مع مربع خط آخر.

أقول نعيد الشكل وليكن ضرب أحد القسمين مرة أولًا ونجعل ول مثل نصف جو وننقص من مربع تف مربع زه مثل سطح ح ونعمل على خط جل سطحًا متوازي الأضلاع ينقص عن تمامه مربعًا ويكون مساويًا لسطح جكهو مع نصف سطح ط وذلك السطح هو ثي وضعفه ثق[107] يساوي فضل مربع جا على سطح ح مع مربع ط .

وإذا نقصنا من فضل مربع جا على سطح ح متمم بع عو ومن سطح ثق سطحي ثف مق

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[99] شق : س ل .
[100] سق : س ت .
[101] تعق : ت ع ف .
[102] سم : ش م .
[103] جد : حـ و .
[104] قج : وحـ .
[105] جد : جب .
[106] قد : ق و .
[107] ثق : حـ ق .

المتساويين بقي متمم تفك مع مربع ست وسطح ط مثل سطح وص أعني سطح جف ونلقي منه مربع ست المشترك يبقى عو مثل متمم تكف وسطح ط < وإذا > زدنا عليهما سطح ح صار سطح ثف مع سطح ح يساوي مربع عا مع سطح ط . وقس عليه أكثر من مرة.

قال نريد أن نقسم خطًا معلومًا بقسمين حتى يكون أحد القسمين في الآخر مرتين أو ثلاث مرات أو كم شئنا مع مربع خط آخر < مثل مربع القسم الآخر مع مربع خط آخر > فنجعله ثلاث مرات. ونجعل الخط بد ونجعل الخط الذي هو مع أحد القسمين في الآخر ثلاث مرات خط ظ والخط الذي هو مع القسم الآخر خط ض ونريد أن نقسم بد قسمين حتى يكون ضرب أحد القسمين في الآخر ثلاث مرات مع مربع ظ مثل مربع القسم الآخر مع مربع ض .

فنزيد في خط دب دف[108] ونجعل دف مرة ونصف مثل بد ونأخذ منه مثل بد وهو

(٢٧٣) دن ونخرج نح ونخرج فج ونعمل مربع هج مثل مربع ض[109] ومربع زج مثل مربع ظ[110] . ونضيف إلى خط فب سطحًا ينقص عن تمامه سطحًا طوله مرتين مثل عرضه ويكون مساويًا لسطح زحبد[111] ولمثلث جهو جميعًا وهو سطح كيسف فخط كل مثل كي و بط مثل مربع طل[112] .

ويصير سطح عقفس المستطيل مرة ونصفًا مثل مربع طل لأن فق مرة ونصف مثل قع ونأخذ من خط لك خط لش ونجعله مثل ثلاثة أرباع لك ونخرج شغ على زاوية قائمة فسطح طلش مثل مثلث عسف وسطح شي هو فضل سطح صعنف[113] على نصف سطح طلدم لأن سطح صعنف[114] مثل نصف سطح طبدم لأن بد ضعف نف ولأن شك[115] مثل ربع بل .

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108 دف : دو .
109 ض : ط .
110 ظ : ص .
111 زحبد : رحـ ب د .
112 طل : و بط مثل مربع طل : ومربع بط مثل طل .
113 صعنف : ص ع ى ف .
114 صعنف : صعيف .
115 شك : شل .

فنلقي سطحي بطدم دمنص[116] المتساويين فيبقى سطح[117] طلدم ونصف سطح طلدم[118] جميعًا مثل سطح زطحم ومثل مثلث جهو لأن نصف سطح طلدم مع[119] سطح كشغ جميعًا مثل سطح صعنف . ونجعل مثلث زجح مشتركًا فسطح طلدم ونصف سطح طلدم مجموعين مع[120] مثلث جزح يعدل مثلث جطم مع مثلث جهو فقد تبيّن أنّا قد قسمنا بد قسمين على ل فصار ضرب دل في[121] بل[122] ثلاث مرات مع مربع زج يعدل مضروب طم في مثله مع مربع هج وذلك ما أردنا أن نبيّن.

ووجدت آخر أشكال كتاب لبني موسى أشكالًا زائدة على مب[123] .

كل خط قسم على نسبة ذات وسط وطرفين فإن مضروب الخط في القسم الأطول مع مربع القسم الاقصر مثل ضعف مربع القسم الأطول. وذلك تبين ويتبيّن من هذه الصورة بالعَلَم ويتبيّن معه من الصورة أن مربع الخط وهو اب مع مربع القسم الأقصر وهو اج ثلاثة أمثال (٢٧٤) مربع القسم[124] الأطول وهو بج .

هذا آخر كتاب نعيم بن محمد بن موسى في الأشكال الهندسية.

[116] دمنص : دم فمه .

[117] سطح : سطحي .

[118] ونصف سطح طلدم : written upside down between the lines.

[119] جهو لأن نصف سطح طلدم مع : written in the margin.

[120] فسطح طلدم ونصف سطح طلدم مجموعين مع : written in the margin only.

[121] قسمنا بد قسمين على ل فصار ضرب دل في : written in the margin only.

[122] بل : بك .

[123] لبني موسى أشكالًا زائدة على مب : written in the margin only.

[124] القسم : written above the line.

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