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Book XVI: A Mediaeval Arabic Addendum To Euclid's *Elements*

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I Introduction

Euclid's *Elements* has been transmitted into many languages since it was first composed in Greek and has provoked extensive discussions in many branches of this transmission. The Arabic transmission was no exception. Once it became available in Arabic during the eighth and ninth centuries, Euclid's mathematical classic was "corrected", summarized, given addenda, reduced to extracts, commented upon in whole and in part, given alternate demonstrations, paraphrased, and much more. These discussions were not limited to any specific time period, but continued to appear regularly until the nineteenth and twentieth centuries.

This paper presents a hitherto unstudied addendum to an edited version of a popular Euclidean treatise. The addendum, entitled Book XVI, comprises nineteen propositions describing techniques for constructing polyhedra within other polyhedra or within spheres. Thus it develops themes related to those of book XV, itself a late Greek addendum to the genuine Euclidean treatise. The contents are, however, in the tradition of Archimedes rather than Euclid. The first part of Book XVI describes construction of semiregular truncated polyhedra inscribed within regular polyhedra (Platonic solids) and within spheres. The addendum ends with what may be the earliest discussion of the construction of a representative example from each of the classes of semiregular polyhedra known today as prisms and antiprisms. Thus, although the editor or copyist who attached this book XVI to the manuscript clearly intended this material as a kind of continuation of the *Elements*, the intention of the original author is less clearly Euclidean.

II Manuscript

Book XVI is appended to a medieval Arabic manuscript the bulk of which is an edited version of the well-known and influential redaction *Taḥrīr Kitāb Uqlīdis* by Naṣīr al-Dīn al-Ṭūsī (597 AH/AD 1201 - 672 AH/AD 1274). This Book XVI is at present known only in this unique copy whose colophon gives the date of copying as

1003 AH/AD 1593/4.[1] The treatise contains 455 pages,[2] of which the last thirteen (pp. 443-455) comprise Book XVI.

The present codex initially appears to be a pastiche of two manuscripts. Pages 1-202 contain a copy of book I from al-Ṭūsī's treatise. The text is written in a flowing, almost scrawling, nastaʿlīq hand with widely spaced lines, ten or eleven lines per page, presumably to facilitate recording comments and notes, implying that it was produced for use in a pedagogical setting.[3] Each folio ends with a guide word to ease the transition to the next folio. This feature is helpful to the modern reader because an extra sheet (occasionally two sheets) for recording additional comments is inserted between nearly every pair of folios.[4] There are extensive annotations in Arabic, although a few notes are in Persian, both interlinearly and in the margins. It sometimes appears that there may be at least two commentators recording these notes because the script is sometimes produced with a significantly finer pen. Stylistic differences also imply multiple commentators, for some of comments are keyed to the text through the use of numbers while others cite passages by using the Arabic formula "qawluhu" (his statement) followed by the first phrase of the passage to be explained or commented upon. Because of damage to the margins, the marginalia are sometimes not complete.

From page 203, beginning near the end of the enunciation of proposition I, 48 and continuing through the remainder of the manuscript, the text is written with approximately twenty normally spaced lines per page. The change takes place within the proposition and at the end of the folio without any sort of repetition which might be expected in a true pastiche. Two possible scenarios suggest themselves. Either the entire manuscript was copied by the same person or someone with very similar writing style received the slightly incomplete first section and copied out the remainder of the *Taḥrīr*, adding Book XVI at the end. The latter appears more likely. The hand used in the second section of the manuscript is also nastʿlīq, but finer, more elegant, and with a somewhat different orthography (no longer writing the letter aleph of the definite article "al-" below the word, as is done routinely in the first section of the manuscript).

Beyond differences in handwriting, there are other variations that incline me to think that the manuscript was prepared by two persons or at least in two distinct

[1]Hyderabad (India), Oriental Manuscripts Library and Research Center [formerly Andhra Pradesh State Central Library], riyāḍī 496. The colophon appears on page 455.

[2]The modern cataloger has adopted pagination, rather than foliation, perhaps because of the frequent interleaving of material.

[3]The pattern of wide spacing is seen in other examples of manuscripts intended for instructional purposes [De Young 1986, 10].

[4]A few of these interleaved sheets have been misplaced by readers. The sheet dealing with the various cases of proposition I, 2, for example, is now located at pages 345-346.

stages. For example, in the second part of the manuscript guide words are no longer used to smooth the transition from one folio to the next. There are occasional interleaved notes in the second part of the manuscript, although not nearly so frequent as in the earlier section. In the first portion of the manuscript, we frequently find that over-lining is used to pick out the letters used to label geometric points. This style is abandoned in the latter section, where over-lining is most typically used to highlight the first words of the enunciation of each proposition. In the second section, proposition numbers (using *abjad* or alpha-numeric notation) appear to be given in red,[5] while they are usually (but not exclusively) appear to be in black in the first section. In the second portion of the manuscript, there is considerable marginalia in the form of commentary notes, but fewer than in the first portion. Most of these notes appear to be in a hand very similar to that used in the copying of the text itself, while the notes in the first section appear to be in two or three different hands.

The manuscript contains an edited version of the text of al-Ṭūsī. This editing process included, for example, altering the standard phrase "aqūlu" (I say) which introduced many of the comments by al-Ṭūsī to the phrase "qāla al-muḥarrir" (the redactor or editor said). Moreover, the commentary notes included within the body of the manuscript are not precisely the same as those found in most manuscripts of the *Taḥrīr*. Some notes are omitted completely, others are present only in the margins of the manuscript, some are modified or abridged, and about two dozen notes are unique to this manuscript. Many of the omissions are of alternate demonstrations which al-Ṭūsī had borrowed from Ibn al-Haytham's *Kitāb fī ḥall shukūk kitāb Uqlīdis* [De Young 2008b].[6] It is probable that the same person who produced the second part of this edited text of the *Taḥrīr* (if different from the person who produced the first part) also appended Book XVI to the manuscript because a note on Thābit's added proposition at the end of book XIII states: "As for the way of drawing the two shapes, and the constructing of the two magnitudes which constitute their sides (edges), the remainder is in book sixteen."[7]

III Diagrams

In the first section of the manuscript, diagrams are typically placed within the text column. Only occasionally do they extend out into the margins. If in the middle of the column, the diagram will either be placed in an empty band stretching across the

[5] That is, they appear less black than the surrounding text in the photocopy available to me.

[6] These alternate demonstrations were also omitted from Quṭb al-Dīn al-Shīrāzī's Persian translation of the *Taḥrīr*. See De Young [2007]. Whether there is any connection between these two observations awaits further investigation.

[7] *Riyāḍī* 496, p. 429.

column or in a small open space bounded on both sides by text. Sometimes the text encroaches on this "white space" so that it completely surrounds the diagram. Such diagrams imply that the manuscript diagrams were constructed during the writing / copying itself and not added later. It appears that the typical methodology of the scribe was to copy out the first portion of the proposition text, then stop to draw the diagram, and afterward finish copying the remainder of the text of the proposition. This methodology also helps to explain why diagrams are so often found at the end of the proposition they represent.

In the second part of the manuscript, diagrams are most frequently placed in the outer margins, although larger diagrams are sometimes inset slightly into the text. This inward extension is usually in the form of a square or rectangular "window" but in a few diagrams the text is filled in up to the margins of the diagram. Occasionally, diagrams in the second part of the manuscript are rotated 90° relative to the text direction. This rotation applies both to the base line of the diagram and to the labels applied to identify its points. In some cases, this rotation appears to be dictated by space considerations; in other cases it appears that only a small adjustment of scale would have made it possible to fit the diagram into the margin without rotation. Diagrams in the first part of the manuscript are not rotated. Whether this lack of rotation reflects the fact that diagrams in book I typically require less space, whether the use of more widely spaced lines itself gave greater placement options for diagrams, or whether the copyist was consciously motivated by pedagogical concerns is at present a moot point.

In general, the diagrams appear to be executed with a straight-edge and compass. Those in the first portion of the manuscript are typically more precise than those in the latter portion, where one detects a certain carelessness – lines frequently do not meet one another neatly at a point and baselines are not always parallel to the bottom of the page or to the line of the text but appear to be tilted a bit. Some lines and a few diagrams appear to be merely sketched, although the majority seem to be carefully drawn. In the second part of the manuscript, labels are frequently placed directly on the diagram points rather than beside them as is typical in the first portion. Specific numerical examples have been inserted into some diagrams in the arithmetical books (books VII-IX). Whether they are introduced by a later reader or by the copyist is impossible to decide based on our current evidence.

IV Authorship

The identity of the author of this edition of the *Taḥrīr* and its addendum, Book XVI, is unknown. No name is mentioned in the commentary notes on al-Ṭūsī's introduction, where one might expect to find the author or source identified, nor is an author or editor named in the colophon. On a flyleaf at the end of the codex, however, there are three notes, in a hand very similar to that of the second part of the treatise itself. The second of these notes states: "The end of the communications of the Mulla <and> Qāḍī, the erudite sun of the religious community and the faith, Muḥammad al-Khafrī."[8]

The fact that the note is on the flyleaf, following the page containing the colophon, already suggests that the statement may be an addition by some later reader. The orientation of the note, not parallel to the bottom of the page but placed at an angle across the page, also suggests that it was not intended to be seen as part of the treatise itself. Still, the reference to al-Khafrī is tantalizing since al-Khafrī was a competent student of mathematical astronomy and cosmography. And the date of copying places the manuscript at about the time when al-Khafrī was active. An additional hint that this treatise might represent the work of al-Khafrī comes from a remark of Saliba [1994, 19] that in al-Khafrī's *Takmila* (literally, completion, but in reality a commentary) to the *Tadhkira* of al-Ṭūsī, he followed the text of al-Ṭūsī word for word until he felt he had something of his own that he wanted to insert. It seems reasonable to assume he might well employ the same technique when dealing with other mathematical texts such as al-Ṭūsī's redaction of Euclid. In light of the report by Saliba, then, it is possible that this treatise may be a commentary of al-Khafrī. If this hypothesis should be correct, this manuscript is the only copy of the commentary that I have been able to identify. Other copies, of course, may exist, perhaps erroneously cataloged as copies of the *Taḥrīr*. These considerations are intriguing, but the evidence is very limited and only circumstantial.

Relatively little is known of the life of al-Khafrī. Even the correct form of his name is a matter of debate. To Western historians, he is known almost exclusively for his work in mathematical astronomy and cosmography.[9] We do know that a small mathematical tract in Arabic on a problem in al-Ṭūsī's *Taḥrīr* has been explicitly attributed to him.[10] But so far as I can determine, none of the standard bio-bibliographical sources credit him with a major commentary or edition of Euclid. I conclude that, without further information, the question of authorship must remain undecided for the present.

[8] *Riyāḍī* 496, p. 456: Ākhar afādāt al-mūllā al-qāḍī al-mutabaḥḥir shams al-milla wa'l-dīn Muḥammad al-Khafrī. The first to call attention to this possible connection to al-Khafrī was Brentjes [1998, 73].

[9] The most complete modern study is Saliba [1994]. See footnote 6 (pp. 36-37 of Saliba's study) for an introduction to the biographical literature discussing al-Khafrī. I have not been able to see the non-Western sources cited in this note.

[10] This work, "Fā'idah al-Fawā'id" (Tehran, Majlis 1805), is mentioned in Sezgin [1974, 113].

V Book XVI: Physical Description

Book XVI follows the edited Books I-XV of al-Ṭūsī's *Taḥrīr*, but is separated from it by a blank page (number 442), on which is written a Persian notice in another hand, placed upside down relative to the main body of the text. This page has been damaged and the statement is now incomplete. From what remains, it seems to have no relation to the appended material that follows it. Book XVI appears to have been copied in the same hand as the previous portion of the treatise. It consists of 19 propositions. Only the first four and the last are explicitly numbered using the typical Arabic alpha-numeric *abjad* numbers.[11] Each of these propositions is also indicated by an over-lining of the first words of the enunciation. The remainder are identifiable only by a slightly larger space between the last words of the previous proposition (easy to recognize because each proposition typically ends with the stereotypical phrase *wa-dhālik mā aradnā an nubayyan* – "that is what we wanted to show") and the first word of the enunciation of the new proposition. Several propositions are followed by short generalizing notes or porisms.

The diagrams that accompany each of its nineteen propositions, too, appear to be executed by the same draftsman as those of the second part of the manuscript. They also exhibit the same style of label orthography and placement of letters within the diagrams. They are placed in the margins at the outer edges of the text column and typically extend somewhat into the text column. There seems to be a preference for placing the diagrams at the bottom of the column when this is practical. Otherwise, they come at the end of the propositions. Several of the diagrams have been considerably damaged by crumbling of the margin and two have been almost completely destroyed. In most cases, however, the general features of the diagram are clear or can be reconstructed from the symmetry of the figure. The labels of geometrical elements within the diagrams follow the typical *abjad* sequence, but the copyist is not consistent in how they are assigned within the diagrams. Sometimes the order of labels follows a clock-wise pattern and sometimes anti-clockwise. Many points in the diagrams are not labeled since the author identifies only those points that are actually used in the demonstration. These diagrams appear not to have been checked carefully against the verbal statements of the text because in at least one case the actual construction of the diagram is inconsistent with the geometric relations described in the proposition.

[11] The ninth proposition is given its *abjad* number in the margin, rather than immediately preceding the first words of the enunciation. It is the only proposition labeled in this way.

VI Book XVI: Mathematical Overview

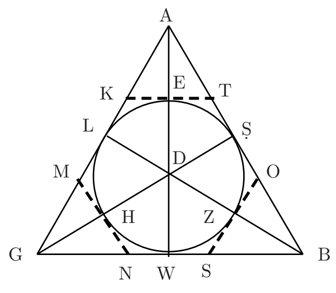
These nineteen propositions deal with the construction of several semiregular polyhedra either within regular polyhedra or within spheres. The discussion can be seen as a kind of extension of book XV of the Arabic transmission of Euclid's text. Book XV gives instruction for inscribing one regular polyhedron within another. Book XVI now turns to less regular polyhedra and shows how they may be inscribed in regular polyhedra.

Proposition I demonstrates a technique that will be applied to construct the truncated versions of each of the regular polyhedra:

> I. To construct an equilateral and equiangular hexagon in an equilateral and equiangular triangle.

To do so, (1) one finds the center of the triangle, (2) extends perpendiculars from that center to the sides of the triangle, (3) connects the center to each vertex of the triangle, (4) marks off on each of these center-vertex lines a distance equal to the perpendiculars, and (5) erects perpendiculars at these points extending to meet the sides of the triangle. An analogous procedure will permit us to create in any equilateral and equiangular polygon another equilateral and equiangular polygon with twice as many sides as the original polygon.

The author could have used an alternative technique, beginning by inscribing a circle within given equilateral triangle $\overline{ABG}$ (*Elements* IV, 4) and drawing tangent lines $\overline{KET}, \overline{OZS}, \overline{MHN}$ (indicated in the drawing by dashed lines) perpendicular to the angle bisector at the point where each angle bisector crosses the circle and extending in either direction to meet the sides of the triangle.

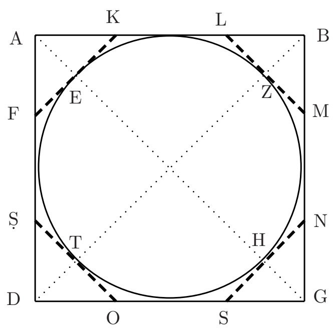


Then the small triangles $\overline{AKT}, \overline{BSO}, \overline{GMN}$ will each be similar to the original triangle, and so will also be equilateral. The side of each of these smaller triangles will be a third of the side of the original triangle. It is then clear that hexagon $\overline{KMNSOT}$ is also equilateral and equiangular. A similar procedure can be used to generate an equilateral and equiangular polygon with twice the number of sides within any equilateral and equiangular polygon.[12] It is not clear why the author chose his technique rather than this procedure.

The author uses his technique in the next propositions in order to construct a semiregular truncated polyhedron within the Platonic solid of the same kind.

> II. To construct in a regular tetrahedron (pyramid) a truncated tetrahedron having eight equilateral faces, four hexagons and four triangles.
> III. To construct in an octahedron a truncated octahedron with fourteen equilateral faces, eight hexagons and six squares.
> IV. To construct in an icosahedron a truncated icosahedron with thirty-two equilateral faces, twenty hexagons and twelve pentagons.
> V. To construct in a cube a truncated cube having fourteen equilateral faces, six octagons and eight triangles.
> VI. To construct in a dodecahedron a truncated dodecahedron with thirty-two equilateral faces, twelve decagons and twenty triangles.

He begins with the three Platonic polyhedra having triangular faces – tetrahedron, octahedron, icosahedron – creating a regular hexagon in each face and forming, through the removal of each vertex, a new triangular face similar to the original face but whose side is one third that of the face. Thus the edges of the newly formed face are one third that of the original polyhedron. He then extends the technique to the remaining two Platonic solids that do not have triangular faces. He first forms an equilateral and equiangular octagon in each square face of the cube. This octagon within the square, too, may be readily constructed using tangents (shown in the diagram below as dashed lines) to a circle inscribed in the square (*Elements* IV, 8).



In this proposition, however, the side of the octagon is not one third of the edge of the regular cube. If we assign a length 1 to each of $\overline{AF}$ and $\overline{AK}$, the sides bounding right angle $\overline{AKF}$ in triangle $\overline{AKF}$, clearly hypotenuse $\overline{FK}$ (a side of the equilateral and equiangular octagon being constructed) will be $\sqrt{2}\ \overline{AF}$ (*Elements* I, 47). A

[12]I thank one of my anonymous reviewers for pointing out this relationship.

similar construction, inscribing a circle within a pentagon (*Elements* IV, 13) and erecting tangents at the points where the circle intersects the bisectors of the angles of the pentagon, could be used to produce a decagon in each pentagonal face of the dodecagon.

Propositions VII with an explanation of a different technique for dividing the regular polygons forming the faces of the octahedron.

> VII. To construct in an octahedron a cuboctahedron with fourteen equilateral faces, eight triangles and six squares.

One bisects each side of the polygonal face and joins each midpoint to the midpoints of the adjacent sides. In this way, a polygon similar to the original polygon but with a side just one half that of the original polygon is formed. To create the new semiregular polyhedron, we need to join the vertices of these smaller polygons.

The next propositions extend this technique to the other Platonic solids.

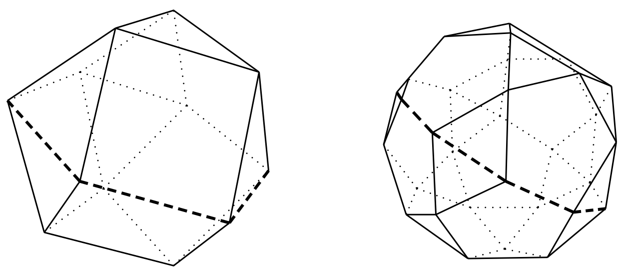
> VIII. To construct in a cube the same cuboctahedron.
>
> COROLLARY: The technique may be used to construct semiregular polyhedra in each of the remaining regular polyhedra.
>
> IX. The side (edge) of this cuboctahedron is the side of the hexagon occurring in the great circle of the sphere in which this cuboctahedron occurs.
>
> X. To construct in a dodecahedron an icosidodecahedron with thirty-two equilateral faces, twenty triangles and twelve pentagons.
>
> XI. To construct in an icosahedron the same icosidodecahedron.
>
> COROLLARY: The side (edge) of the icosidodecahedron is the side of the decagon occurring in the great circle of the sphere in which this icosidodecahedron occurs.

In this new series of polyhedra, the author omits the tetrahedron. He does not state a reason for this omission, but we might speculate that it lies in the fact that the new polyhedron produced from the tetrahedron in this series is not semiregular but rather the regular octahedron. The inscription of a regular octahedron in a tetrahedron was described in Book XV (proposition 2) appended to the *Elements* already in late antiquity. Thus it is perhaps not surprising that it should be omitted here.

The cube and octahedron are "dual" polyhedra, as are the dodecahedron and the icosahedron. In each "dual", the number of vertices of the first is equal to the number of faces of the second, and vice versa. Thus, when we apply the new technique to the cube and octahedron in propositions VII and VIII respectively, we form in each case a semi-regular cuboctahedron made up of eight triangular and six square faces. Similarly, when we apply the technique to the dodecahedron and icosahedron in propositions X and XI respectively, we form in each case an icosidodecahedron having twenty triangular faces and twelve pentagonal faces.

Proposition IX gives the value of the side of the cuboctahedron formed: it is the side of a regular hexagon inscribed in the great circle of a sphere that circumscribes the two polyhedra. This is equivalent to finding the equatorial polygon, the polygon formed by the edges of the polyhedron when cut by the plane of a great circle of the circumscribing sphere.[13] A porism at the end of proposition XI states that the side (edge) of the icosidodecahedron is the side of a decagon inscribed in the great circle of a sphere that circumscribes the two polyhedra. In other words, the equatorial polygon of the icosidodecahedron is a decagon. The demonstration, if we wished to follow it out, would parallel the argument given in proposition IX.

Another porism to proposition XI notes that if we cut the cuboctahedron and the icosidodecahedron along their equatorial planes and then rotate one half of the polyhedron relative to the other through a sixth or a tenth of a circle respectively, we form two new semi-regular polyhedra now known as the triangular orthobicupola (Johnson solid J_{27}) and the pentagonal orthobirotunda (Johnson solid J_{34}) respectively.[14] Another way to describe these polyhedra is that instead of having a triangle opposite a square or a pentagon respectively along the equatorial plane, we now have triangle opposite triangle, square opposite square, and pentagon opposite pentagon. These two polyhedra are illustrated below, the triangular orthobicupola on the left and the pentagonal orthobirotunda on the right. In each, the section of the equatorial plane in the visible part of the polyhedron is indicated with a dashed line. Edges not visible to the observer are indicated by dotted lines.



This note seems to be the earliest recognition of these polyhedra as independent entities.

The author then shows how each of the truncated polyhedra formed in this series may be inscribed in a sphere.

> XII. To construct a truncated tetrahedron, half of whose faces are hexagons and half are triangles, in a given sphere.

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[13]The various features of these equatorial polygons are explored by Coxeter [1973, 18-19].

[14]Diagrams based on illustrations from http://en.wikipedia.org/wiki/Triangular_orthobicupola and http://en.wikipedia.org/wiki/Pentagonal_orthobirotunda.

The procedure is worked out in detail for the truncated tetrahedron. One begins by inscribing the regular tetrahedron in a sphere. This inscription procedure is assumed in Book XVI, but a technique was given already in *Elements* XIII, 13-17 and an alternative procedure using analysis and synthesis was developed in late antiquity by Pappus, *Collectionis*, problems 54-58.[15] With the regular polyhedron inscribed in the sphere, we construct within it the truncated tetrahedron as in proposition II. After connecting each vertex of the truncated tetrahedron with the center of the circumscribing sphere, we extend these lines until they reach the surface of the sphere itself. We connect these points on the sphere to produce the edges of a new and somewhat larger truncated tetrahedron that is now circumscribed by the given sphere. The author notes that we may do the same for any of the other semi-regular polyhedra constructed in the previous propositions.

Next come several rather elementary lemmas leading up to the construction of two new classes of semiregular polyhedra in propositions XVIII and XIX.

> XIII. Given two intersecting planes, there being extended in each plane a perpendicular falling on a single point on their common section, if two other perpendiculars in those planes fall on another point in the common section, the angles between the second pair of perpendiculars will be equal to the angles between the first pair.
>
> XIV. If two lines be perpendiculars falling on another line and the three be in a single plane, and there be extended from the meeting of the two perpendiculars and the line two other perpendiculars to that line in the elevation of the plane such that they bound, together with each of the first perpendiculars, two angles equal to one another, then they (the second pair of perpendiculars) are in the same plane.
>
> XV. The ratio of the diameter of the circle to the side of the equilateral polygon occurring in it is as the ratio of the diameter of any circle to the side of a similar polygon occurring in that circle.
>
> XVI. If there be two lines according to the ratio of the diameter of a circle and the side of the equilateral polygon occurring in it, the shorter line is the side of that polygon occurring within the circle of which the longer line is the diameter.
>
> XVII. To construct in a circle a quadrilateral similar to a rectangular parallelogram.

The author concludes his discussion with the construction of two new semiregular polyhedra which do not depend on the regular Archimedean solids.

> XVIII. To construct in a sphere a semiregular polyhedron having equilateral faces, two of which are a specified polygon occurring within parallel and equal circles and the remainder are squares.

[15] For the Greek text of Pappus see Hultsch [1876-1878, III, 142-162]. For a summary of the technique used by Pappus, see the notes to *Elements* XIII, 13-17 in Heath [1956].

XIX. To construct in a sphere a semiregular polyhedron having equilateral faces, two of which are a specified polygon occurring within parallel and equal circles and the remainder are equilateral triangles.

What is being constructed in proposition XVIII is a polyhedral prism, inscribed in a sphere. To accomplish this task, we inscribe in two equal and parallel circles cutting a sphere, one on either side of a great circle, the same regular polygon, positioned so that the vertices of one are directly opposite the vertices of the other. Then we connect each vertex with its corresponding vertex to form a series of squares. The result is a semiregular prism. The specific example worked out in this proposition is a decagonal prism, but a porism at the end of the proposition notes that there are an infinite number of such polyhedral prisms that can be formed using this technique.

In proposition XIX we form a polyhedral antiprism in a sphere. The procedure is in many ways analogous to that of the previous proposition. We begin with two equal and parallel circles cutting the sphere, one on either side of a great circle, and inscribe the same polygon in each of the two circles. The essential difference between the prism and the antiprism is that in the latter we align the vertices of one polygon so that each is directly opposite the midpoint of the arc subtended by the side of the polygon in the opposite circle. We then connect the vertices together to form equilateral triangles, rather than squares as in the case of the prism. This proposition works out the specific case of a square antiprism, but, like the previous proposition, it is followed by a porism noting that the number of polyhedral antiprisms that can be formed using this technique is infinite.

VII Discussions of Regular and Semiregular Polyhedra Before Book XVII

Regular polyhedra, that is, convex polyhedra each of whose faces is a similar equilateral and equiangular polygon and each of whose angles is congruent, have long been known. Both Heath [1981, I, 158-162] and Vitrac [1990-2001, IV, 95-106] discuss the limited, fragmentary, and ultimately inconclusive literary evidence, mainly a statement of Proclus, that these polyhedra were discovered by the ancient Pythagoreans. Sachs [1917] has argued forcefully that this attribution to the Pythagorean tradition is not credible. Thus the early history of the Platonic solids rests almost completely on the famous scholion to book XIII, where the cube, pyramid and dodecahedron are attributed to Pythagoras but the octahedron and icosahedron are ascribed to Theaetetus. Waterhouse [1972-1973] argues that the seeming improbability of the attribution to someone as late as Theaetetus is understandable if we remember that one cannot recognize the existence of "regular solids" until one has developed an abstract definition that will set these solids apart from all others – a definition that he argues developed within Plato's academy. Thus he concludes that the scholion is likely to represent a valid historical fact. These solids are often called "Platonic" solids, although their connection to Plato is somewhat tenuous, deriving mainly from his association of each of them to one of the four elements (the dodecahedron was used for the universe itself) in *Timaeus*, 55C - 56B [Heath 1981, I, 294-295]. Tradition has sometimes pointed to Theaetetus as the first to develop the geometrical understanding of the regular polyhedra [Heath 1981, I, 212; Vitrac, 1990-2001, IV, 98]. The geometrical construction of these polyhedra, and the proof that there could be only five of them, which is systematically developed in propositions 13-18 of *Elements* XIII, are believed to rest on the theoretical work of Theaetetus.

Semiregular polyhedra, those convex polyhedra bounded by equilateral and equiangular but not similar polygons whose edges are all equal and such that any polygon of one type is completely surrounded by polygons of the second type, also have a long history. Archimedes undertook a systematic discussion of thirteen different types of these polyhedra. This work is not extant, but Pappus included a summary of these results in his *Collectiones*.[16] Pappus has arranged his report of Archimedes results in terms of increasing number of sides of the semi-regular polyhedra. To what extent his treatise matches Archimedes original discussion of these polyhedra we do not know. The work of Pappus does not seem to have been directly known in Arabic, although there remains the possibility that it entered the Arabic tradition indirectly through some secondary source.

Archimedes work is also described very briefly in a note in Heron's *Definitiones*. The Greek text, however, has become corrupt. According to Heron's report, Archimedes credited Plato with knowledge of at least one semi-regular fourteen-sided polyhedron. One was clearly the cuboctahedron, but the other cannot now be identified because the Greek text contains an error in its description of the polyhedron [Heath 1981, I, 295]. Waterhouse [1972-1973, 219-221] employing a careful textual analysis, argues that the basic information included in Heron's report – that Plato knew the "tesserakaidecaedron" (the cuboctahedron) is probably reliable. Why did Plato then not develop other Archimedean solids which are similarly formed by truncating or cutting off the corners of other regular solids? Waterhouse [1972-1973] suggests that this "tesserakaidecaedron" might have been introduced as a "counter-example" to the "Platonic solids". This "counter-example" adds credence to the suggestion that the abstract concept of regular solid figures was developed within the Academy.

There were discussions of regular polyhedra quite early in the Arabic tradition.

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[16] Heath [1981, II, 98-101] provides a brief English summary. The original Greek of Pappus can be found in Hultsch [1876-1878, I, 352,354]. A transcript of Pappus's Greek text, with English translation, can be found in Thomson [1941, II, 195-197]. Both the Greek text and English translation, along with small perspective drawings of the Archimedean solids, are available on the internet at http://www.cs.drexel.edu/ crorres/Archimedes/Solids/Pappus.html.

Thābit ibn Qurra (221 AH/AD 836 – 288 AH/AD 901), for example, is credited with a treatise discussing the comparison of the sides of the five regular polyhedra (*Elements* XIII, 18 in Heiberg's Greek edition and XIII, 21 in the Arabic transmission).[17] He is also credited with a treatise discussing the construction of a fourteen-sided solid – presumably a discussion of the cuboctahedron and its salient features.[18] Thus it would appear that some knowledge of Archimedean solids had infiltrated the Arab world although the precise pathway is still impossible to delineate.

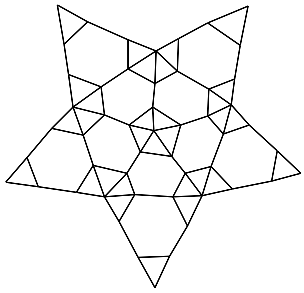
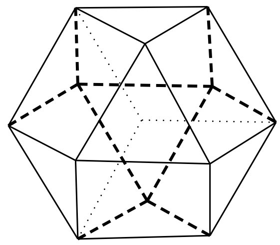
Abū'l-Wafā' al-Buzjānī, an Arab mathematician of the 4th/9th century, has discussed some examples of construction of regular and semiregular polyhedra in the twelfth chapter of his treatise *Kitāb fīmā yaḥtāǧu ilaihi ṣ-ṣāniʿ min aʿmāl al-handasa*.[19] His approach to the problem is quite different from that of Pappus, since he is interested primarily in construction techniques. His formulation is typically in terms of dividing the surface of a sphere into equilateral and equiangular triangles, quadrilaterals, pentagons. Despite this formulation, however, the diagrams represent various semi-regular polyhedral figures. I show two examples below: the truncated icosahedron (left) and the cuboctahedron (right).[20] The dashed lines in the diagrams represent edges not visible to the observer and the dotted lines in the diagram of the cuboctahedron represent lines incorrectly included in the diagram by the copyist.

[17]Sezgin [1974, 271-272] notes the existence of a manuscript of this treatise in Cairo. I have been unable to confirm this information.

[18]The Arabic title is "Maqāla fī ʿamal šakl muǧassam ḏī arba ʿašra qāʿida tuḥīṭu bihī kura maʿlūma." [Bessel-Hagen and Spies 1932] have published the text and a German translation. Aghaniya Chavoshi [2007] has published a French translation. I have not been able to see these works.

[19]There exists at least one Arabic manuscript that may be complete, but the Milan Arabic fragment studied by Suter [1922] is incomplete and does not include the discussion of the polyhedra. Chapter XII of the Persian summary studied by Woepcke [1855] is at present the best source of information available. A Persian translation of the Arabic text appears to exist in manuscript form. Presumably it is this Persian rendition which, with the title *Abu al-Wafâ al-Buzjâni, Ketabé Nejârat: Edition d'une version persane ancienne avec traduction française et commentaires*, is now in final preparation by Aghayani Chavoshi. Professor Aghayani Chavosi has very kindly provided me a pre-print of his French translation of chapter 11 of this treatise. A detailed comparison of these two Persian versions must await a future study. Sezgin [1974, 324] indicates that there was at least one extant commentary composed in Arabic. It's precise relation to these Arabic and Persian versions is yet to be determined.

[20]Diagrams based on BNF, Persan 169, fol. 178a and BNF Persan 169, fol. 177b respectively. A comparison with the same diagrams from Book XVI, propositions IV and VIII respectively shows that the copyist of Persan 169 had considerably better control over perspective drawing.

The twenty one propositions of book XII of the *Aʿmāl al-handasa* of Abū'l-Wafā', as recorded by Woepcke [1855, 352-358], are as follows:[21]

1. To trace out a great circle on a sphere.
2. To trace out two great circles on a sphere at right angles to one another.
3. To trace out three great circles on a sphere such that each cuts the others at right angles.[22]
4. To construct a great circle passing through two specified points.
5. To divide the surface of a sphere into four equilateral and equiangular triangles [equivalent to constructing a circumscribed regular tetrahedron]. We construct three great circles at right angles to one another on the sphere. This forms eight triangles on the surface of the given sphere. One chooses any one of these triangles. Then there are three triangles that lie opposite the vertices of the chosen triangle. Joining the center points of these four triangles along great circles, one divides the surface of the given sphere into the desired four triangles.[23]
6. An alternate procedure for dividing a sphere into four equilateral and equiangular triangles. Set out line $\overline{AB}$ equal to the diameter of the sphere. Mark off $\overline{AC}$ equal to 1/3 $\overline{AB}$. Erect at $\overline{C}$ a perpendicular meeting at $\overline{D}$ the semicircle drawn on $\overline{AB}$ as diameter. Placing one foot of a compass at a chosen point (as pole) of the sphere and with the compass opening equal to distance $\overline{DB}$, draw a circle. Divide the smaller circle into three equal parts. These three points plus the pole point are the four points needed to divide the surface of

[21] Woepcke did not translate the entire treatise, but only the enunciations of the propositions. When there are alternative constructions, he supplies a summary of each technique. His study also does not include the diagrams that accompany these propositions.

[22] The octahedron is not specifically mentioned by Abū'l-Wafā' in this chapter. Woepcke [1855, 352 note 1] notes that the polyhedron is implicit in this construction, since the intersections of great circles in this proposition define the vertices of the octahedron.

[23] It is clear that if we now draw planes through each set of three points, we will construct a regular tetrahedron inscribed in the sphere. Abū-l-Wafā' does not explicitly take this step, however.

the sphere into four equal parts.[24]

7. To divide the surface of a sphere into six equilateral and equiangular quadrilaterals [equivalent to constructing a circumscribed cube]. We join the centers of the eight triangles formed in proposition 5 along great circle arcs.
8. An alternate procedure for dividing a sphere into six equilateral and equiangular quadrilaterals.[25]
9. To divide the surface of a sphere into twenty equilateral and equiangular triangles [equivalent to constructing a circumscribed regular icosahedron]. We trace a great circle on the sphere and choose two points, $\overline{E}, \overline{Z}$ to be the poles. We divide the circle into ten equal parts, $\overline{AB}, \overline{BC}, \overline{CD}$ etc. With the compass opening equal to one of these parts, and taking $\overline{A}, \overline{B}$ as centers, we draw two small circles cutting one another on the side toward pole $\overline{E}$ at $\overline{X}$; taking $\overline{B}, \overline{C}$ as centers and drawing two small circles cutting one another on the side toward pole $\overline{Z}$ at $\overline{K}$. We do the same process going all around the great circle. When we finish, we will have five points $\overline{X}$ and five points $\overline{K}$. These, together with the two poles will be twelve points which, when joined along the arcs of great circles, will divide the sphere into twenty triangles.
10. An alternative procedure for dividing a sphere into twenty equilateral and equiangular triangles.[26]
11. To divide the surface of a sphere into twelve equilateral and equiangular pentagons [equivalent to constructing a circumscribed regular dodecahedron]. The procedure is to divide the surface of the sphere into twenty triangles as in proposition 9. We join the centers of these triangles along arcs of great circles and in so doing we divide the surface as required.
12. An alternative procedure for dividing a sphere into twelve equilateral and equiangular pentagons. We set out line $\overline{AB}$, the diameter of the sphere, and divide $\overline{AB}$ into three equal parts, $\overline{AC}, \overline{CD}, \overline{DB}$. At $\overline{B}$ we erect a perpendicular meeting at $\overline{E}$ the semicircle drawn around $\overline{D}$ as center with radius $\overline{DA}$. We extend $\overline{AB}$ in the direction of $\overline{B}$ to $\overline{H}$ such that $\overline{BH}$ is half of $\overline{BE}$. We mark off from $\overline{HA}$ the distance $\overline{HT}$ equal to $\overline{HE}$. Then $\overline{BT}$ will be the chord corresponding to the side of the spherical pentagon needed to divide the surface of the sphere. Now we take point $\overline{I}$ at random on the surface of the sphere. We describe about it a small circle with the opening of the compass set equal to $\overline{BT}$ and divide this circle into three equal parts at $\overline{K}, \overline{L}, \overline{M}$. With these points as centers and with the same compass opening, we describe circles which we also divide into three equal parts in the same way. We choose in each circle a point $\overline{I}$ for one of the three points of division. In the end, we obtain twenty points $\overline{I}$

[24]The procedure outlined is essentially that used in *Elements* XIII, 13.

[25]The procedure is essentially that used by Euclid in *Elements* XIII, 14.

[26]The procedure here is essentially the same as Euclid's in *Elements* XIII, 16

which, when joined along great circle arcs, divide the sphere as required.

13. An alternative procedure for dividing the surface of a sphere into twenty equilateral and equiangular triangles [equivalent to construcing a circumscribed regular icosihedron]. We presume the construction in proposition 11. We join the midpoints of the pentagons formed there along the arcs of great circles. The result is the desired solution.
14. To divide the surface of a sphere into fourteen parts, of which six are quadrilaterals and eight are triangles [equivalent to constructing a circumscribed cuboctahedron]. We construct three great circles on the sphere, each at right angles to the other, thus forming eight triangles. We take the midpoints of the sides of these triangles and join them along the arcs of great circles. We obtain then eight triangles placed respectively in the middle of each original triangle and six quadrilaterals at the points of intersection of the great circles, the vertices of the eight original triangles.
15. An alternative procedure for dividing the surface of a sphere into fourteen parts, six quadrilaterals and eight triangles . We trace on the sphere six quadrilaterals as in proposition 7. We take the midpoints of the sides and join these points along arcs of a great circle. We obtains then six quadrilaterals situated in the middle of the original quadrilateral faces and eight triangles which are situated at the vertices of the original quadrilaterals.
16. To trace on a sphere twelve pentagons and twenty triangles [equivalent to constructing a circumscribed icosidodecahedron]. We divide the sphere into twenty triangles as in proposition 9.[27] We take the midpoints of their sides and join these points along the arcs of great circles.
17. To trace on a sphere twelve pentagons and twenty hexagons [equivalent to constructing a circumscribed truncated icosahedron]. We divide the sphere into twenty triangles as in proposition 9 and divide the sides of each triangle into thirds. We connect these divisions along arcs of great circles such that in each original triangle there is a hexagon and at the vertex of each original triangle there are placed five small triangles forming a pentagon.
18. An alternate procedure for dividing the surface of a sphere into twelve equilateral and equiangular pentagons [equivalent to constructing a circumscribed regular dodecahedron]. We divide the sphere into twelve pentagons and twenty triangles as in proposition 16. Then we join the centers of the triangles along arcs of great circles.[28]
19. An alternate procedure for tracing on a sphere twelve pentagons and twenty hexagons [equivalent to constructing a circumscribed truncated dodecahedron].

[27] We begin with an icosahedron.

[28] In this construction, we begin with an icosidodecahedron. This technique for creating a regular dodecahedron is not mentioned in Book XVI.

We divide the sphere into twelve pentagons as in proposition 11. Then we join the midpoints of their sides along the arcs of great circles.

20. To divide a sphere into six quadrilaterals and eight hexagons [equivalent to constructing a circumscribed truncated octahedron]. We divide the sphere into eight triangles by intersecting three great circles at right angles to one another. We divide the sides of the each triangle into three equal parts and join these divisions along the arcs of great circles. We obtain eight hexagons in each of the original triangles and six quadrilaterals situated at the vertices of the triangles or the points of intersection of the three great circles.
21. To divide the surface of a sphere into four triangles and four hexagons [equivalent to constructing a circumscribed truncated tetrahedron]. We divide the sphere into four triangles as in proposition 5 and divide the sides of each triangle into three equal portions. We connect the points of division along the arcs of great circles. We obtain in the center of each original triangle a hexagon and four triangles situated at the vertices of the original triangles.

The careful reader will notice immediately a number of important points of difference between the discussion of Abūʾ-l-Wafāʾ and that presented in Book XVI. One of the most obvious has already been mentioned – the fact that Abūʾ-l-Wafāʾ is constructing his figures by dividing the surface of a sphere. Further comparison will show that the two treatises discuss the truncated polyhedra in different order. Abūʾ-l-Wafāʾ has arranged them in order of increasing number of faces, while Book XVI has placed them so that we deal with triangular faces (tetrahedron, octahedron, icosahedron), square faces (cube) and pentagonal faces (dodecahedron). Furthermore, Book XVI carries the discussion further than does Abūʾ-l-Wafāʾ, mentioning both the triangular and pentagonal orthobirotunda and giving the construction of the semiregular prism and antiprism in full. Another remarkable difference is that Abūʾ-l-Wafāʾ describes the derivation of the regular dodecahedron from an icosidodecahedron. This procedure is not found in Book XVI. These many differences need not mean that the two works were completely independent of one another, of course. The author of Book XVI may well have known the earlier work of Abūʾ-l-Wafāʾ. But the differences do suggest that if the author of Book XVI knew and used the work of Abūʾ-l-Wafāʾ, he was not just repeating but was reworking and extending the earlier discussion.[29]

[29] Because of the similarity of subject matter between these two treatises, one of my referees questioned whether Abūʾ-l-Wafāʾ might be the author of Book XVI. This possibility seems unlikely to me, given the many differences in style and approach between the two treatises. A more definitive answer must await further study of the original treatise of Abūʾ-l-Wafāʾ.

VIII Later Discussions of Archimedean Polyhedra

Book XVI is certainly not the only attempt to add to or extend the *Elements* during the medieval period. There are at least two other important medieval extensions of the discussion of regular polyhedra. One is the extensive re-writing of book XV in the Arabic redaction of the *Elements* by Muḥyī al-Dīn al-Maghribī (d. 682 AH / AD 1283). It includes theorems comparing the sides, faces, surface areas, and volumes of the five regular polyhedra.[30] The Latin version of Euclid by Campanus of Novara (mid-13th century) was probably the most important edition of Euclid circulating in manuscript form. It incorporates additional propositions in both book XIV and book XV in which he works out additional relationships among the regular polyhedra [Busard 2005, 607-610]. Although Campanus frequently borrowed material from other mathematicians,[Busard 2005, 32-38] these added propositions have not yet been traced to earlier sources and so may represent his own emendation of the text.

The influential Latin edition of the *Elements* by Christoph Clavius (1538-1612) also included a section labeled "Book XVI", which Clavius borrowed, with attribution, from Francisco Flussate Candalla (1502?-1594?). It is entitled "In which it is explained how various of the regular solids may be mutually inscribed and a comparison of their sides."[31] This "Book XVI" occupies pages 610-637 of Clavius's 1612 edition of his commentary on Euclid.

Candalla's Book XVI was divided into three chapters:

1. Relations of the sides and diameters of regular polyhedra to one another and comparison of the volumes of regular polyhedra
2. Inscribing each of the five regular polyhedra within a sphere, using the techniques of Pappus
3. On the comparison of the five regular polyhedra in terms of volumes and angles

Clavius's treatment of polyhedra is not identical with the Arabic material developed by al-Maghribī or with the Arabic Book XVI presented here. The section describing the techniques of Pappus for inscribing each of the regular polyhedra in a sphere certainly derives in some measure from the Archimedean tradition. But the development of semi-regular solids is outside the scope of this Latin Book XVI. Thus, although they share the same title, the two works are quite different and

[30] This discussion of the regular polyhedra has been edited and translated by Hogendijk [1993].

[31] "Elementum sextumdecimum. Quo variæ solidorum regularium sibi mutuo inscriptorum, & laterum eorundem comparationes explicantur, à Francisco Flussate Candalla adiectum, & de quinque corporibus." The author is François de Foix, Compte de Candalle. His "Book XVI" was appended to his edition of the *Elements* (Paris: Jean Le Royer, 1566), which was re-edited in 1578 and 1602. [Heath 1956, I, 104].

appear unrelated to one another except for the fact that each uses some portion of the Archimedean discussion of polyhedra. Each of these treatises, in extending the propositions of the Book XV traditionally appended to Euclid's text, produced different propositions and used different approaches to the problems they set out to address. Although independent of one other, when taken together, they show something of the diversity and fecundity of the Euclidean tradition during the medieval period.

Clavius's work was especially important because of its wide influence. The first full English translation of the *Elements*, by Henry Billingsley (London, 1570), was based on Clavius [Archibald 1950]. Although I have not been able to see a copy of this translation, it appears that, at least initially, it contained also Clavius's appended Book XVI [Xu 2005, 16]. Clavius's Book XVI was also incorporated into the English translation of Isaac Barrow (London, 1660), beginning with the 1714 edition [Xu 2005, 11]. Although the Billingsley translation served as the basis for the first Chinese translation of books VII-XV, it appears that the appendix on the Archimedean solids was not rendered into Chinese by Alexander Wylie and Li Shanlan. Whether this omission was because the appended material was removed from later editions of Billingsley or whether Wylie and Li Shanlan simply decided not to translate it is not yet clear to me. By the mid-nineteenth century, presumably, anyone with some experience in the Western mathematical tradition would have realized that Candalla's work was both limited and outdated.

By the time the appendix of Candalla was incorporated into Barrow's edition of the *Elements*, it had already been superseded by the systematic study of regular and semiregular solids carried out by Johannes Kepler. As a young man, Kepler had happened, by accident, to notice that the circles representing the orbits of Saturn and Jupiter were related to one another through a quasi-triangular construction. This observation set him on a Pythagorean quest for relations between the orbital circles of the other planets and the plane geometric figures. Eventually he derived a cosmological system which one of the five Platonic solids was placed between each pair of the spheres of the six Copernican planets. His efforts led to publication of *Mysterium Cosmographicum* (1596), including one of the most evocative images of Renaissance cosmography showing the universe as a great goblet within which the planetary spheres are interspersed with and separated by the five Platonic solids [Koyré 1973, 146]. In his last major publication, *Harmonice Mundi*, Kepler [1619] developed a systematic study of convex polyhedra and extended the study considerably beyond the limits known up to that time. It is his nomenclature and classification system that continues to be used today in many discussions of polyhedra.

Conclusion

On one level, Book XVI offers an insight into one of the less studied currents of mathematical scholarship in the early medieval Islamic tradition. In it we see what seems to be the earliest recognition of some classes of Archimedean solids such as the bicupola, the birotunda, and the antiprisms. On another level, this study of the anonymous Book XVI leaves, perhaps, more questions than answers. We do not know the name of the author of the Book XVI material, nor can we be sure of the identity of the editor who added it to the *Taḥrīr* with the intention of somehow completing Euclid's treatise. While it seems in some ways to complement parts of Euclid's discussion, it has also several important disjunctions as well. In fact, the addition of Book XVI often seems more like a kind of intellectual xenograft, joining a discussion of Archimedean solids with a Euclidean treatise. How the editor might have resolved these tensions within his own intellectual tradition is still a mystery. At the very least, the existence of Book XVI reveals another facet of the remarkable fecundity of the mathematical tradition of medieval Islam.

Appendix: Translation and Commentary

The Arabic text is unique, so far as we know. Thus there is no possibility of establishing a textual history or stemma. My editing has been limited, then, to reconstructions of portions of the text damaged by crumbling margins and providing tentative correction of a few copyist errors. I have expanded the abbreviations used by the copyist and standardized some of the spelling. The Arabic text, as is usually the case, contains no punctuation nor paragraphing. All punctuation and paragraphing in the edition are my own, introduced in order to clarify the flow of ideas within the text. The punctuation and paragraphing used in the English translation generally parallels that of the Arabic edition.

When translating, I have endeavored to follow as literally as possible to the Arabic text. Sometimes this policy would lead to a translation that is too concise to be clear. In these cases, I have added words or phrases to the translation in order to produce a meaningful statement in English. These words are enclosed in pointed brackets <>. Sometimes the medieval Arabic technical terminology does not correspond exactly to current mathematical usage. I have then adopted the more modern terminology in my translation, and explain the original meaning of the Arabic terms in the commentary notes to the propositions. Occasionally, I have also found it necessary to add to the translation an explanatory term to clarify those places where the Arabic grammar, because it differs from English, leaves the translation vague or unclear. These explanations are enclosed in parentheses ( ). The text has been damaged at several points. In most cases, it is only a few letters at the end of the line that are missing and the words can be reconstructed with little doubt. I have indicated my reconstructions by enclosing the words in square brackets [ ] without distinguishing whether all or only part of the word is missing.

The geometrical data contained within the diagrams that accompany the text has been extracted and preserved using DRaFT, a software tool developed under the leadership of Professor Ken Saito (Osaka Prefecture University).[32] The software allows us to capture the salient geometrical data from the original diagram and use it to create a permanent record. Using this data, we are able to reconstitute the essential features of the original diagrams for each proposition (although sometimes on a smaller scale). A number of the manuscript diagrams have been damaged. I indicate the ragged edge of the damage using dotted lines. In some cases, it is feasible to reconstruct parts of the diagram based on symmetry arguments and the surviving portions of the diagram. I indicate such reconstructed portions using a dashed line. Some cannot be reconstructed based on the surviving portions of the diagram and so are left incomplete. In the manuscript, the diagrams typically come near the end of the proposition. I have opted for the more modern placement after the enunciation of the problem or proposition. These medieval diagrams are often difficult for readers unaccustomed to a lack of perspective to visualize readily. Therefore, in the notes to each proposition I have redrawn its diagram using more modern perspective techniques.

Each proposition is followed by commentary and notes. General comments discussing the proposition as a whole are placed first. Specific comments referring to individual statements within the translated text are referenced within the text by a numeral in parentheses. This numeral corresponds to the numbered note at the end of the proposition.

[32]The software and accompanying tools are available gratis. They may be downloaded from Professor Saito's website (http://www.greekmath.org/diagram/). For additional description of the software and its potential application, see Saito [2006, 92-94] and De Young [2008b].

المقالة السادسة عشر
وهى التى الحقها بلكتاب
وقد استخرجها بقريحتى وقدم أنها تسعة عشر شكلًا

ا

نريد أن نرسم مسدسًا متساوى الأضلاع والزوايا في مثلث كذلك.

مثلًا في مثلث ا ب ج .

فلنجد مركز المثلث وهو د ونخرج منه أعمدة على أضلاع المثلث. فيكون متساوية.

وننصف الأضلاع المثلث ونصل خطوط ا د ، ب د ، ج د ونفصل د ه ، د ز ، د ح مساوية للأعمدة ونخرج من نقط ه ، ز ، ح أعمدة على الخطوط [التي] نخرجها من الجانبتين إلى أضلاع المثلث.

ولا يمكن أن يقع نقطة ط [فوق] نقطة ص ولا أن يقع تحتها لأنّ ص د ، د ه متساويان وزاوية ه قائمة. فيحصل محيط مسدس ط ك م ن س ع ولنصل خط ط د .

فنقول (إن) زاويتا ا ب د ، ب ا د متساويتان لتساوي ساقي د ب ، د ا .

وزاويتا ز ، ه في مثلثي ب ز ع ، ط ه ا قائمتان وخطا ب ز ، ه ا متساويان. فخطا ب ع، ط ا متساويان. فيبقى ع ص ، ص ط متساويان.

ولكون زاويتي ص ، ه في مثلثي ط ص د ، ط ه د قائمتين وضلعي ص د ، د ه متساويين، فإذا اسقطنا مربعي ص د ، د ه من مربع ط د بقي مربعا ص ط ، ط ه متساويين. فخطا ص ط ، ط ه متساويتان.

لتساوي أضلاع مثلثي ا ب د ، ا د ج يكون زاويتا ا متساويتيين زاويتا ه في مثلثي ا ط ه، ا ه ك قائمتان وضلع ا ه مشارك فضلعا ط ه ، ه ك مساويان.

وقد ثبت أن ع ص ، ص ط متساويان و ص ط ، ط ه متساويان. فمجموع ع ط ، ط ك متساويان.

وبمثل ذلك تبيّن أن جميع أضلاع المسدس متساوية.

ولأنّ زوايا ب ع ز ، ا ط ه ، ا ك ه في مثلث ب ع ز ، ا ط ه ، ا ك ه متساوية كان زوايا س ع ط ، ع ط ك ، ط ك م متساوية.

وبمثل ذلك تبيّن أن جميع زوايا المسدس متساوية.

وذلك ما أردناه.
ويمكن بمثل هذا العمل نرسم الثمن في المربع والمعشر في المخمس.

ب

نريد أن نرسم في مخروط متساوي أضلاع القواعد مجسمًا ذا ثماني قواعد مساويات الأضلاع أربع مسدسات وأربع مثلثات ونبيّن [أنّه يقع] في كرة وأن ضلعه ثلث ضلع المخروط.

فليكن المخروط ا ب ج [ونرسم] في كل قاعدة من قواعده مسدسًا متساوي الأضلاع والزوايا. [فتحصل] أربع مسدسات.

ولأن كل زاوية مجسمة من زواياه الأربع [مركبة] من ثلاثة[33] زوايا مسطحة تحصل عند كل زاوية مجسمة مثلث. فتحصل أربع مثلثات.

ولأن زوايا المخروط مماسة للكرة التي يقع المخروط فيها تكون أبعادها عن مركز الكرة متساية.

فإذا وصلنا خطوطا بين زوايا المخروط والمركز يحصل مثلثات متساويات الأضلاع النظائر. فيكون زواياها النظرة متساوية.

وإن وصلنا بين مركز الكرة وزوايا المجسم تحصل مثلثات أضلاعها[34] الخطوط الواصلة بين مركز الكرة وزوايا المخروط والخطوط الواصلة بين المركز وزوايا المجسم.

والخطوط الواصلة بين زوايا المخروط وزوايا المجسم أعني أضلاع المثلثات الباقية من قواعد المخروط بعد رسم المسدسات وهي متساوية فيكون ضلعان وزاوية بينهما في تلك المثلثات متساوية. فالأضلاع الباقية في تلك المثلثات متساوية أعني الخطوط الواصلة بين مركز الكرة [. . .]

. . .] ببعد إحدى الزوايا وادرناه مرّ بجميع الزوايا فالمجسم يقع في كرة.

ولأنّ زاويتين من زوايا المثلثات الباقية من قواعد المخروط بعد رسم المسدس فهما متساوية والزاوية الثالثة زاوية المثلث المتساوي الأضلاع أعني ثلثي قائمة يكون كل منها ثلثي قائمة. فأضلعها متساوية.

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[33] ثلاثة] ثلث

[34] أضلاعها: في الهامش في النسخة

فضلع المسدس أعني قاعدة المرسوم ثلث ضلع المثلث أعني قاعدة المخروط. وذلك ما أردناه.

ج

نريد أن نرسم في ذي ثماني قواعد مجسمًا ذا أربع عشرة قاعدة متساوية الأضلاع ثمانية منها [مسدسات] وستة منها مربعات.

فلكن ذو الثماني قواعد ا ب ج .

ونرسم في [كل] قاعدة من قواعده مسدسًا فيحصل ثماني مسدسات.

ويحصل عند كل زاوية من الزوايا الست مربع لأنا إذا وصلنا بين مركز الكرة وبين كل زاوية من زوايا ذي الثماني قواعد يحصل مثلثات متساويات الزوايا على التناظر.

فإذا أخرجنا أعمدة من زوايا ذي أربعة أضلاع مرّ هذا الخط في داخله على هذا[35] الخط تحصل أربع مثلثات أحد أضلاعها متساوية وهي البقايا من أضلاع قواعد ذي الثماني قواعد.

والزوايا التي عند العمود قائمة والزوايا التي عند زاوية المخروط متساوية. فأضلاع المثلثات متساوية على التناظر. فالأعمدة تقع على نقطة واحدة.

والخط الواصل بين مركز الكرة والزاوية عمود على تلك الخطوط واقع على فصلها المشترك. فالجميع في سطح واحد.

ولأن المثلثات الحادثة من تلك الأعمدة وأضلاع ذي أربعة الأضلاع متساوية الأضلاع متساوية[36] على التناظر. فزواياها متساوية على التناظر.

ولأن كل زاوية من زوايا ذي أربعة الأضلاع مركبة من إثنين منها يكون متساوية. فهي مربعات.

وبمثل ما مرّ يتبيّن أن المجسم يقع في كرة وضلعه ثلث ضلع ذي الثماني قواعد.

وذلك ما أردناه.

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[35] هذا] هذ

[36] متساوية: نقص من النسخة

د

نريد أن نرسم في ذي عشرين قاعدة مجسمًا ذا إثنين وثلثين قاعدة متساوية الأضلاع عشرون منها مسدسات وإثنا عشرة منها[37] مخمسات.

[فلنصف] المرئي من ذي عشرين قاعدة ا ب ج د ه .

ونرسم في [كل قاعدة] من قواعد ذي العترين قاعدة مسدسًا فتحصل عشرون [مسدسات] وتحصل عند كل زاوية من زوايا ذي العشين قاعدة مخمس لأنّ كل زاوية منها مركبة من خمس زوايا[38] مسطحة.

وبمثل ما مرّ يتبيّن أن مخمسًا متساوية الزوايا[39] وأن أضلاعها في سطح واحد وأن المجسم يقع في كرة وأن ضلعه ثلث ضلع ذي العشرين قاعدة.

وذلك ما أردناه.

ه

نريد أن نرسم في مكعب مجسمًا ذا أربع عشرة قاعدة ستة مثمنات وثمانية مثلثات.

فليكن المكعب ا ب ج د .

ونرسم في كل قاعدة من قواعده مثمنًا تحصل ستة مثمنات.

ويحصل عند كل زاوية مجسمة من زواياه الثماني مثلثًا لأن كل واحدة مركبة من ثلث زوايا مسطحة.

وبمثل ما مرّ يتبيّن أن المجسم واقع في الكرة.

وذلك ما أردناه.

[37] منها: نقص من النسخة

[38] زوايا] زوا

[39] الزوايا] لزوايا

و

نريد أن نرسم في ذي إثنى عشرة قاعدة مجسمًا ذا إثنين وثلثين قاعدة متساوية الأضلاع إثنا عشرة منها معشرات وعشرون منها مثلثات.

وليكن ذو إثنى عشرة قاعدة ا ب ج د ه .

ونرسم في كل قاعدة من قواعده معشرًا. فيحصل إثنا عشر معشرًا. ويحصل عند كل زاوية من زواياه العشرين مثلثًا.

وبمثل ما مرّ يتبيّن أنه يقع في كرة.

وذلك ما أردناه.

ز

نريد أن نرسم في ذي ثماني قواعد مجسمًا ذا اربع عشرة قاعدة متساوية الأضلاع ثمانية مثلثات وستة مربعات.

وليكن ذو ثماني قواعد ا ب ج .

فننصف جميع أضلاعه ونصل الخطوط. فيحصل في كل قاعدة ذي الثماني قواعد مثلث متساوي الأضلاع لأن كل ضلع منه نصف ضلع ذي الثماني قواعد.

ويحصل عند كل زاوية من الزوايا الست التي لذي الثماني قواعد ذو أربعة أضلاع.

ويتبيّن بمثل ما مرّ أنها مربعات. فيحصل المجسم.

وبمثل ما مرّ يتبيّن أنه يقع في كرة.

وذلك ما أردناه.

ح

نريد أن نرسم هذا[40] المجسم في مكعب.

[40] هذا] هذ

وليكن المكعب ا ب ج د .

فننصف جميع أضلاه ونصل الخطوط. فيحصل عند كل زاوية مجسم مثلث متساوي الأضلاع لأن كل واحد منها وتر زاوية قائمة وأضلاع تلك القوائم متساويات.

ويحصل في كل واحدة منها تمام زاويتين كل منهما نصف قائمة من قائمتين.

ولما كان في المكعب ثماني زوايا مجسمة وست قواعد يكون المثلثات ثمانية والمربعات ستة.

ويتبيّن بمثل ما مرّ أنه يقع في كرة.

وذلك ما أردناه.

ط

ضلع هذا المجسم ضلع المسدس الواقع في أعظم دائرة الكرة التي تقع هذا المجسم فيها.

وليكن لبيانه ا ب ج د ه أربع قواعد من قواعده.

ونصل ا د ، ب ج .

وإذا وصلنا بين مركز الكرة ولنفرضه م وبين ا ، ب ، ج ، د ، ه يحصل مثلثات م ا ه ، م ب ه، م ج ه ، م د ه متساويات زوايا م . [وإذا] خرجنا منه نقط ا ، ب ، ج ، د أعمدة على خط م ه تجتمع عند نقطة [واحدة]. فذو أربعة أضلاع ا ب ج د في سطح واحد.

وليكن تلك النقطة [و فلأنّ] مثلثي ا ب و ، د ج و متساويتي الزوايا النظائر وكذلك [مثلثي] [ا] د و ، ب ج و تساوي زوايا ذي أربعة أضلاع ا ب ج د فهو متوازي الأضلاع.

فإذا وصلنا بين ا ج مرّ بنقطة و . فخط ه و م في سطح مثلث ا ه ج .

فإذا اخرجنا السطح مرّ بالمركز وكذلك سطح خط ه ج مع الخط الذي يلاقيه يمرّ بالمركز.

والسطحان واحد. وإلا اخرجنا من نقطة على خط ه ج خطين في السطحين إلى المركز فهما احاطا بسطح. هذا خلف.[41] وكذلك السطوح المارة بالأضلاع الباقية من مسدس ا ه ج .

فمسدس ا ه ج واقع في أعظم دائرة الكرّة.

وذلك ما أردناه.

[41] هذا خلف] هف

ي

نريد أن نرسم في ذي إثنى عشرة قاعدة مجسمًا ذا إثنين وثلثين قاعدة مساوية الأضلاع عشرون مثلثات[42] وإثنا عشر مخمسات[43] ونبيّن أنّه يقع في كرة.

وليكن ذا إثنى عشرة قاعدة $\overline{\text{ا ب ج د ه}}$.

فننصف جميع أضلاعه ونصل الخطوط. فيحصل عند كل زاوية مجسمة مثلث وفي كل مخمس مخمس لأن كل زاوية من زواياه تمام زاويتين مجموعهما فضل قائمتين على زاوية المخمس من قائمتين.

ولما كان زوايا ذي إثنى عشرة قاعدة عشرين، فمثلثات هذا[44] المجسم عشرون ومخمساته إثنى عشر.

وبمثل ما مرّ يتبيّن أنّه يقع في كرة.

وذلك ما أردناه.

يا

نريد أن نرسم هذا المجسم في ذي عشرين قاعدة.

وليكن النصف المرئي من ذي عشرين قاعدة $\overline{\text{ا ب ج د ه}}$.

فننصف جميع أضلاعه ونصل الخطوط. فيحصل في كل [قاعدته] مثلث وعند كل زاوية مخمس.

ولما كان زوايا ذي عشرين قاعدة إثنى عشرة فمخمساته إثنا عشر وكان مثلثاته عشرين فالكل كما ذكر.

وبمثل ما مرّ يتبيّن أنه يقع في كرة.

وذلك ما أردناه.

وضلع هذا[45] المجسم ضلع المعشر الواقع في أعظم دائرة الكرة بمثل ما مرّ في الشكل المتقدّم.

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42 مثلثات] مثلثا

43 مخمسات] مخمسا

44 هذا] هذ

45 هذا] هذ

وإذا قطع هذان المجسمان الدائرتين المنصفتين وأطبق أضلاع القواعد على أضلاع القواعد النظائر يحصل مجسمان اخران واقعان في الكرّة.

يب

نريد أن نرسم مجسمًا ذا ثماني قواعد متساويات الأضلاع نصفها مسدسات ونصفها مثلثات في كرة مفروضة.

وليكن الكرة ا ب ج ولنرسم فيها مخروطا ذا أربع قواعد مثلثات متساويات الأضلاع وليكن مثلث ا ب ج أحد قواعده.

ثمّ نرسم في المخروط هذا المجسم. ثمّ نصل بين مركز الكرة وبين زوايا المجسم بخطوط ونخرجها إلى محيط الكرة ونصل بين أطراف تلك الخطوط. فيحصل المجسم المطلوب.

وذلك لأن الزوايا التي حدثت في مركز الكرة يوترها أضلاع المجسمين.[46]

وأضلاع المجسم الأصغر متساوية. فتلك الزوايا متساوية. فأضلاع المجسم الأعظم أيضًا متساوية.

ولأن المثلثات الحادثة من تلك الخطوط مع أضلاع المجسم الأصغر متساوي الساقين وكذا المثلثات الحادثة عنها مع أضلاع المجسم الأعظم. فالزوايا التي على قواعد المثلثات متساوية.

فأضلاع المجسمين متوازية. فزوايا قواعدهما متساوية.

والسطوح المارة بأضلاع [المجسمين] متوازية. فأضلاع كل قاعدة من قواعد المجسم الأعظم في [الكرة مفروضة].

وذلك ما أردناه.

ويمكن بمثل هذا[47] العمل رسم جميع [مجسمات] التي رسمناها في كرة مفروضة.

يج

إذا تقاطع سطحان [ويخرج] فيهما عمودان على فصلهما المشترك من نقطة واحدة فزاويتيهما

[46] المجسمين] الخمسين (مكوب كذا في النسخة ويصحّها في الهامش)

[47] هذا] هذ

[48] الفصل] النصل

مساوية لكل زاوية حدثت من عمودين على الفصل[48] المشترك في ذينك السطحين.

فليكن السطحان ا ب ، ج د وفصلهما المشترك ه ز وليكن ا ه ، ج ه عمودين على ه ز وكذلك ح ز ، ز ط .

فنقول لكون زاويتي ا ه ز ، ح ز ه قائمتين يكون خطا ا ه ، ح ز متوازيين.

وبمثل ذلك يتبيّن أن خطي ج ه ، ط ز متوازيان. فأضلاع زاويتي ا ه ج ، ح ز ط متوازية. فالزاويتان متساويتان.

وذلك ما أردناه.

يد

إذا كان خطان عمودين على خط والثلاثة في سطح واحد وأخرج من ملتقي العمودين والخط عمودان على ذلك الخط في سمك السطح وأحاطا مع العمودين الأوليين بزاويتين متساويتين فهما في سطح واحد.

مثلًا ا ب ج د سطح و ا ب ، ج د عمودان على خط ب ج في ذلك السطح وخطا ب ه ، ج ز عمودان على خط ب ج وفي السمك من السطح وأحاطا مع خطي ا ب ، ج د بزاويتين متساويتين.

فنقول (إن) خطا ب ه ، ج ز في سطح واحد.

فإذا مرّ سطح بخطي ب ه ، ب ج مرّ بخط ج ز . وإلا مرّ في أحد جانبتيه.

فإذا فرضنا سطحا مارا بخطي ج د ، ج ز وذلك السطح كان خط ب ج عمودا على هذا السطح لكون زاويتي ب ج د ، ب ج ز قائمتين. فالفصل المشترك بين هذا السطح وبين السطح المار بخطي ب ه ، ب ج كان عمودا على خط ب ج . فالزاوية التي بين الفصل المشترك وخط د ج مساوية لزاوية ا ب ه .

وكان زاوية د ج ز مساوية لها. فيلزم تساوي الجزء والكل. هذا خلف.[49] وذلك ما أردناه.

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[49] هذا خلف] هف.

يه

نسبة قطر الدائرة إلى ضلع الشكل المتساوي الأضلاع الواقع فيها كنسبة قطر كل دائرة إلى ضلع ذلك الشكل الواقع في تلك الدائرة.

مثلًا ا ب ج دائرة و ا ب قطرها و ب ج ضلع شكل متساوي الأضلاع واقع فيها و د ه ز دائرة أخرى و د ه قطرها و ه ز[50] ضلع الشكل الواقع فيها.

فنقول (إن) نسبة ا ب إلى ب ج كنسبة د ه إلى ه ز .

ونصل خطي ا ج ، د ز . فيكون في مثلثي ا ب ج ، د ه ز زاويتا ا ، د متساويتين. وكذلك زاويتا ج ، ز لكونهما واقعتين في نصفي الدائرة. وكذلك الزاويتان الباقيتان.

فأضلاع المثلثين متناسبة. فنسبة ا ب إلى د ه كنسبة ب ج إلى ه ز .

وبالأبدال نسبة ا ب إلى ب ج كنسبة د ه إلى ه ز .

وذلك ما أردناه.

يو

إذا كان خطان على نسبة قطر دائرة وضلع شكل متساوي الأضلاع واقع فيها فالخط الأقصر ضلع ذلك الشكل الواقع في الدائرة التي يكون الخط الأطول قطرها.

فليكن ا قطر دائرة و ب ضلع [معشر]ها و ج ، د على تلك النسبة والأقصر د .

فنقول (إن) د ضلع معشر دائرة خط ج قطرها.

وإلا فليكن ضلع معشرها أطول منه أو أقصر. وهو ه .[51] فنسبة ا ب كنسبة ج ه .

وكان نسبة ج د تلك النسة. فـ د ه متساويان. هذا خلف.[52]

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[50] و ه ز] و د ز

[51] ه: نقص من اكنسغة

[52] هذا خلف] هف.

يز

نريد أن نرسم في دائرة ذا أربعة أضلاع شبيهًا بذي أربعة أضلاع متوازي الأضلاع قائم الزوايا.

فليكن الدائرة ا ب ج وذو أربعة الأضلاع و ز ح ط ونصل و ح ، ز ط ويتبيّن أن القطرين مناصفان.

ولنجد مركز الدائرة وهو ه ولنعمل فيه زاوية ا ه ب مثل زاوية و ي ز ونخرج ا ه ، ب ه إلى ج ، د ونصل ا ب ، ب ج ، ج د ، د ا .

فنقول (إن) في مثلثي ا ه ب ، و ي ز زاوتا ه ، ي متساويتان. فيبقى مجموع الزاويتين الباقيتين من أحدهما متساوية لمجموع الباقيتين من الآخر.

ولكون ساقي ا ه ، ه ب متساويين يكون زاويتا ا ، ب متساويتين. وكان زاويتا و ، ز متساويتين. فيكون كل من زاويتي ا ، ب مساوية لزاويتي و ، ز .

فزوايا مثلثي ا ه ب ، و ي ز متساوية. فالمثلثان متشابهان. وكذلك المثلثات الثالث الباقية فالسطحان متشابهان.

وذلك ما أردناه.

يح

نريد أن نرسم في كرة مجسمًا ذا قواعد متساويان الأضلاع إثنان منها شكلان مفروضان واقعان في دائرة واحدة والبواقي مربعات.

وليكن الشكلان معشرين ودائرة ا ب ج أعظم دائرة الكرة وخط و ه قطر دائرة و ه ز ضلع معشرها. وليكن عمودا عليه ونتمّ ذا أربعة أضلاع و ه ز ح متوازي الأضلاع قائم الزوايا ونرسم في دائرة ا ب ج[53] ذا أربعة أضلاع ا ب ج د شبيهًا به وليكن ا د نظر و ه . ولننصف [ا د] على [ط] و ب ج على ي . ونخرج من نقطتي ط ، ي عمودين على سطح الدائرة ونخرجهما في طرفيهما إلى محيط الكرة. فالعمود الخارج من ط مع خط ا د في سطح واحد. وكذلك العمود الخارج من ي مع خط ب ج في سطح واحد.

والفصلان المشتركان من السطحين وسطح الكرة دائرتان.

[53] ا ب ج] + دائرة.

ولنصل لبيانه خط ط ي وهو عمود على خطي ا د ، ب ج . فيمرّ بمركز الدائرة أعني مركز الكرة وعلى العمودين أيضًا. فيكون عمود على سطحي العمودين وخطي ا د ، ب ج .

وإذا اخرجنا من مركز الكرة خطوطًا إلى نهاياتهما حصلت مثلثات قوائم الزواليا وهي زوايا ط. وأوتارها متساوية وهي أنصاف أقطار الكرة وأحد أضلاع القوائم مشتركة وهي خط ط ك . فجميع أضلاع البواقي متساوية.

فيكون الفصل المشترك بين هذا السطح وسطح الكرّة دائرة. ومركزها نقطة ط ولذلك السطح المار بالعمود الآخر وخط ب ج دائرة مركزها ي ويكون خطا ا ب ، د ج معشرهما.

ونرسم في الدائرتين معشرين ونجعل مبدئهما نقطتي ا ، ب . فيكون خط ا ب واصلًا بين زاويتين منهما.

وإذا وصلنا في سطحي الدائرتين خطوطًا بين المركزين وزوايا المعشرين كانت أعمدة على خط ط ي وفي السمك من سطح ذي أربعة أضلاع ا ب ج د . فالخطوط التي يحيط مع خطي ا ط ، ب ي بالزوايا المتساوية يكون في سطح واحد.

فإذا وصلنا بين زوايا المعشرين بخطوط كانت موازية لخط [ط] ي فتلك الخطوط أيضًا أعمدة على سطحي الدائرتين.

فالزوايا [التي حدثت] منها ومن أضلاع العشرين قوائم ويكون متوازية. فالحاصل [منها] ومن أضلاع المعشرين مربعات.

وذلك ما أردناه.

وقد [يتبيّن] من هذا أن الأشكال المتساوية الأضلاع المرسومة في الكرة غير متناهية.

يط

نريد أن نرسم في كرة مجسمًا ذا قواعد متساويات الأضلاع إثنان منها شكلان مفروضان واقعان في دائرة واحدة والبواقي مثلثات.

وليكن الشكلان مربعتين وليكن مربع ا ب ج في دائرة ا ب .

وننصف قوس ا ب على د[54] ونصل وتر ا د . ونرسم على خط ا ب نصف دائرة ونرسم وتر ا همثل ا د ونصل ب ه .

وليكن و ز مثل ا ج و ز ح مثل ب ه عمودا على و ز ونتمّ ذا أربعة أضلاع و ز ح .

[54] د: نقص من النسخة.

وليكن ط ي ل أعظم دائرة يقع في الكرة ونرسم فيها ذا أربعة أضلاع ط ك ل ي شبيهًا بذي أربعة أضلاع و ز ح وليكن ك ط نظير ز و و ط ي نظير ز ح ·

و نخرج عمودي ك ن ، ل س على سطح الدائرة و نخرج سطحي ك ط ن ، ي ل س حتى يحصل في الكرة دائرتا ط ك ، ي ل ·

ونرسم في دائرة ط ك مربع ط ع ك ونفصل قوس ي م [من] الكرة ونرسم منها مربع م ف ونفصل قوس ط ع من الدائرة ونصل ص م · وتبيّن أنه مساو لـ ط ي ونصل ط م ، ط ص ·

نقول (إن) نسبة ص م أعني ط ي إلى ط ك كنسبة ب ه إلى ا ج أعني نسبة ز ح إلى ز و·

ونسبة ط ك إلى ط ع كنسبة ا ج إلى ا ب ونسبة ط ك إلى ط ص كنسبة ا ج إلى ا ه أعني ا د · فنسبة ص م إلى ط ع كنسبة ب ه إلى ا ب ·

وإلى ص ط كنسبة ب ه إلى ه ا وزاويتا ص ، ه في مثلثي ط ص م ، ا ه ب قائمتان· فالمثلثان متشابهان·

نسبة ص م إلى ط م كنسبة ب ه إلى ا ب وكانت نسبة ص م إلى ط ع كذلك· فـ ط ع، ط م متساويان·

ونصل م ع وتبيّن أنّه مساو لـ ط ع · فمثلث ط م ع متساوي الأضلاع·

وإذا وصلنا بين نقطة الزويا حصل الشكل المطلوب·

وأنواع هذا المجسم أيضًا غير متناهية·

تمّ الكتاب بحسن توفيق الّه تعالي وتقدس

وفرعت من تحرير فما خلاصة

التحرير يوم السبت ٨ شهر صفر

ختم بالحرّ والطفر هذا السلطية

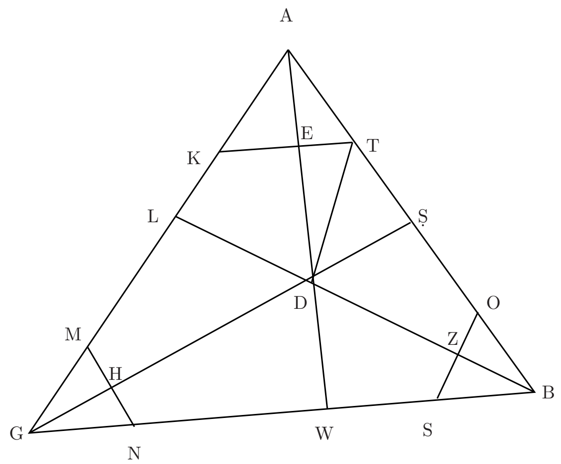
قروين سنة ١٠٠٣

BOOK XVI

It is that which I have appended to the treatise. Through my talent it was derived and it has continued up to nineteen propositions.

Proposition I

We want to draw an equiangular and equilateral hexagon in a triangle likewise (that is, having similar characteristics), for example in triangle $\overline{ABG}$.



Let us find the center of the triangle, namely $\overline{D}$ (1) and extend perpendiculars from it to the sides of the triangle. (2) They are equal to one another and bisect the sides of the triangle.

We connect lines $\overline{AD}, \overline{BD}, \overline{GD}$ and mark off $\overline{DE}, \overline{DZ}, \overline{DH}$ equal to the perpendiculars. We extend from points $\overline{E}, \overline{Z}, \overline{H}$ perpendiculars to the lines ($\overline{AD}, \overline{BD}, \overline{GD}$) which <perpendiculars> we extend on both sides to <meet> the sides of the triangle (3).

It is not possible that point $\overline{T}$ fall [on] (4) point $\overline{\text{Ṣ}}$ nor below it, since $\overline{\text{Ṣ}D}, \overline{DE}$ are equal to one another and angle $\overline{E}$ is right.(5) Then the circumference (or boundary) of hexagon $\overline{TKMNSO}$ is produced.(6) And let us connect line $\overline{TD}$.

We say that angles $\overline{ABD}, \overline{BAD}$ are equal to one another on account of the equality of sides $\overline{DB}, \overline{DA}$ to one another.(7)

But angles, $\overline{Z}, \overline{E}$ in triangle $\overline{BZO}, \overline{TEA}$ are right and lines $\overline{BZ}, \overline{EA}$ are equal to one another. Thus lines $\overline{BO}, \overline{TA}$ are equal to one another. There remain $\overline{O\text{Ṣ}}, \overline{\text{Ṣ}T}$ <which are> equal to one another. (8)

And on account of angles $\overline{\text{Ṣ}}, \overline{E}$ in triangles $\overline{T\text{Ṣ}D}, \overline{TED}$ being right and sides, $\overline{\text{Ṣ}D}, \overline{DE}$ <being> equal to one another, if we remove the squares on $\overline{\text{Ṣ}D}, \overline{DE}$ from the square on $\overline{TD}$, there remain the squares on $\overline{\text{Ṣ}T}, \overline{TE}$, <which are> equal to one another. Thus lines $\overline{\text{Ṣ}T}, \overline{TE}$ are equal to one another (9).

<And> on account of the equality of the sides of triangles $\overline{ABD}, \overline{ADG}$, the angles at $\overline{A}$ are equal to one another. The angles at $\overline{E}$ in triangles $\overline{ATE}, \overline{AEK}$ being right and the side AE being shared, sides $\overline{TE}, \overline{EK}$ are equal to one another.

But it was established that $\overline{O\text{Ṣ}}, \overline{\text{Ṣ}T}$ are equal to one another and $\overline{\text{Ṣ}T}, \overline{TE}$ are equal to one another. Thus, the whole of $\overline{OT}, \overline{TK}$ are equal to one another.

In the same way, it may be shown that all of sides of the hexagon are equal to one another.

But because angles $\overline{BOZ}, \overline{ATE}, \overline{AKE}$ in triangles $\overline{BOZ}, \overline{ATE}, \overline{AKE}$ are equal to one another, angles $\overline{SOT}, \overline{OTK}, \overline{TKM}$ are equal to one another. (10)

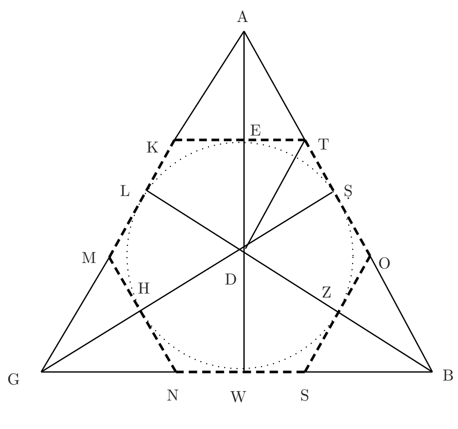
In the same way, it may be shown that all the angles of the hexagon are equal to one another.

That is what we wanted.

It is possible, on the example of this construction, to draw an octagon in a square and a decagon in a pentagon. (11)

Commentary and Notes

In my reconstruction of the diagram using more modern projection,[55] I have outlined the face of the constructed hexagon in dashed lines. As remarked earlier, the described technique is equivalent to inscribing a circle within the equilateral triangle $\overline{ABG}$ and drawing a tangent line perpendicular to the angle bisector at the point where each angle bisector crosses the circle and extending in either direction to meet the sides of triangle. In this reconstructed diagram, I have indicated the inscribed circle using a dotted line. There is no circle in the original diagram.



[55]Based on a diagram from:

http://www.geocities.com/apollonius_theocritos/Pythagoras_files/Tetraktys.gif.

This technique is used in creating truncated versions of each of the regular (Platonic) polyhedra inscribed within a regular polyhedron of the same class (that is, truncated tetrahedron within regular tetrahedron, etc.) in the next five propositions.

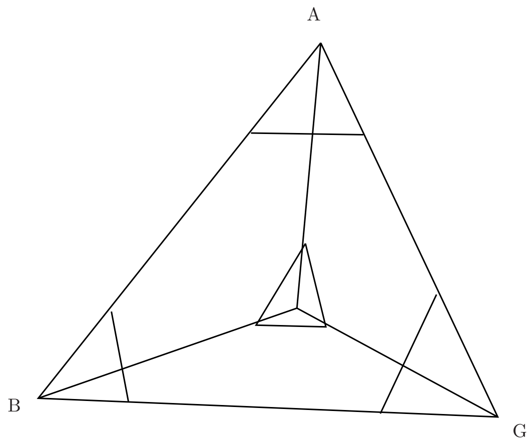
1. The center is where the bisectors of the three sides meet. This definition is not directly given in the *Elements*. See also Plato, *Timaeus*, 54, D and E and the discussion of this passage by Heath [1956, II, 97-99] in his notes to proposition IV, 10. Of course, since we are given that the triangle is equilateral, the angle bisectors and side bisectors are the same.
2. The perpendiculars constructed from this center are not identified explicitly, although used later in the demonstration. From the diagram, we know that they are $\overline{DL}, \overline{DW}, \overline{D\d{S}}$.
3. These constructed perpendiculars are also not identified at this point. The diagram shows that they constitute lines $\overline{TEK}, \overline{MHN}, \overline{SZO}$.
4. This word has been lost due to damage. Only the initial stroke survives but it is not clear what letter was intended. Since it is contrasted with the term "below", one would expect either "on" or "above". The argument, however, seems to require us to read it as "on", rather than "above".
5. Since $\overline{T}$ is the point where the perpendicular from $\overline{E}$ meets side $\overline{AB}$, it may fall on point $\overline{\d{S}}$ (the midpoint) or above it (on the side toward $\overline{A}$ or below it (on the side toward $\overline{B}$. Because $\overline{E}$ was constructed between $\overline{D}$ and $\overline{A}$, though, the perpendicular can only meet $\overline{AB}$ between $\overline{\d{S}}$ and $\overline{A}$. The argument is highly compressed. Assume $\overline{T}$ does fall on $\overline{\d{S}}$. Then, in triangle $\overline{ET(\d{S})D}$, angle $\overline{E}$ is right, since $\overline{ET}$ is perpendicular to $\overline{DA}$. Since $\overline{\d{S}D}$ and $\overline{DE}$ are equal by construction, angle $\overline{D}$ must also be right. But this is impossible because a triangle cannot contain two right angles. Next, assume that $\overline{T}$ falls beyond $\overline{\d{S}}$ toward $\overline{B}$ – for example at point $\overline{O}$. Now angle $\overline{ATE}$ in triangle $\overline{ATE}$ is acute because angle $\overline{AET}$ is right. Thus angle $\overline{ETB}$ is obtuse. Then in triangle $\overline{ET(O)D}$, angle $\overline{DET}$ is right and the angle at $\overline{T}$ is obtuse, which is impossible in a right triangle.
6. It seems clear that we are intended to make the same kind of argument in relation to points $\overline{K}, \overline{M}, \overline{N}, \overline{S}, \overline{O}$.
7. Because $\overline{D\d{S}}$ is perpendicular to $\overline{AB}$ and $\overline{\d{S}}$ bisects $\overline{AB}$, we have in triangles $\overline{AD\d{S}}$, $\overline{D\d{S}B}$ right angles at $\overline{\d{S}}$, sides $\overline{A\d{S}} = \overline{\d{S}B}$ and $\overline{D\d{S}} = \overline{D\d{S}}$. Therefore $\overline{AD} = \overline{DB}$.
8. They are equal to one another because $\overline{\d{S}}$ bisects side $\overline{AB}$. If equals be subtracted from equals, the remainders are equal.
9. *Elements* I, 47.
10. *Elements* I, 13.
11. A marginal gloss states: It is possible to draw an equilateral hexagon in <an equilateral> triangle by dividing into thirds the sides of the triangle and con-

necting the lines so that there is formed at the side of each angle of the triangle an equilateral triangle. He only employed the mentioned technique in order to generalize to the construction of an octagon in a square and the construction of a decagon in a pentagon, or rather a general construction for any equilateral and equiangular polygon in each equiangular and equilateral polygon falling in a circle, the number of its (the circumscribing polygon) sides being half the number of its (the inscribed polygon) sides.

Proposition II

We want to draw, in a tetrahedron (1) having equilateral faces, a polyhedron (2) having eight equilateral faces – four hexagons and four triangles – and to show [that it occurs] within a sphere and that its side (edge) is a third of the side (edge) of the tetrahedron.

Let the tetrahedron be $\overline{ABG}$, and [let us draw] in each of its faces an equilateral and equiangular hexagon. Thus, [there are produced] four hexagons.



But because each of its four vertices (3) is [made up of] three planar angles, there is produced at each vertex a triangle. (4) Thus, there are produced four triangles.

But because the vertices of the tetrahedron are tangent to the sphere in which the tetrahedron occurs, their distances from the center of the sphere are equal to one another. (5)

Now, if we connect lines between the vertices of the tetrahedron and the center, there are produced triangles whose corresponding sides are equal to one another. Thus, their corresponding angles are equal to one another.

And if we connect between the center <of the sphere> and the vertices of the polyhedron, there are produced triangles whose sides are the lines connecting between the center of the sphere and the vertices of the tetrahedron and the lines connecting between the center and the vertices of the polyhedron.

But the lines connecting between the vertices of the tetrahedron and the vertices of the polyhedron – I mean, the sides of the triangles remaining from the faces of the tetrahedron after drawing the hexagons in them – being equal to one another, (6) two sides and the angle between them in <each of> these triangles are equal to one another. Thus the remaining sides in the triangles are equal to one another – I mean the lines connecting between the center of the sphere (7) [and the vertices of the polyhedron.

Let us construct a semi-circle whose center is on the line connecting the vertex of the polyhedron and the center of the sphere] with a distance (or radius) of one of the vertices of the eight-sided polyhedron <from the center> and we revolve it <about the line>, it passes through all the vertices <of the polyhedron>. (8) Thus, the polyhedron occurs inside a sphere.

Now, because two angles of <each of> the triangles remaining from the faces of the tetrahedron after drawing the hexagon are equal to one another and the third angle is an angle of the equilateral triangle – I mean, two thirds of a right angle – each of them is two thirds of a right angle. Thus, their sides are equal to one another.

Therefore, the side of the hexagon – I mean <the edge> of the face of the drawn <polyhedron> – is a third of the side of the triangle – I mean, the face of the tetrahedron.

That is what we wanted.

Commentary and Notes

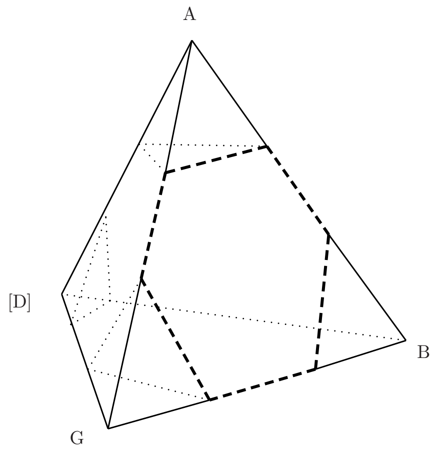
As we can see in the diagram, this tetrahedron has four vertices and four faces, but only one of the faces is labeled. This is a standard practice for the author of these propositions. If asked the reason why, he would probably point out that each face of a regular polyhedron is equivalent to every other face. For this reason, the geometric operations are described only for one face. It is presumed that we will complete exactly the same operations in each of the other faces.

In this proposition, we want to construct within a regular tetrahedron a truncated tetrahedron. To do so, we follow the technique developed in proposition I. We construct in each face of the tetrahedron a hexagon. Then we remove each of the vertices of the tetrahedron along a plane passing through the edges of the newly formed hexagons that lie adjacent to each vertex. Because the solid angle at each vertex was constructed from three planar angles, there remains an equilateral and equiangular triangle face where each vertex once stood.

This diagram shows the tetrahedron in more modern perspective.[56] The dashed lines outline the hexagon in face $\overline{ABG}$. The dotted lines indicate the position of

[56] Diagram based on drawing from http://www.synearth.net/afullerex%20Folder/tructetra9.9.JPG.

the hidden edges of the tetrahedron and the planes of truncation for removal of the vertices of the tetrahedron.

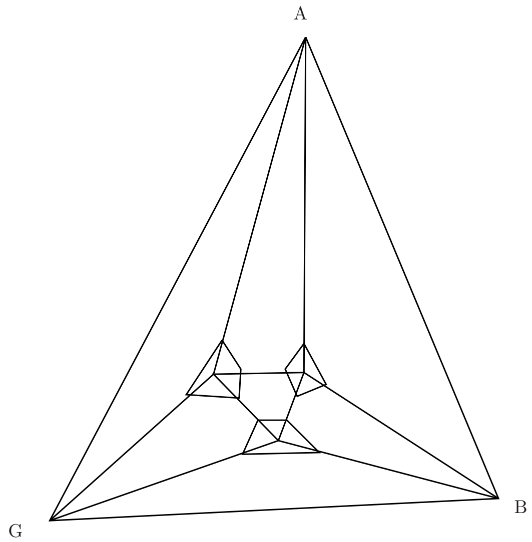


1. The traditional Arabic terminology does not correspond to modern mathematical usage. The term *makhrūṭ* used in this proposition literally means a cone. In more contemporary terminology, we are here dealing with a regular tetrahedron. I have used the latter term in my translation.
2. The generic term *mujassam* (solid) is used in the Arabic Euclidean tradition for any three-dimensional figure. I translate it in this treatise as polyhedron.
3. The author uses the term *zāwiya mujassama* (solid angle) in this context. I shall use the term vertex when the solid angle is actually a portion of a polyhedron and not merely three non-coplanar lines.
4. These triangles are formed by the removal of the vertices of the tetrahedron along a plane passing through the edges of the hexagons closest to the vertices. These removed vertices are themselves regular tetrahedra, similar to the regular tetrahedron with which we began the problem.
5. *Elements* XIII, 13.
6. Proposition I.
7. The text appears to be corrupt at this point. It appears that the copyist has omitted a segment (a line?) of text. I have suggested a provisional reconstruction in square brackets.
8. Euclid also used the technique of rotating a semicircle in *Elements* XIII, 13, but Euclid's demonstration requires construction of a sphere equal to a given sphere. Euclid then shows that the constructed sphere does indeed circumscribe the tetrahedron. In the present proposition, however, the author shows, based on equality of triangles, that the vertices of the newly formed polyhedron are equidistant from the center of the sphere and hence touch the spherical surface.

Proposition III

We want to draw, in an octahedron, a polyhedron having fourteen equilateral faces, eight [hexagons] and six squares, <and to show that the polyhedron occurs in a sphere, and that the edge of the polyhedron is a third of the edge of the octahedron>. (1)

Let the octahedron be $\overline{ABG}$.



We draw in [each] of its faces a hexagon, (2) so that there are produced eight hexagons.

And there is produced, at each of its six vertices, a square (3) because if we connect between the center of the sphere (4) and each of the vertices of the octahedron there are produced equiangular triangles according to the regularity of the characteristics <of the octahedron>. (5)

Then, if we extend perpendiculars from the vertices of the four-sided <figure>, through whose interior this line passes, to this line there are produced four triangles, <any> one of their sides being equal to another, namely the remainders from the sides of the faces of the octahedron.

But the angles at the foot of the perpendiculars are right and the angles which are at the vertex of the pyramid are equal to one another. (6) Thus the sides of the triangles are equal to one another according to their mutual correspondence. Thus, the <four> perpendiculars fall on the same point.

But the line connecting from the center of the sphere and the angle of the perpendicular to these lines <is> falling on their common section. Thus all of them are in a single plane.

But because the triangles resulting from these perpendiculars and the sides of the four-sided figure are equal to one another, the sides are <equal to one another> according to their mutual correspondence. Thus their angles are equal to one another according to mutual correspondence.

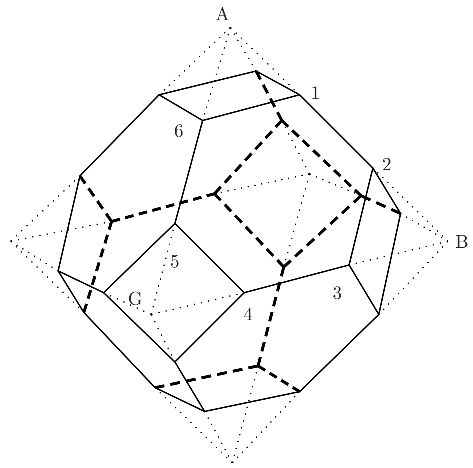
But because each vertex of the four-sided <figure> is composed of two of them (that is, two of the equal angles mentioned above), they are equal to one another. Thus they are squares.

And on the example of what preceded, it may be shown that the polyhedron occurs in (or is circumscribed by) a sphere, and <that> its side (edge) is a third of the side (edge) of the octahedron.

That is what we wanted.

Commentary and Notes

In this proposition we construct a truncated octahedron within a regular octahedron. To do so, we use the procedure developed in proposition I. We construct in each face of the octahedron an equilateral and equiangular hexagon. We then remove the vertices of the original octahedron along a plane passing through the sides of the hexagons closest to each vertex. Because there are four sides bordering each vertex, the face resulting from removal of the vertex is square.



This reconstructed diagram shows the figure in more modern perspective.[57] I have used dotted lines to indicate the removed vertices of the regular tetrahedron. The dashed lines indicate edges not visible to the observer looking at this solid figure. Following the convention of the author, I label only one face $\overline{ABG}$. I have also labeled the six vertices of the hexagon formed in face $\overline{ABG}$ with the numerals 1-6.

The diagram in the manuscript is incomplete, since squares have been drawn at only three of the six vertices. Therefore, only one of the hexagons can be seen in its entirety.

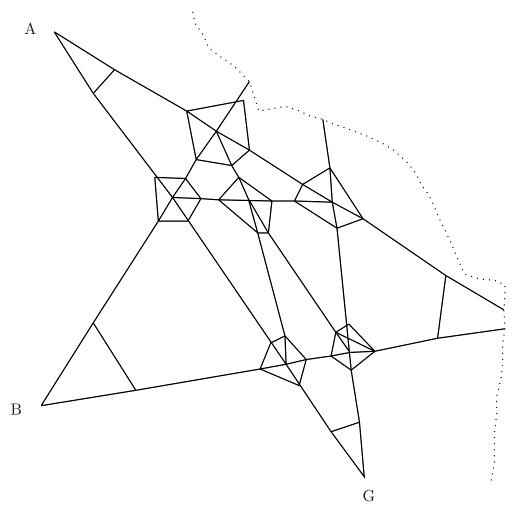
[57] Diagram based on a drawing from
http://www.ac-noumea.nc/maths/amc/polyhedr/polyh_draw_.htm

1. The author / copyist omits the last part of the enunciation, although he makes reference to it at the conclusion of the proposition. Perhaps this statement is omitted because the author does not prove the relationships here but refers the reader to the previous proposition.
2. Proposition I.
3. These squares are produced from the removal of the vertices. Although he calls these polygons "squares" (*murabbaʿāt*), the remainder of the proposition proves that they are indeed true squares as defined by Euclid. Perhaps it is for that reason that he refers to these polygons as "four-sided figures" (*dhū arbʿa aḍlāʿ*) during the remainder of the proposition.
4. The intended point could be the center of either the sphere circumscribing the given octahedron or the inscribed sphere (which circumscribes the truncated octahedron) – the centers of both spheres will coincide.
5. This is the only proposition in which the author explicitly appeals to "mutual correspondence" (*tanāẓur*). Perhaps he wants to emphasize that whatever is true for the angles and sides of one triangular face will be true of the angles and sides of the other faces as well.
6. The Arabic term is *makhrūṭ* (literally, cone). The author earlier used this term to denominate the tetrahedron. He now uses the same term for a pyramid, half of the octahedron. Perhaps he wishes to emphasize that he uses the technique employed in proposition II.

Proposition IV

We want to draw, in an icosahedron, a polyhedron of thirty-two equilateral faces, twenty of them hexagons and twelve of them pentagons, <and to show that the polyhedron occurs in a sphere and that its side is a third of the icosahedron> (1).

Let the visible [half] of the icosahedron be $\overline{ABGDE}$. (2)



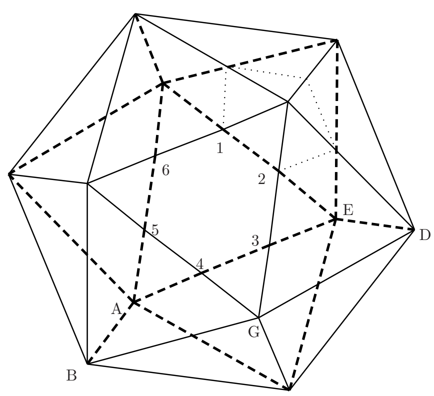
We draw in [each face] of the icosahedron a hexagon. (3) Thus there are produced twenty [hexagons] and there is produced at each vertex of the icosahedron a pentagon because each of its vertices is composed of five planar <angles>.

And on the example of what preceded, it may be shown that a pentagon is equiangular and that its sides are in the same plane (4) and that the polyhedron occurs in a sphere and that its side (edge) is a third of the side (edge) of the icosahedron. (5)

That is what we wanted.

Commentary and Notes

We are forming a truncated icosahedron inscribed within a regular icosahedron. Following the technique used in the previous propositions, we construct in each triangular face of the regular icosahedron an equilateral and equiangular hexagon. The diagram in the manuscript has been constructed with a vertex first view. This has the effect of skewing the perspective on the hexagonal face. In constructing my perspective drawing, I have used a side first view in order to show the hexagon as equilateral and equiangular.[58] The vertices of one such hexagon are indicated in the perspective drawing using numerals 1-6. We remove each vertex from the regular icosahedron along a plane passing through the the sides of the hexagons that lie closest to each vertex. Each newly formed face will then be a regular pentagon because each vertex is a solid angle made up of five planar angles. One of these pentagonal faces is indicated in the perspective drawing using dotted lines. The dashed lines indicate edges invisible to the observer in the chosen perspective. Typically, the author / copyist labels one of the faces of the solid figure displayed in the proposition diagram. In this diagram, though, he has labeled the vertices that would be visible to the observer if he were to take a viewpoint from directly above one of the vertices, in this perspective, the lowest or the bottom point. These labeled vertices are not in the same face, even though they are all in the same plane.



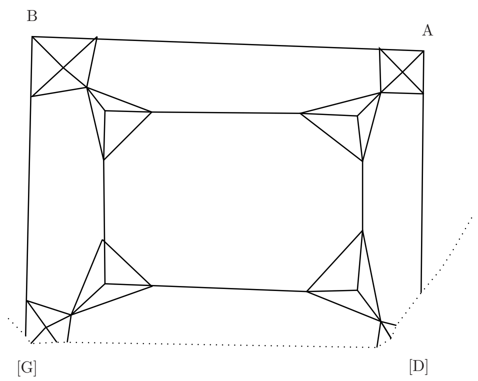
[58]Diagram based on a drawing in http://www.cadimage.net/postimages/soccer-2.jpg

1. The author / copyist has omitted the last portion of the enunciation, although the conclusion to the proposition makes clear that these relations are also part of the proposition.
2. Only when dealing with the icosahedron does the author specifically draw the visible half (*al-niṣf al-mar'iy*) rather than the entire polyhedron. He will follow the same pattern in proposition XI.
3. Proposition I.
4. Proposition III.
5. Proposition II.

Proposition V

We want to draw in a cube a polyhedron <having> fourteen <equilateral> (1) faces, six <of them> octagons and eight <of them> triangles, <and to show that the polyhedron occurs in a sphere>. (2)

Let the cube be $\overline{ABGD}$.



We draw in each of its faces an octagon, (3) so that there are formed six octagons.

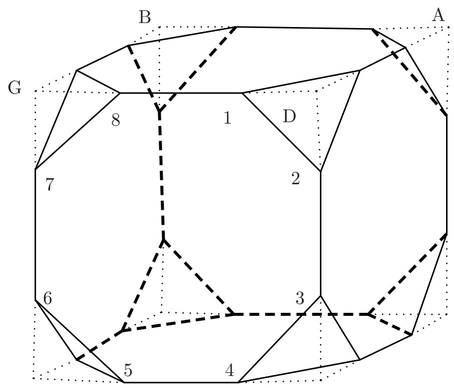
And there is produced at each of its eight vertices a triangle because each <vertex> is composed of three planar angles.

And on the example of what preceded, it may be shown that the polyhedron occurs in a sphere. (4)

This is what we wanted.

Commentary and Notes

In this proposition, we construct a truncated cube within a regular cube. The technique is similar to that developed in proposition I. Thus, we draw an octagon in each of the square faces of the regular cube. Then we remove each vertex from the cube along a plane passing through the edges of the octagon adjacent to each vertex. Since three faces come together at each vertex of the cube, the part removed will be a small tetrahedron. Thus each of the newly formed faces of the truncated cube will be triangular. Because of the construction technique, each triangular face will be equilateral and equiangular.



The diagram in the manuscript has been drawn as though looking directly at one face of the cube. As is his custom, the author labels only this one face. In the perspective drawing, I have indicated the removed vertices of the cube by dotted lines, whether or not that vertex is visible to the observer. The dashed lines indicate edges of the truncated cube invisible when looking with the perspective of the drawing. I have labeled the vertices of one of the constructed octahedrons using numerals 1-8.[59]

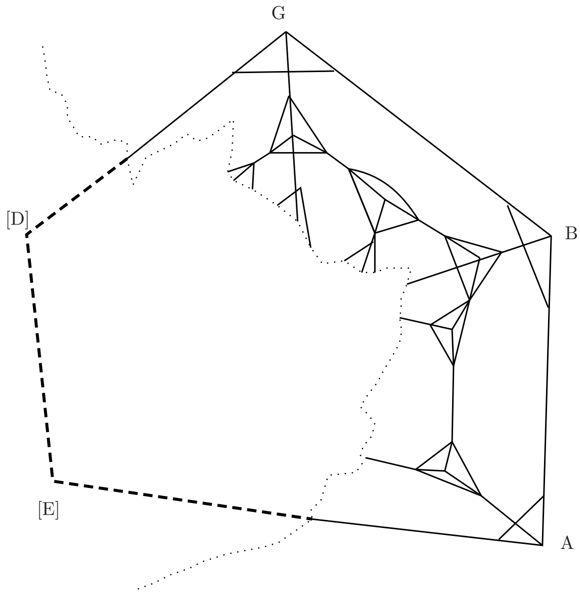
1. The author / copyist has omitted this qualifying term. Since the proposition follows the same pattern as preceding propositions, I suggest that the term should have been included. Its absence may well be a copying error.
2. The author / copyist has omitted the last part of the enunciation, although he again refers to this result at the end of the proposition.
3. Proposition I.
4. Proposition II.

Proposition VI

We want to draw inside a dodecahedron a polyhedron <having> thirty-two equilateral faces, twelve of them decagons and twenty of them triangles, <and to show that the polyhedron occurs in a sphere>. (1)

Let the dodecahedron be $\overline{ABGDE}$.

[59]Diagram based on a drawing in http://www.ac-noumea.nc/maths/amc/polyhedr/polyh_draw_.htm



We draw in each face a decagon. (2) Thus there are produced twelve decagons. And there is produced at each of its twenty vertices a triangle.

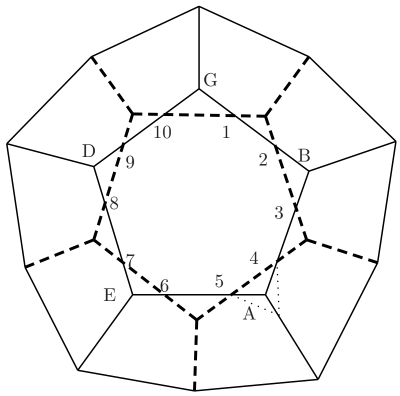
And on the example of what preceded, it may be shown that it occurs in a sphere. (3)

That is what we wanted.

Commentary and Notes

In this proposition we construct a truncated dodecahedron within a regular dodecahedron. We construct in each pentagonal face of the regular dodecahedron a decagon using a technique similar to that developed in proposition I. In the perspective drawing,[60] the vertices of one of these decagons are indicated by numerals 1-10. Then we remove each vertex of the regular dodecahedron along a plane passing through the sides of the decagon adjacent to the vertex. Since three faces of the regular dodecahedron converge at each vertex, the newly formed faces will be equilateral triangles. The removal of vertex A has been indicated in the perspective drawing using dotted lines. In constructing the drawing, I have followed the convention of the author in adopting a point of view from directly above one of the pentagonal faces. Dashed lines are used to indicate edges invisible to the observer in the perspective of the drawing.

[60]Diagram based on a drawing in http://www.math.fau.edu/locke/Graphs/dodecahedron.jpg

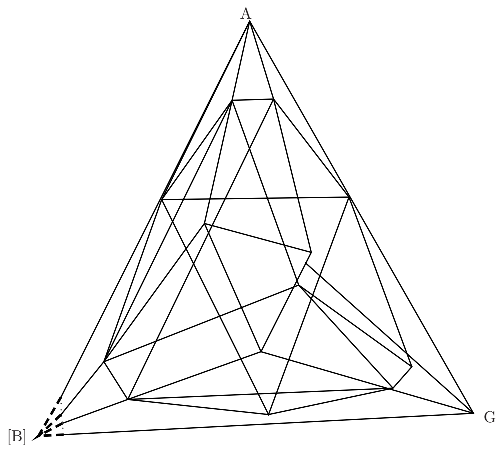


1. In this proposition the author / copyist has again omitted the concluding statement from the enunciation of the problem, although he refers to it in the conclusion of the proposition. It is possible that the omission is made because the full demonstration is not given, but the reader is referred to the earlier proposition for the argument.
2. Proposition I.
3. Proposition II.

Proposition VII

We want to draw inside an octahedron a figure (polyhedron) <having> fourteen equilateral faces, eight triangles and six squares <and to show that it occurs in a sphere>.

Let the octahedron be $\overline{ABG}$.



We bisect each of its sides (edges) and connect the lines. Thus there is produced in each face of the octahedron an equilateral triangle (1) because each of its sides is half the side (edge) of the octahedron.

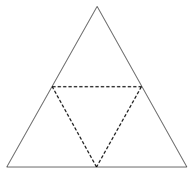
And there is formed at each of the six vertices of the octahedron a quadrilateral. It may be shown, on the example of what preceded, that they are squares. (2) Thus, the <desired> polyhedron is produced.

And, on the example of what preceded, it may be shown that it occurs in a sphere. (3)

That is what we wanted.

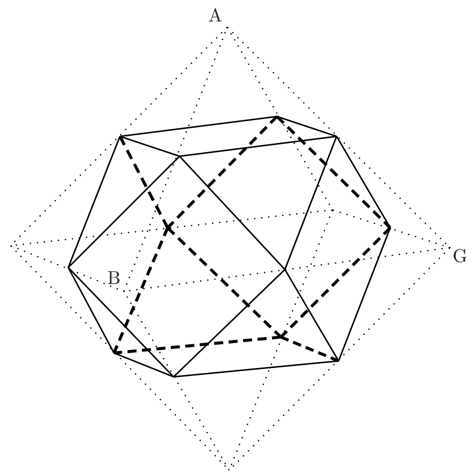
Commentary and Notes

This proposition introduces a new technique which will be used also in propositions VIII and X-XI. We bisect the midpoints of the sides (edges) of each face of the octahedron and join these midpoints with the midpoints of the adjacent faces. Thus we form an equilateral and equiangular triangle, whose side is half that of a side (edge) of the octahedron, in each of its faces.



In this proposition we construct the semiregular cuboctahedron within a regular octahedron. To do so, we bisect the sides of each triangular face of the octahedron, forming eight triangular faces. We then remove the vertices of the octahedron along a plane passing through the sides adjacent to each vertex. Since four faces converge at each vertex, the six newly formed faces will be squares. In the perspective drawing, I have indicated the edges of the tetrahedron using dotted lines, whether or not the edges are visible to the observer. The edges of the cuboctahedron are drawn with a solid line if they are visible to the observer. If invisible, they are drawn with a dashed line.[61] As is the custom of the author, only one face of the octahedron is actually labeled.

[61] Diagram based on a drawing from http://www.ac-noumea.nc/maths/amc/polyhedr/polyh_draw_.htm

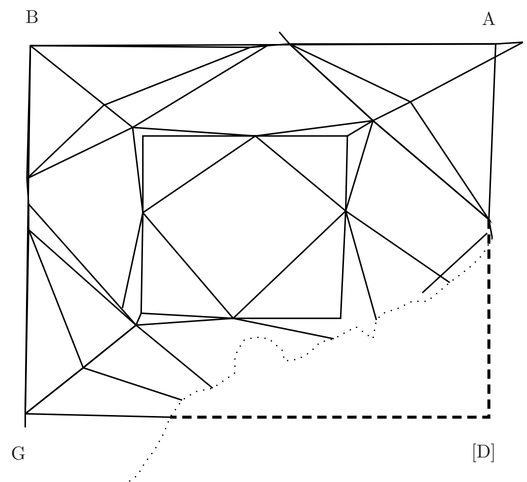


1. There will actually be formed four equilateral triangles, each with sides equal to half the side of the octagon. The author has in mind, however, the central triangle bounded by the lines connecting the midpoints of the adjacent sides of the octagon face. The central triangle in each face will become a face of the desired polyhedron.
2. Proposition III.
3. Proposition II.

Proposition VIII

We want to draw this polyhedron inside a cube. (1)

Let the cube be $\overline{ABGD}$.



Now, we bisect each of its sides and we connect the lines. (2) Thus, there is formed, at each vertex of the polyhedron, an equilateral triangle because each one of them (3) is subtending a right angle (4) and the sides of these right angles are equal to one another. (5)

And there is formed in each one (6) the complement of the two angles, each of which is half a right angle, from two right angles.

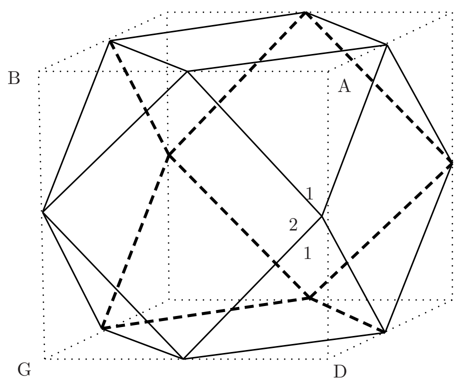
And since there are in the cube eight vertices and six faces, there are eight triangles and six squares.

And it may be shown, on the example of what preceded, that it occurs in a sphere. (7)

That is what we wanted.

Commentary and Notes

This proposition is the correlate of proposition VII. The cube and octahedron are called "dual" polyhedra – the cube has eight vertices and six faces, while the octahedron has six vertices and eight faces. Thus the same semi-regular polyhedron, a cuboctahedron, will be formed within each of these regular figures when we apply the bisecting technique introduced in proposition VII. In that proposition, we formed squares when removing the solid angles of the octahedron and an equilateral triangle in each face. In this proposition, we form equilateral triangles at each of the solid angles of the cube and a square in each face.



The drawing in the manuscript is made from the viewpoint of looking at one face of the cube straight on. In my perspective drawing, I have outlined edges of the cube using dotted lines, whether or not the edges of the face are visible to the observer. In constructing the cuboctahedron, I use solid lines when the edges are visible to an observer and dashed lines when they are invisible.[62]

1. That is, we want to draw the same cuboctahedron produced in proposition VII.

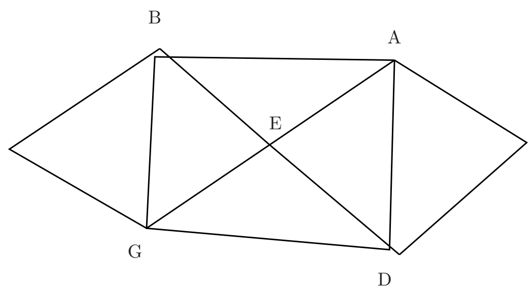
[62]Diagram based on a drawing from http://www.ac-noumea.nc/maths/amc/polyhedr/polyh_draw_.htm

2. He means that we should connect the lines between midpoints of adjacent, not opposite sides, of each face of the cube.
3. That is, each side of one of the triangular faces of the desired polyhedron.
4. That is, one of the three planar right angles making up the solid right angle of the cube.
5. That is, in each of the three planar right triangles, each of the two sides of the right angle is half the side (edge) of the given regular cube.
6. This is, at first reading, a confusing statement. I understand the author to mean that each of the right angles in the square constructed in the face of the cube is complemented by a pair of angles equal to one another whose sum equals the difference between two right angles (a straight line) and the newly constructed right angle. In the perspective drawing, I have inserted "2" in one of the right angles of the square and "1" in each of the pair of complementary angles between the right angle and the edge of the cube. We find a similar argument involving complementary angles used in proposition X.
7. Proposition II.

Proposition IX

The side (or edge) of this polyhedron (1) is the side of the hexagon occurring in a great circle of the sphere in which this polyhedron occurs.

For its demonstration, let $\overline{ABGDE}$ be four of its faces.



We join $\overline{AD}, \overline{BG}$.

And if we join the center of the sphere (2) – let us specify it as $\overline{M}$ (3) – and $\overline{A}, \overline{B}, \overline{G}, \overline{D}, \overline{E}$ there are formed triangles $\overline{MAE}, \overline{MBE}, \overline{MGE}, \overline{MDE}$, the angles <at> $\overline{M}$ <being> equal to one another. [If] we extend from points $\overline{A}, \overline{B}, \overline{G}, \overline{D}$ perpendiculars to line $\overline{ME}$ they come together at a [single] point. Thus quadrilateral $\overline{ABGD}$ is in a single plane. (4)

Let this point be [$\overline{W}$. Then, because] triangles $\overline{ABW}, \overline{DGW}$ have the corresponding angles equal to one another (5) and likewise [triangles] $\overline{[A]DW}, \overline{BGW}$, the angles of quadrilateral $\overline{ABGD}$ are equal to one another. Thus it is a parallelogram. (6)

Now, if we join <a line> between $\overline{A}, \overline{G}$, it passes through point $\overline{W}$. Thus line $\overline{EWM}$ is in the plane of triangle $\overline{AEG}$. (7)

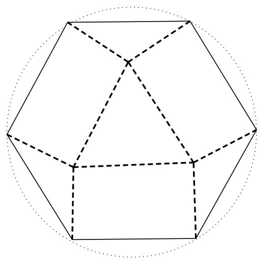
Now, if we extend the plane passing through the center (point $\overline{M}$) and likewise, the plane of line $\overline{EG}$, together with the line which meets it, passes through the center, the two planes are the same. If not, we extend from a point on line $\overline{EG}$, lines in the two planes to the center. Then the two of them surround an area. This is impossible.

Likewise for the planes passing through the remaining sides of hexagon $\overline{AEG}$. Therefore, hexagon $\overline{AEG}$ falls within a great circle of the sphere.

That is what we wanted.

Commentary and Notes

In proposition VII, the side (edge) of the newly formed triangular face of the cuboctahedron is half the side (edge) of the face of the octahedron. But in proposition VIII, the side (edge) of the newly formed square face of the cuboctahedron is not half the side (edge) of the face of the cube. How is this newly formed side (edge) related to the side (edge) of the cube? This proposition shows that it is equal to the side of the hexagon occurring in the great circle of a sphere that circumscribes the cuboctahedron. The result is immediately obvious when we consider the equatorial polygon of the cuboctahedron.[63]

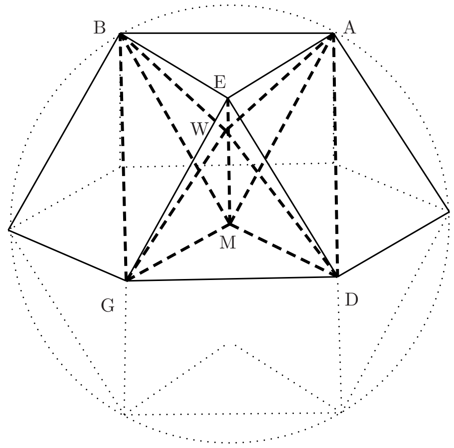


In the perspective drawing, I have outlined some (but not all) of the faces of the cuboctahedron.[64] The edges of the four faces used in this demonstration are in solid lines, the other edges are in dotted lines. The constructed lines are indicated by dashed lines.

[63]For a discussion of the equatorial polygons associated with various semi-regular polyhedra, see Coxeter [1973, 17-20].

[64]My diagrams are based on a drawing from

http://www.gaia-orionis.co.uk/Journeys_Overview.php



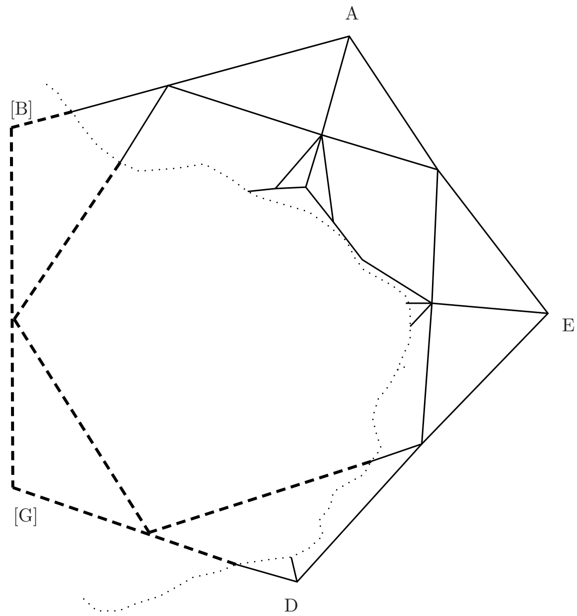
1. The reference is to the cuboctahedron constructed in the previous two propositions.
2. That is, the sphere circumscribing the cuboctahedron. Because a regular hexagon is made from six equilateral triangles, the side of the hexagon is also the length of the radius of the circle.
3. In ancient Greek mathematics, the letter labels are typically assigned in alphabetical order as the elements are introduced into the demonstration, and even if this pattern is not completely followed, the lettering will be "compact", not omitting any letters.[65] This procedure is often imported into the Arabic Euclidean tradition as well. In this proposition we encounter an exception to the rule – and it is sufficiently exceptional that it immediately stands out to someone accustomed to reading Euclidean treatises. One might speculate that the author intended the letter $\overline{M}$ to represent the Arabic term for center (*markaz*) but there is little evidence to support the hypothesis since the center of the triangle in proposition I is not labeled $\overline{M}$.
4. Proposition III.
5. Someone has written above the line: "Its demonstration was undertaken in proposition III". The statement does not appear to be the hand of the copyist.
6. *Elements* I, 27.
7. The triangle and the line have at least two points in common ($\overline{E}$ and $\overline{W}$) so that they share a common section. This is only possible if they lie in the same plane.

VIII.1 Proposition X

We want to draw, inside a dodecagon, a polyhedron <having> thirty-two equilateral faces, twenty triangles and twelve pentagons and show that it occurs in a sphere.

[65] Netz [1998, 35].

Let the dodecagon be $\overline{ABGDE}$.



We bisect each of its sides and connect the lines. (1) Then there is produced at each solid angle a triangle and in each pentagon, a pentagon because each of its angles (2) is the complement of two angles whose sum is the difference between two right angles and the angle of the pentagon. (3)

But since the vertices of the dodecagon are twenty, the triangular <faces> of this <new> polyhedron are twenty and its pentagonal <face*s* > twelve.

On the example of what preceded, it maybe shown that it occurs in a sphere. (4)

That is what we wanted.

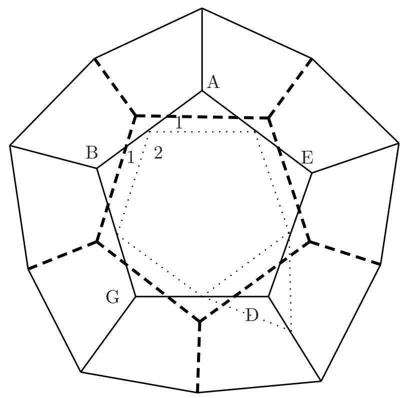
Commentary and Notes

In this proposition, we construct an icosidodecahedron using the technique of bisecting the sides (edges) of each face of the dodecahedron. This produces a new face similar to each of the original faces. We remove each vertex of the dodecahedron along a plane that passes through each one of the edges of the new faces that lie adjacent to the vertex. Since three pentagonal faces converge at each vertex, the newly formed faces will be equilateral triangles.

The perspective diagram shows the regular dodecahedron with which we begin the proposition.[66] The solid lines represent edges that can be seen by the observer, dashed lines represent edges not visible to the observer. Within the dodecahedron, the new pentagonal face has been drawn in face $\overline{ABGDE}$ using dotted lines. The

[66]Diagram based on a drawing from http://www.math.fau.edu/locke/Graphpic.htm.

removal of the vertex at $\overline{A}$, creating a triangular face is also outlined in dotted lines. The same procedures can be used for every face and vertex.

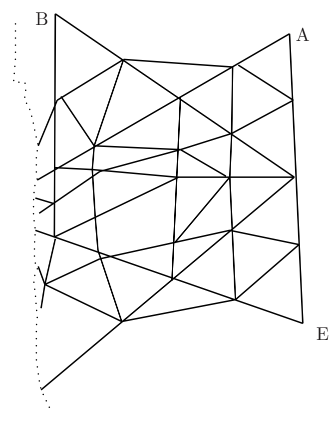


1. He intends that we should connect each midpoint only with the midpoints of the adjacent sides of the pentagonal face.
2. He refers to the angles of the newly formed pentagon within the pentagonal face.
3. The angle of the newly formed pentagon (indicated in the perspective diagram by the number "2") is less than two right angles by the amount of a pair of angles (indicated in the diagram by number "1") – and their sum is the difference between the angle of the pentagon and two right angles (a straight line). A similar argument involving complementary angles was used in proposition VIII.
4. Proposition II.

Proposition XI

We want to draw this polyhedron (1) in an icosahedron.

Let the visible half (2) of the icosahedron be $\overline{ABGDE}$.



We bisect each of its edges and join the lines. (3) Thus there is produced in each <face of the icosahedron> a triangle, (4) and at each vertex a pentagon.

But since the vertices of the icosahedron are twelve, there are twelve pentagons and its triangles are twenty. Thus, everything is as we have mentioned <in the previous proposition>.

And on the example of what preceded, it may be shown that it (the thirty-two sided polyhedron) occurs in a sphere (6).

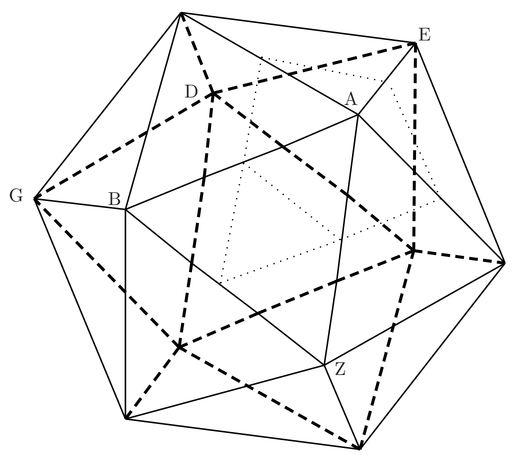
That is what we wanted.

The edge of this polyhedron is the edge of the decagon occurring in a great circle of the sphere, on the example of what occurred in the preceding proposition (7).

And when these two polyhedra (8) are cut by two bisecting circles (9), and the sides (edges) of the faces <of one of the polyhedra> be compared to the sides (edges) of the corresponding faces <of the other polyhedron>, there are formed two other polyhedra occurring in the sphere (10).

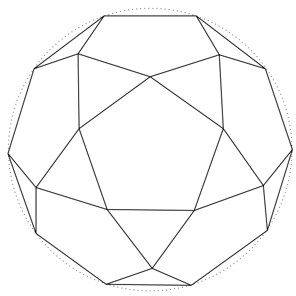
Notes and Commentary

This proposition is a correlative to proposition X. LIke the cube and octahedron, the icosihedron and dodecahedron are “dual” polyhedra. Because the number of faces of the icosahedron equals the number of vertices of the dodecahedron and vice versa, we saw that in the dodecahedron of proposition X we create pentagons in the pentagonal faces and triangles at the vertices. In this proposition, we begin with an icosahedron and create a triangle in each of the triangular faces and pentagons at each vertex (because five triangular faces converge at each vertex of the icosahedron). In the perspective drawing, one of these new triangular faces is shown in dotted lines in face $\overline{ABZ}$.[67] The new pentagonal face formed by removing one of the vertices of the icosahedron is indicated at vertex $\overline{A}$, again using dotted lines. The same procedures would be carried out in each face and at each vertex of the icosahedron.



[67]Diagram based on a drawing from http://www.scio.org.uk/links

1. That is, we want to construct an icosadodecahedron within the icosahedron.
2. Only when dealing with the icosahedron does the author specifically represent only the visible half (*al-niṣf al-marʾiy*) rather than the entire polyhedron. He follows the same procedure in proposition IV.
3. He means the lines joining the midpoints of the edges in each face.
4. As we saw in proposition VII, there are actually four equilateral triangles formed, but he is interested only in the triangle formed between the lines connecting the midpoints of the sides of the triangular faces.
5. Proposition II.
6. He refers to proposition IX, where we show that the edge of the cuboctahedron is equal to the edge of the hexagon occurring in a great circle of the circumscribing sphere. The argument in the present proposition follows a parallel development. When we look at the perspective drawing, the conclusion appears to be obvious.[68]



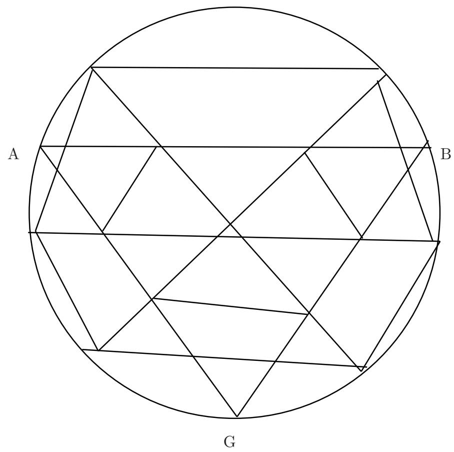
7. He means the icosadodecahedrons formed in proposition X and in this proposition.
8. I interpret this to refer to the great circles mentioned in the enunciations of these propositions, since any great circle divides its sphere in half.
9. This is a cryptic statement that is not easy to decode. I believe that the dual forms used in the Arabic refer to the cuboctahedrons of propositions VII-VIII and the icosidodecahedrons of propositions X-XI. In the case of the cuboctahedron, if we cut the polyhedron along its equatorial plane and rotate one half through a sixth of a turn relative to the other half, so that the faces move over one position we form a new figure, a triangular orthobicupola. Similarly in the case of the icosidodecahedron, if we cut the polyhedron along its equatorial plane and turn one half a tenth turn relative to the other, we form a pentagonal orthobirotunda.

[68]Diagram based on a drawing from
http://www.mathartfun.com/shopsite_sc/store/html/PolyhedraAbout.html

Proposition XII

We want to draw an eight-sided polyhedron <having> equilateral faces, half of them hexagons and half of them triangles, in a given sphere.

Let the sphere be $\overline{ABG}$ and let us draw in it a tetrahedron having four equilateral triangular faces. Let triangle $\overline{ABG}$ be one of its faces.



We draw in the tetrahedron this polyhedron. Then we connect between the center of the sphere and the vertices of the polyhedron by lines and extend them to the circumference of the sphere. We connect between the extremities of these lines. Thus there is produced the desired polyhedron.

That is because the sides (edges) of the two polyhedra subtend the angles produced at the center of the sphere.

But the sides (edges) of the smaller polyhedron are equal to one another. (1) Thus, these angles are equal to one another. Therefore, the sides (edges) of the larger polyhedron are also equal to one another. (2)

But because the triangles produced from these lines, together with the sides (edges) of the smaller polyhedron, are isosceles, likewise the triangles formed from them with the sides of the larger polyhedron <are isosceles>. Therefore, the angles which are at the bases of the triangles are equal to one another. (3)

Now, the sides (edges) of two polyhedra are [parallel to one another]. (4) Thus, the angles of their bases are equal to one another. (5)

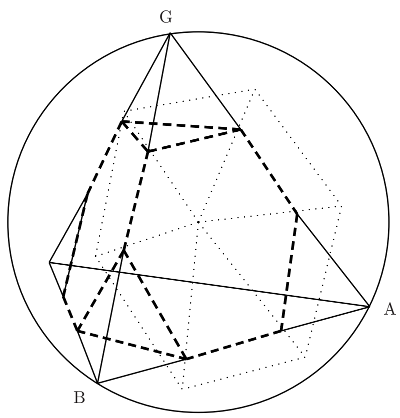
But the planes passing through the sides (edges) [of the two polyhedra] are parallel to one another. Therefore, the sides (edges) of each face of the larger polyhedron are in [the specified sphere].

That is what we wanted.

It is possible, on the example of this procedure, to draw all <the polyhedra> which we have drawn <so far> within a given sphere.

Commentary and Notes

In this proposition, we construct a truncated tetrahedron in a sphere. In proposition II, the author had claimed that the truncated tetrahedron can be inscribed in a sphere. He now shows how this can be done. That is, we are going to construct within the regular tetrahedron the truncated tetrahedron. Euclid, in book XIII, has shown that the regular solids can be inscribed in a sphere, but does not provide the procedure for doing so. Pappus in *Collections III*, problems 54-58, has shown such a technique. Our text assumes that the reader will already know this procedure. Our text then assumes the construction of a truncated solid within this circumscribed Platonic solid. To do so, we can follow the procedure explained in proposition II. We then project the vertices outward to fall on the sphere and connect these vertices to find the expanded version of the truncated polyhedron. In the perspective drawing, we have a truncated tetrahedron, outlined in dashed lines, inscribed in a regular tetrahedron. The dotted lines show the projection of one hexagonal face of the truncated tetrahedron onto the sphere.[69]



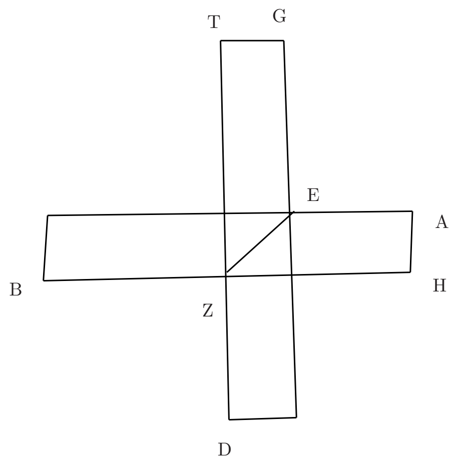
1. The equality comes from the construction of the regular tetrahedron.
2. *Elements* VI, 6. They are equal because the vertices are equidistant from the center of the sphere. Thus the extensions of these straight lines to the surface of the sphere are also equal.
3. *Elements* VI, 5.
4. He means the edges of the original truncated tetrahedron and the edges of the newly produced truncated tetrahedron. *Elements* XI, 7.
5. *Elements* I, 7.
6. *Elements* XI, 15.

[69]Diagram based on a drawing from http://www.synearth.net/afullerex%20Folder/tructetra9.9.JPG.

Proposition XIII

If, two planes intersecting one another, [there be extended] in the two of them two perpendiculars <falling> at a single point on their common section, then their two angles are equal to each angle produced from the two perpendiculars <falling> on the common section in these two planes.

Let the two planes be $\overline{AB}, \overline{GD}$ and <let> their common section <be> $\overline{EZ}$. Let $\overline{AE}, \overline{GE}$ be two perpendiculars falling on $\overline{EZ}$, and likewise $\overline{HZ}, \overline{ZT}$.



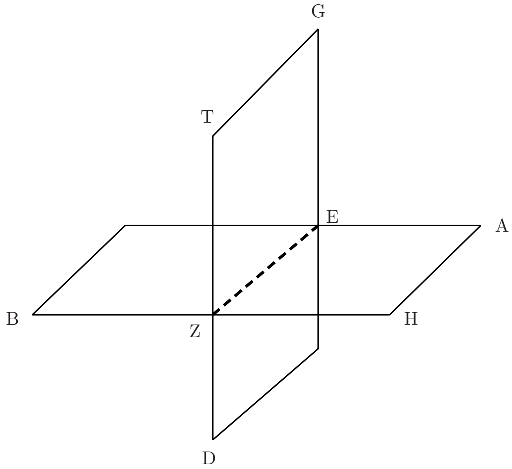
Now we say: On account of angles $\overline{AEZ}, \overline{HZE}$ being right, lines $\overline{AE}, \overline{HZ}$ are parallel to one another (1). On that pattern, it may be shown that lines $\overline{GE}, \overline{TZ}$ are parallel to one another.

Thus, the sides of angles $\overline{AEG}, \overline{HZT}$ are parallel to one another. Therefore, the two angles are equal to one another. (2)

That is what we wanted.

Commentary and Notes

The conclusion of the proposition, that angles whose sides are parallel to one another are equal angles, is clearly stated. On must wonder why the author chose such a convoluted formulation for the enunciation of the problem. The present enunciation reads much more like a summary of the procedure for demonstrating a result. Although the argument is straightforward, the lack of perspective in the diagram may make it a little difficult at first to follow the logic using the diagram. I have redrawn the diagram in more modern perspective terms. In this redrawn diagram, I use a dashed line to indicate the common section between the two planes.

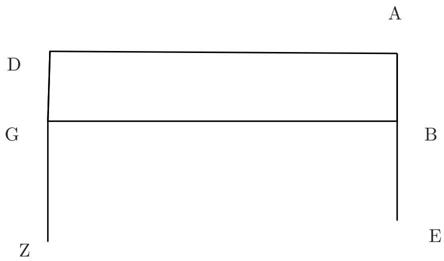


1. *Elements* I, 28.
2. *Elements* XI, 10.

Proposition XIV

If two lines be perpendiculars <falling> on a line and the three <lines> be in single plane, and there be extended from the meeting of the two perpendiculars and the line two perpendiculars to that line in the elevation of the plane and the two surround, together with the first two perpendiculars, two angles equal to one another, the two of them are in the same plane.

For example, <let> $\overline{ABGD}$ be a plane and $\overline{AB}, \overline{GD}$ be two perpendiculars <falling> on line $\overline{BG}$ in that plane and <let> lines $\overline{BE}, \overline{GZ}$ be two perpendiculars <falling> on line $\overline{BG}$ and <let them be> in the elevation of the plane, and the two of them surround, together with lines $\overline{AB}, \overline{GD}$, two angles equal to one another.



We say that lines $\overline{BE}, \overline{GZ}$ are in the same plane.

For if a plane passes through lines $\overline{BE}, \overline{BG}$, it passes through line $\overline{GZ}$. But if not, it passes to one side of it (that is, line $\overline{GZ}$).

Now, if we specify <another> plane passing through lines $\overline{GD}, \overline{GZ}$ and that plane, line $\overline{BG}$ is a perpendicular <falling> on this plane, an account of angles $\overline{BGD}, \overline{BGZ}$ being right (1), then the common section between this plane and the plane passing through lines $\overline{BE}, \overline{BG}$ is perpendicular to line $\overline{BG}$ (2). Thus the angle which is between the common section and line $\overline{DG}$ is equal to angle $\overline{ABE}$.

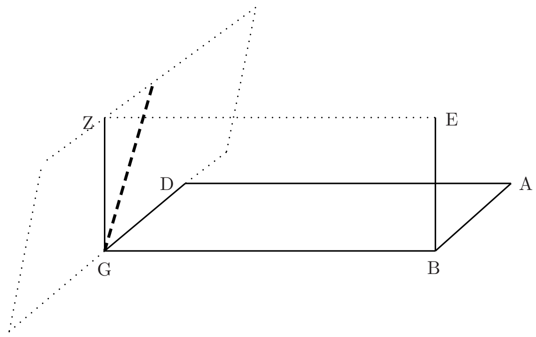
But angle $\overline{DGZ}$ was equal to it. Therefore, it is necessary that the part and the whole be equal to one another (3). This is impossible.

That is what we wanted.

Commentary and Notes

This proposition is built on the previous proposition. It is, in a sense, a converse to the preceding proposition. In this proposition, we argue from the fact of equal angles to the conclusion that the lines forming the sides of the angles are in the same plane. In proposition XIII, we showed that if the sides of the angles are parallel to one another, the two angles will be equal to one another.

In the perspective drawing, I have used dotted lines to indicate the presence of planes that are assumed in the demonstration but not specifically defined in the proposition itself. The common section between the two planes is indicated using a dashed line.

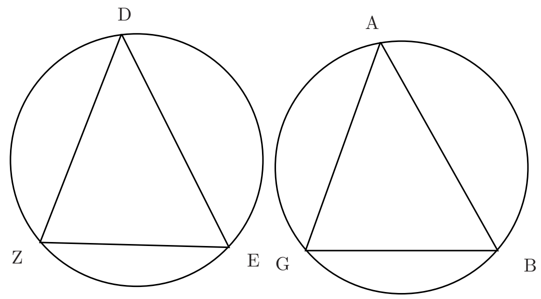


1. *Elements* XI, 4.
2. The common section mentioned in this statement is either line GZ or a line to one side of it.
3. That is, if the plane through BE, BG passes to one side of GZ, the angle of the common section will be greater or less than right angle DGZ, even though it has been argued that it must be equal to it.

Proposition XV

The ratio of the diameter of the circle to the side of an equilateral polygon occurring within it is as the ratio of the diameter of any <other> circle to the side of that <same> polygon occurring within that circle.

For example, <let> $\overline{ABG}$ be a circle, $\overline{AB}$ its diameter, $\overline{BG}$ the side of an equilateral polygon occurring within it; $\overline{DEZ}$ is another circle, $\overline{DE}$ its diameter, $\overline{EZ}$ the side of the polygon occurring within it.



We say <that> the ratio of $\overline{AB}$ to $\overline{BG}$ is as the ratio of $\overline{DE}$ to $\overline{EZ}$.

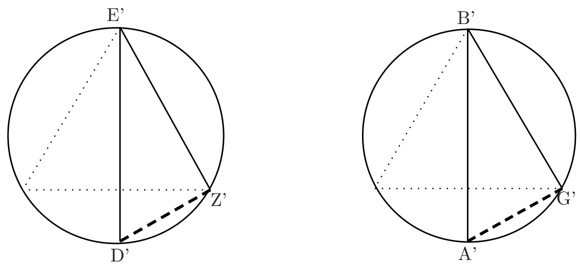
We connect lines $\overline{AG}, \overline{DZ}$. Then, in triangles $\overline{ABG}, \overline{DEZ}$ angles $\overline{A}, \overline{D}$ are equal to one another (1). Likewise, angles $\overline{G}, \overline{Z}$, on account of the two of them occurring in two semi-circles (2). Likewise, the two remaining angles <are equal to one another>. Thus the sides of the two triangles are proportional. Therefore, the ratio of $\overline{AB}$ to $\overline{DE}$ is as the ratio of $\overline{BG}$ to $\overline{EZ}$. (3)

But by alternation, the ratio of $\overline{AB}$ to $\overline{BG}$ is as the ratio of $\overline{DE}$ to $\overline{EZ}$.

That is what we wanted.

Commentary and Notes

The diagram for this proposition is improperly constructed so that it does not accurately portray the geometrical situation described in the text, since it does not include the diameters of the circles, which are specifically mentioned in the proposition. I have reconstructed the diagram according to the text of the proposition and give that reconstruction below. The constructed lines are indicated as dashed lines. I have also included the outline of the equilateral figure (based on the diagram in the manuscript, it should be a triangle), in dotted lines.



The proposition as it stands does not demonstrate similarity of figures which should be required in order to conclude that parts stand in a ratio to corresponding parts. In order to complete the proof, we need further information that will allow us to show that the figures are equiangular in addition to being equilateral.

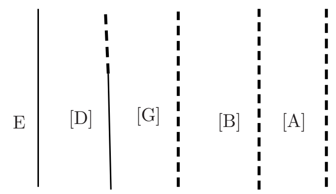
1. The angles at $\overline{A}$ and $\overline{D}$ are equal because they represent angles formed between the line connecting the center of the circle to the vertex of the polygon and the side of the given equilateral polygon.

2. *Elements* III, 31 .
3. *Elements* VI, 4.

Proposition XVI

If there be two lines according to the ratio of the diameter of a circle and the side of an equilateral polygon occurring within it, then the shorter line is the side of that polygon occurring within the circle of which the longer <line> is its diameter.

Let $\overline{A}$ be the diameter of the circle, $\overline{B}$ the side of its [decagon], and $\overline{G}, \overline{D}$ are according to this ratio, and the shorter is $\overline{D}$.



We say <that> $\overline{D}$ is the side of a decagon of the circle <of which> $\overline{G}$ is a diameter.

If not, let the side of its decagon, namely $\overline{E}$, be longer or shorter than it. (1) Then the ratio of $\overline{A}, \overline{B}$ is as the ratio of $\overline{G}, \overline{E}$. (2)

But $\overline{G}, \overline{D}$ is this ratio. So $\overline{D}, \overline{E}$ are equal to one another. (3) That is a contradiction.

Commentary and Notes

This proposition is a converse of the previous proposition.

The diagram for this proposition has been almost completely destroyed. Based on the surviving fragment taken in isolation, it is impossible to guess whether the diagram lines would have been drawn with differing lengths (even if not really approximating the expressed ratio) or whether the lines would have been constructed with uniform lengths (and so clearly not conforming to the verbal statement of the proposition). If we look to other portions of the manuscript where straight lines are used to represent magnitudes, as in books V and VII-IX, we see that the copyist is inconsistent in his practice. In book V, many diagram lines differ in length, while in books VII-IX, the tendency (but by no means a universal practice) is to produce all the lines the same length. Just about the only feature of the diagram of which we are quite certain is that the lines were arranged vertically. I indicate the reconstructed portion using dashed lines and labels in square brackets. I have opted, in my reconstruction, to make all the lines the same length.

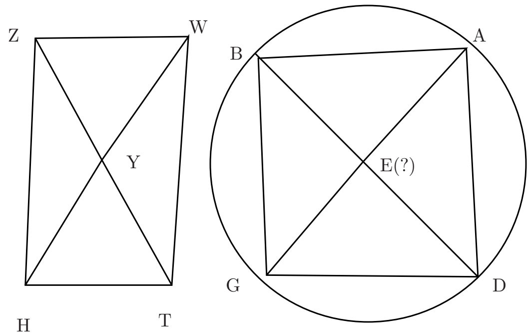
1. The pronoun here clearly refers to line $\overline{D}$.
2. Proposition XV.

3. *Elements* V, 11 and 9.

Proposition XVII

We want to draw inside a circle a quadrilateral similar to a right-angled parallelogram.

Let the circle be $\overline{ABG}$ and the quadrilateral $\overline{WZHT}$. We connect $\overline{WH}, \overline{ZT}$ and it may be shown that the two diameters bisect one another (1).



Let us find the center of the circle, namely $\overline{E}$ (2). Let us construct within it (the circle) angle $\overline{AEB}$ equal to angle $\overline{WYZ}$ (3) and we extend $\overline{AE}, \overline{BE}$ to $\overline{G}, \overline{D}$ and we connect $\overline{AB}, \overline{BG}, \overline{GD}, \overline{DA}$.

We say <that> in triangles $\overline{AEB}, \overline{WYZ}$ angles $\overline{E}, \overline{Y}$ are equal to one another. Thus there remains the sum of the two remaining angles from one of them is equal to the sum of the remaining <angles> from the other.

But on account of sides $\overline{AE}, \overline{EB}$ being equal to one another, angles $\overline{A}, \overline{B}$ are equal to on another. But angles $\overline{W}, \overline{Z}$ are equal to one another (4). Thus each of angles $\overline{A}, \overline{B}$ are equal to $\overline{W}, \overline{Z}$.

Now, the angles of triangles $\overline{AEB}, \overline{WYZ}$ are equal to one another. Thus, the two triangles are similar to one another. (5) Likewise, the three remaining triangles <are similar to one another>. Therefore, the areas are similar to one another.

That is what we wanted.

Commentary and Notes

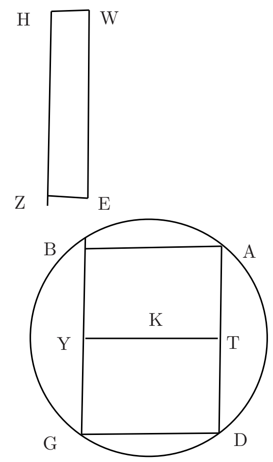
1. We could, for example, construct a circle to circumscribe the right-angled parallelogram. The center of the circle will be where the diagonals meet. Each half-diagonal will be equal to every other half-diagonal, since each is a radius of the circle. This will only happen, though, in the case of the right-angled parallelogram. Also, because we have a parallelogram, in triangles $\overline{ZWY}, \overline{YTH}$ angle $\overline{Z}$ equals angle $\overline{T}$ and angle $\overline{W}$ equals angle $\overline{H}$. $\overline{HT}$ equals $\overline{ZW}$ because the figure is a parallelogram. Thus triangles $\overline{ZWY}$ and $\overline{YTH}$ are congruent, with respective sides and angles equal. Thus $\overline{Y}$ is the midpoint of the two diagonals of the parallelogram.

2. *Elements* III, 1.
3. The construction assumes that we first insert a diameter – either $\overline{AG}$ or $\overline{BD}$ – into the circle and locate point E on it. Then we can use *Elements* I, 23 to construct the angle.
4. The abstract enunciation of the problem specifies that the figure is right-angled. And we know that all right angles are equal.
5. *Elements* VI, definition 1.

Proposition XVIII

We want to draw in a sphere a polyhedron <having> equilateral faces, two of which are specified figures occurring in a single circle (1) and the remainder are squares.

Let the two figures be decagons, and <let> circle $\overline{ABG}$ be a great circle of the sphere; <let> line $\overline{WE}$ be the diameter of the circle and $\overline{EZ}$ a side of its decagon. (2) Let it ($\overline{EZ}$) be perpendicular to it ($\overline{WE}$). We complete right-angled parallelogram $\overline{WEZH}$. We draw in circle $\overline{ABG}$ quadrilateral $\overline{ABGD}$ similar to it. (3). Let $\overline{AD}$ correspond to $\overline{WE}$. And let us bisect [$\overline{AD}$] at [$\overline{T}$] and $\overline{BG}$ at $\overline{Y}$. We extend from points $\overline{T}, \overline{Y}$ perpendiculars to the plane of the circle and we extend them from their extremities to the circumference of the sphere. Then the perpendicular extended from $\overline{T}$, together with the line $\overline{AD}$ is in one plane. Likewise, the perpendicular extended from $\overline{Y}$, together with line $\overline{BG}$, is in one plane. (4)



But the two common sections from the two planes and the surface (5) of the sphere are two circles.

Let us connect, for the sake of its demonstration, line $\overline{TY}$, namely a perpendicular to lines $\overline{AD}, \overline{BG}$. It passes through the center of the circle – I mean, the center of the sphere (6) – and it falls on the two perpendiculars also. Thus, it is perpendicular to the planes of the two perpendiculars and lines $\overline{AD}, \overline{BG}$.

But if we extend from the center of the sphere lines to their extremities, (7) there are produced right triangles, namely the angles at $\overline{T}$. Their chords are equal to one another and they are halves of diameters of the sphere. And one of the sides of the right angles, namely line $\overline{TK}$, is shared. Thus the remaining sides are equal to one another.

Now, the common section between this plane and the surface of the sphere is a circle and its center is point $\overline{T}$. And on account of that, the plane passing through the other perpendicular and line $\overline{BG}$ of a circle whose center is $\overline{Y}$ and lines $\overline{AB}, \overline{DG}$ are the decagons of the two of them.

If we draw in the two circles two decagons and we make their beginning points $\overline{A}, \overline{B}$, then line $\overline{AB}$ is a connection between their two angles.

But if we join, in the planes of the two circles, lines between the centers and the angles of the decagons, they are perpendicular to line $\overline{TY}$ and above the plane of quadrilateral $\overline{ABGD}$. Thus the lines which, together with lines $\overline{AT}, \overline{BY}$, surround the equal angles are in a single plane.

Now, if we join the angles of the two decagons with lines, they will be parallel to line $\overline{[T]Y}$. Thus these lines, too, are perpendicular to the planes of the two circles.

Now, the angles [produced] from them and from the sides of the decagon are right and they (the perpendiculars) are parallel to one another. Thus there are formed from them and from the sides of the two decagons, squares. (8)

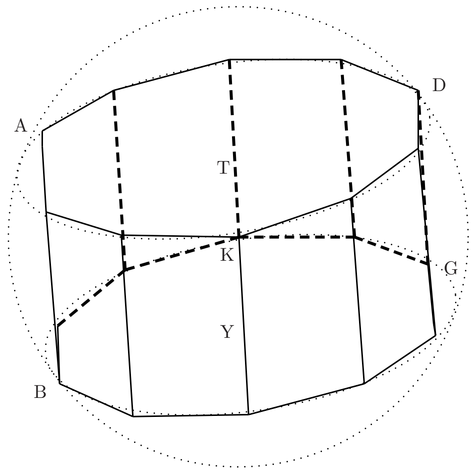
That is what we wanted.

And it [may be shown] from this that equilateral figures drawn in a sphere are limitless.

Commentary and Notes

In this proposition, we construct a decagonal prism.[70] The figure consists of two planes parallel to a great circle such that the two planes cut the sphere forming equal circles. In these two circles we construct our desired equilateral figures – in this case decagons. We arrange these figures so that the vertices of one lie directly over the vertices of the other in the other plane and connect the two vertices by lines between the two planes. We show that these connecting lines are perpendicular to the planes of the circles and that they are all equal to one another. The semi-regular polyhedron formed in this procedure is not an Archimedean solid.

[70] Diagram based on a drawing from
http://streaming.stat.iastate.edu/ dicook/geometric-data/polyhedra/prism/



1. The two specified polygons must be the same.
2. Proposition XVI.
3. Proposition XVII.
4. Proposition XIV.
5. The term *saṭḥ* typically is translated as plane or area, depending on the context. Here, however, it must mean the surface (and in that sense, an area) of the sphere.
6. Since $\overline{ABGD}$ is a great circle, it passes through the center of the sphere.
7. That is, the extremities of the perpendiculars or lines $\overline{AD}, \overline{BG}$.
8. The ratio of $\overline{WE}$ to $\overline{EZ}$ is given as the ratio of the diameter of the circle to the side of the decagon inscribed within that circle. Since we have constructed $\overline{ABGD}$ to be similar to $\overline{WEZH}$, the ratios of $\overline{DA}$ to $\overline{AB}$ or $\overline{GB}$ to $\overline{BA}$ are also in the ratio of the diameter to the side of the decagon. If $\overline{AD}, \overline{BG}$ are the diameters of the two circles, then $\overline{AB}, \overline{GD}$ are equal to the sides of the decagons inscribed in these circles. In that case, the quadrilaterals formed by joining the vertices of the two polyhedra (decagons, in this case) are squares.[71]

Proposition XIX

We want to draw inside a sphere a polyhedron having equilateral faces, two of which are a specified figure occurring in the same circle (1) and the remainder <being> triangles.

Let the two shapes be squares and let square $\overline{ABG}$ be in circle $\overline{AB}$. (2)

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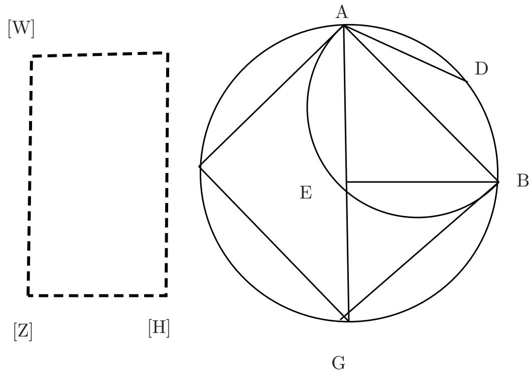
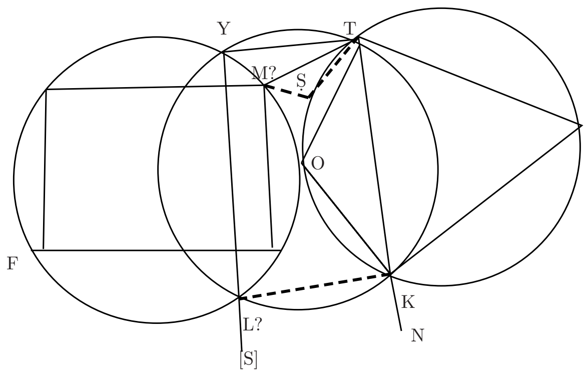
[71] I thank one of my referees for helping to explicate the geometry of this proposition.



We bisect arc $\overline{AB}$ at $\overline{D}$ and we connect chord $\overline{AD}$. We draw on line $\overline{AB}$ a semicircle and we draw chord $\overline{AE}$ equal to $\overline{AD}$ and we connect $\overline{BE}$.

Let $\overline{WZ}$ be equal to $\overline{AG}$ and <let> $\overline{ZH}$, a perpendicular to $\overline{WZ}$, be equal to $\overline{BE}$. We complete quadrilateral $\overline{WZH}$.

Let $\overline{TYL}$ be a great circle that occurs in the sphere. We draw in it quadrilateral $\overline{TKLY}$ (3) similar to quadrilateral $\overline{WZH}$ (2). Let $\overline{KT}$ correspond to $\overline{WZ}$ and $\overline{TY}$ correspond to $\overline{ZH}$.



We extend perpendiculars $\overline{KN}, \overline{LS}$ to the plane of the circle and we extend planes $\overline{KTN}$, $\overline{YLS}$ until there are formed in the sphere circles $\overline{TK}, \overline{YL}$.

We draw in circle $\overline{TK}$ square $\overline{TOK}$. We mark off arc $\overline{YM}$ from the sphere. We draw from it square $\overline{MF}$. We mark off arc $\overline{TO}$ [from] the circle and we connect $\overline{\text{Ṣ}M}$ (4). It may be shown that it is equal to $\overline{TY}$. We connect $\overline{TM}, \overline{T\text{Ṣ}}$.

We say <that> the ratio of $\overline{\text{Ṣ}M}$ – I mean $\overline{TY}$ – to $\overline{TK}$ is as the ratio of $\overline{BE}$ to $\overline{AG}$ – I mean, the ratio of $\overline{ZH}$ to $\overline{ZW}$.

But the ratio of $\overline{TK}$ to $\overline{TO}$ is as the ratio of $\overline{AG}$ to $\overline{AB}$ and the ratio of $\overline{TK}$ to $\overline{T\text{Ṣ}}$ is as the ratio of $\overline{AG}$ to $\overline{AE}$, I mean $\overline{AD}$. So the ratio of $\overline{\text{Ṣ}M}$ to $\overline{TO}$ is as the ratio of $\overline{BE}$ to $\overline{AB}$

But <its ratio> to $\overline{\text{Ṣ}T}$ is as the ratio of $\overline{BE}$ to $\overline{EA}$ and angles $\overline{\text{Ṣ}}, \overline{E}$ in triangles $\overline{T\text{Ṣ}M}, \overline{AEB}$ are right angles. Thus the two triangles are similar.

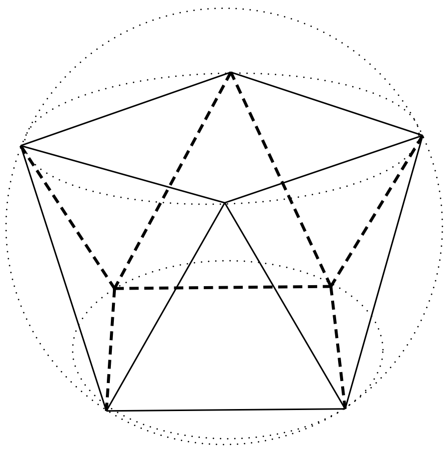
The ratio of $\overline{\text{Ṣ}M}$ to $\overline{TM}$ is as the ratio of $\overline{BE}$ to $\overline{AB}$. But the ratio of $\overline{\text{Ṣ}M}$ to $\overline{TO}$ is likewise. Thus $\overline{TO}, \overline{TM}$ are equal to one another.

But we connect $\overline{MO}$ and it may be shown that it is equal to $\overline{TO}$. Thus triangle $\overline{TMO}$ is equilateral. And if we connect between the points of the angles, the desired figure is found.

And the species of this polyhedra also are limitless.

Commentary and Notes

In this proposition, we construct a square antiprism.[72]



The proposition includes a pair of complementary diagrams. These have been separated on two successive pages in the current copy of the manuscript. In their current positions, these diagrams have been placed in the reverse order from the way their elements are introduced in the text. I believe that they were originally placed side by side in the exemplar from which the copyist is working. The copyist, however, neglected to leave sufficient room for the entire diagram and, beginning his construction from the right-hand side of the diagram, was forced to move the left-hand diagram to a later position in the proposition. I have not found any other example of a reversal of two diagrams elsewhere in the manuscript. I have placed the two diagrams in the proper order in my translation. I have also provided a provisional reconstruction of the missing section of the first diagram which has now been destroyed through damage to the margin of the manuscript.

This proposition uses a technique in some ways similar to that employed in the previous proposition. We divide a sphere using two planes parallel to a great circle and at equal distances on either side of it so that there are produced in the sphere two circles. In each of these circles we draw the desired figure – in this case a square. If we position these figures so that the vertices of one are directly above the other, we will produce a situation like that in proposition XVIII, in which the lines between

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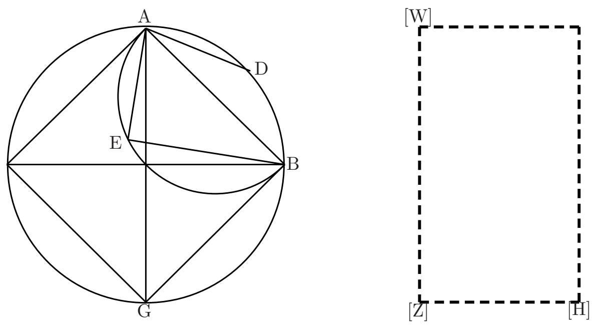
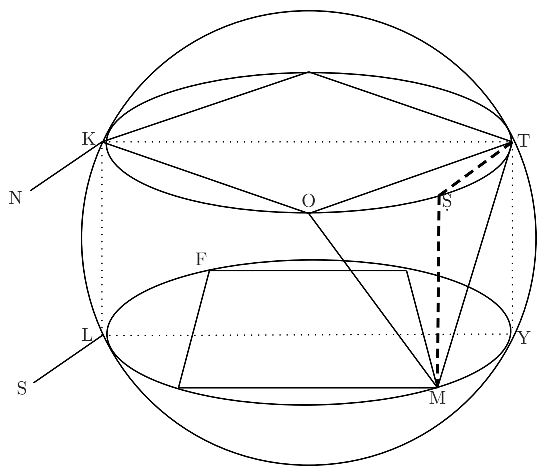
[72] Diagram based on a drawing from

http://www.ac-noumea.nc/maths/amc/polyhedr/convex2_.htm the vertices of the two squares will be perpendicular, forming rectangular figures. In the present case, however, we want to form equilateral triangles between the two circles. Thus we must rotate one of the figures such that its vertices lie opposite the midpoint of the arc of the other circle cut off by the corresponding side of the opposite figure. The situation is illustrated in the perspective diagram above. The result produced in this construction is a semiregular polyhedron that is not one of the classic Archimedean solids. As in the case of the prisms, an infinite series of such polyhedra can be formed.

The proposition opens with a preliminary construction through which we establish the relationship between $\overline{WZ}$ and $\overline{ZH}$ in terms of the relationship between $\overline{AG}$ and $\overline{BE}$ in the circle. $\overline{WZ}$ and $\overline{ZH}$ give us the relation between the diameter of the sphere and the distance between the two circles, as in proposition XVIII.



Having established these relationships, we now can proceed to the construction of the polyhedron itself. We begin with a great circle of the sphere within which we construct quadrilateral $\overline{YTKL}$ equal to quadrilateral $\overline{WZH}$. We erect perpendiculars at $\overline{K}$ and $\overline{L}$, creating planes $\overline{YLS}$ and $\overline{TKN}$ which cut the sphere to produce two circles. Within these circles, we construct squares $\overline{TOK}$ and $\overline{MF}$.



Now, if we compare the constructions in the first part of the demonstration with those in the second part, we can immediately see several relationships:

$\overline{TK} : \overline{TO} :: \overline{AG} : \overline{AB}$
$\overline{TK} : \overline{TṢ} :: \overline{AG} : \overline{AE}\ (= \overline{AD})$
$\overline{ṢM} : \overline{TO} :: \overline{BE} : \overline{AB}$

Thus we see that

$$\overline{ṢM} : \overline{ṢT} :: \overline{BE} : \overline{EA}$$

Now, angle $\overline{E}$ of triangle $\overline{AEB}$ is right because it occurs in a semicircle (*Elements* III, 31) and in triangle $\overline{TṢM}$ angle $\overline{Ṣ}$ is right because $\overline{Ṣ}$ was constructed to be the midpoint of arc $\overline{TO}$ and hence it is directly above point $\overline{M}$. Therefore triangles $\overline{TṢM}$ and $\overline{AEB}$ are similar (*Elements* VI, 6), so that

$$\overline{ṢM} : \overline{TM} :: \overline{BE} : \overline{AB}$$

But since

$$\overline{ṢM} : \overline{TO} :: \overline{BE} : \overline{AB}$$

it is clear that

$$\overline{TM} = \overline{TO}$$

In the same way, we can show that $\overline{MO} = \overline{TO}$. Thus triangle $\overline{TMO}$ is equilateral. The same argument can be used for each triangle constructed between the two circles and their inscribed squares.

1. The two polygons must be the same.
2. The intent in this statement is not just that the square is somehow enclosed inside the circle, but that it is circumscribed by the circle.
3. Proposition XVIII.
4. Line $\overline{KL}$, needed for construction of this quadrilateral, is not present in the existing diagram. It has been added using dashed lines.
5. Lines $\overline{ṢM}$ and $\overline{TṢ}$ are missing from the diagram as found in the manuscript. I have added them using dashed lines.

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