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Editing a Collection of Diagrams Ascribed to Al-Ḥajjāj: An Initial Case Study

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Abstract

This paper present an initial case study in editing diagrams from medieval mathematical manuscripts. The collection of alternative diagrams ascribed to "Ḥajjāj" — presumably al-Ḥajjāj ibn Yūsuf ibn Maṭar, whose translation of Euclid's *Elements* into Arabic is not extant in its pristine form, because it offers a reasonablly small number of diagrams to study and these diagrams seem likely to be of some interest to historians of mathematics. In the introduction to the edition, these diagrams are situated in relation to the known features of the Arabic transmission of the *Elements* attributed to al-Ḥajjāj. Two appendices discuss several technical issues likely to confront anyone who wishes to edit medieval mathematical diagrams and survey some recent contributions to the study of historical mathematical diagrams.

I Introduction

An earlier study, De Young [2005], discussed some general characteristics of a collection of alternative diagrams found in the margins of Princeton University Library, Yahuda 4848, an Arabic manuscript of Naṣīr al-Dīn al-Ṭūsī's *Taḥrīr Kitāb Uqlīdis fī-l-Uṣūl*. The majority of these alternative diagrams are explicitly attributed to "Ḥajjāj" — apparently referring to al-Ḥajjāj ibn Yūsuf ibn Maṭar (c. 169 / 786 – c. 217 / 833), who is credited with the earliest translation of Euclid's *Elements* into Arabic. These alternative diagrams imply that at some points al-Ḥajjāj had used diagrams different from those currently known in the Arabic Euclidean transmission. Frustratingly, a number of the alternative diagrams in Yahuda 4848 are now incomplete (because the manuscript pages are crumbling away) and are sometimes difficult to read because the red ink used to draw the diagram lines often does not reproduce well in the black and white preservation microfilm available for study. The serendipitous discovery of another manuscript (Bayerische Staatsbibliothek, cod. arab. 2697) containing a nearly identical collection of alternative diagrams and in a much better state of preservation immediately suggested the possibility to produce an edition of these alternative diagrams. And this newly discovered manuscript is available online in high-quality color scans so that the role of color in the diagrams can be unambiguously studied.

The importance of these alternative diagrams for historians of medieval mathematics can be understood against the background of the medieval transmission of the *Elements*. The remarkable complexity of this medieval transmission from Greek into Arabic and subsequently into Latin and Hebrew has been succinctly summarized by Brentjes [2001, 39-47] and I shall only summarize the outlines of the Greek – Arabic phase here. The standard

interpretation of the medieval Arabic transmission has been derived from the report of the biobibliographer, Ibn al-Nadīm. According to his report, Euclid's treatise was first rendered into Arabic by al-Ḥajjāj, who also translated the *Almagest* of Ptolemy. Later in his life, he produced a second version, in which he apparently manipulated the text — corrected errors, filled in gaps, removed unnecessary material, etc. A second translation was prepared by Isḥāq ibn Ḥunayn (215 / 830 – 298 / 910). His translation, in turn, was revised by the mathematician Thābit ibn Qurra (221 / 836 – 288 / 901). Of these four versions, only the last appears to be extant. But it now exists in two distinct forms (sometimes designated as Group A and Group B manuscripts [De Young, 1984; 2004]) that differ in the order of their definitions and propositions as well as through use of differing technical terminology. The relation of these two surviving Arabic textual traditions to the other versions reported by Ibn al-Nadīm remains a matter of continuing discussion.

The actual Arabic translation of al-Ḥajjāj, apart from a few brief purported quotations and several reports in the secondary transmission, has almost completely disappeared from the historical transmission. It is in the context of this "lost" transmission of al-Ḥajjāj that this collection of diagrams acquires its historical interest. The fact that a collection of diagrams explicitly associated with the name Ḥajjāj passed into the secondary Arabic transmission suggests that some scholars apparently considered them to represent an important alternative to the mainstream Euclidean tradition. And in light of our still inadequate understanding of the translation of al-Ḥajjāj, it is hoped that an edition of this collection of alternative diagrams can shed a new ray of light on the still open historical questions about the origins of the Arabic transmission of the *Elements*.

Apart from De Young's earlier study of these alternative diagrams, the focus of scholarship relating to al-Ḥajjāj has so far been almost entirely textual. Brentjes [1993; 1994; 1997; 2000; 2006], in a series of extended philological and textual studies examining both alternative formulations and interpolations of material preserved in several early Arabic commentaries on the *Elements* and has elucidated some of the early influences of the translation of al-Ḥajjāj. In addition, De Young [1991; 2002-2003] has studied two collections of quotations attributed to al-Ḥajjāj, as well as the reports by Naṣīr al-Dīn al-Ṭūsī describing how the structure of the Ḥajjāj transmission differed from the transmission ascribed to Isḥāq – Thābit [De Young, 2003, 134–138]. Both Brentjes [2001] and De Young [2004] have also investigated the possible continuing influence of the Ḥajjāj Arabic version on the Latin transmission, as has Busard, the editor of the main Latin translations from Arabic [1968, 1–7; 1977, 1–13; 1983, 2–7; 1984, xii–xv; 1992, 11–30; 2001, 7–12; 2005, 2–28]. Meanwhile, Lévy [1997a; 1997b; 2005] has initiated a study of the medieval Hebrew transmission of the *Elements*, which also provides evidence for the continuing influence of al-Ḥajjāj. And as we shall see in the conclusion, it is mainly through diagrams that we are able to trace the influence of al-Ḥajjāj in the later history of the transmission of the *Elements* to Europe via Latin and Hebrew.

II Characteristics of the manuscripts

II.1 Yahuda 4848

The colophon of manuscript Yahuda 4848 states that Muḥammad ibn Sulaymān ibn ʿAlī al-Asadī completed the copying in Ramaḍān of year 736 of the Islamic hijra calendar, or 1336 CE. Thus the manuscript was copied less than a century after the text was completed in 646 / 1246 The text, occupying folios 10b-78b of the present codex, is acephalous. It opens abruptly with the last lines of proposition I, 29. The text is written in a small, cramped naskhi hand with 29 (occasionally reduced to 22–24) lines per page. The manuscript appears to be in fragile condition and deterioration of the margins has resulted in partial destruction of some marginalia, including some of the alternative diagrams discussed in this paper. Considerable spotting and discoloration of the paper, perhaps from water damage in the past, is noticeable in the top third of nearly every folio. This discoloration makes the text difficult and sometimes impossible to decipher. Typically these water-damaged areas are easier to read directly from the microfilm than from paper prints made from the microfilm.

The diagrams accompanying al-Ṭūsī's text are usually situated in small rectangular spaces set into the text along the outer margins of the codex, although in a few cases they are placed within the text itself. They are placed near the end of each proposition, as is traditional in medieval Arabic Euclidean manuscripts. These diagrams were apparently constructed during the copying process itself.[1] The line segments representing magnitudes or numbers are typically placed vertically — unless there is only one line segment, in which case it will be placed horizontally. The alternative diagrams in the margins, however, appear to be oriented more or less horizontally. The alternative diagrams collected in the margins typically appear to be produced without the aid of drawing instruments, unlike the diagrams in the text itself. These alternative diagrams are rarely numbered but if the proposition number is given, it is in the *abjad* (alphanumeric) form. Usually the alternative diagram will be placed close to the proposition it should accompany, but one must always compare the alternative diagram to the diagrams in the main text on the page to be certain of the proposition it was intended to accompany.

The diagrams, at least in the main text, are constructed in two colors — red and black.[2] The typical convention when two colors are used in diagram construction is that the diagram lines are drawn with red ink and the labels of the diagram points are written in black. Use of

[1] The text crowds up against the diagrams, often invading the diagram area so that little or no white space is left between diagram and text. Such architecture is only possible when the diagrams are constructed during the copying process itself [Hall, 1996, 11; De Young, 2005, 164–167; De Young, 2012, 22–26]. The copyist apparently wrote one or more lines of text, then stopped to draw the diagram before filling in the remaining text of the proposition.

[2] I thank the Princeton University Library Rare Books and Special Collections curatorial staff for this information.

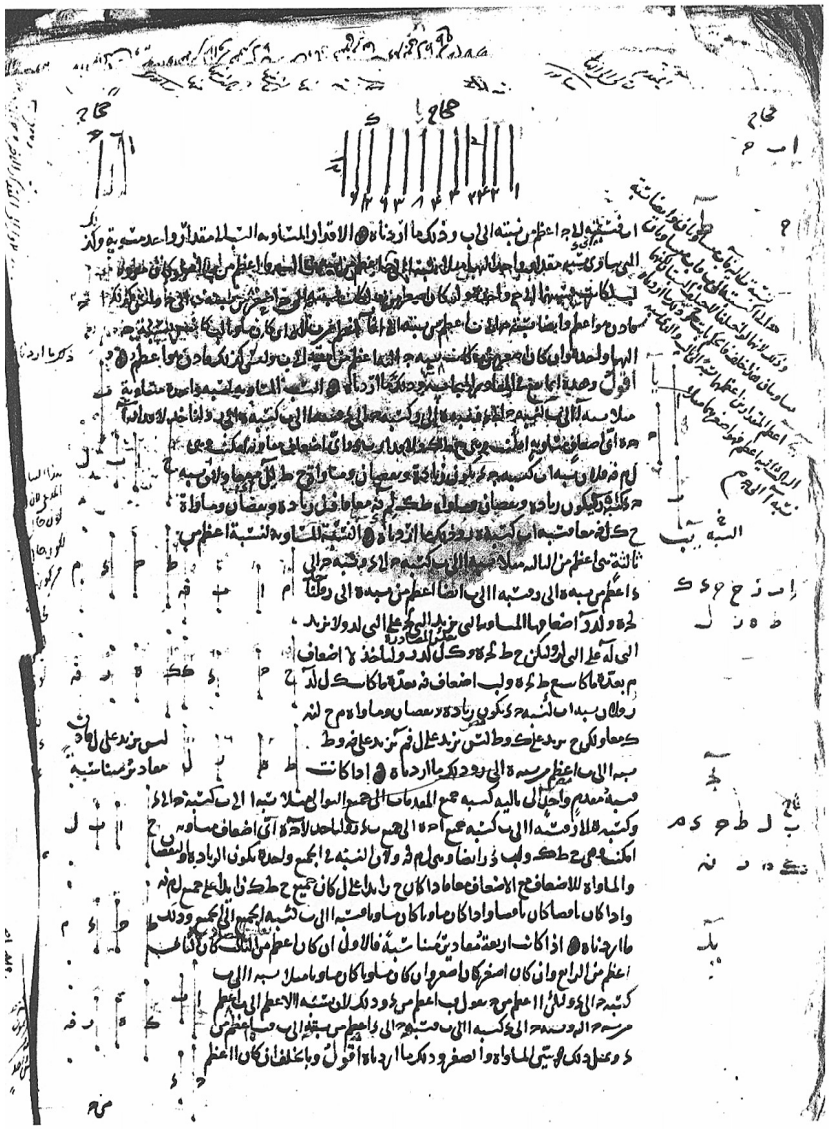

Plate 1: Sample page from Princeton University Library, Yahuda 4848 (Garrett 358Y), folio 28b. Reproduced by permission Manuscripts Division, Department of Rare Books and Special Collections, Princeton University Library. At the top center of the page is a bold diagram with clear ascription to al-Ḥajjāj and labeled as belonging to proposition 11. (Compare to the same diagram in Plate 3.) All five alternative diagrams on this page are frustratingly incomplete and the diagram at the lower right has been partially lost due to damage to the margins.

Plate 2: Sample page from Princeton University Library, Yahuda 4848 (Garrett 358Y), folio 45a. Reproduced by permission Manuscripts Division, Department of Rare Books and Special Collections, Princeton University Library. The two alternative diagrams at the top of the page show the numerals apparently written in red ink. The three alternative diagrams in the left margin are poorly preserved and at least two have been damaged by the disintegration of the margins.

red ink to construct mathematical diagrams is quite common during the medieval period.[3] In the text of al-Ṭūsī we find two different kinds of diagrams, those belonging to the original Euclidean proposition and those belonging to alternative demonstrations or added cases inserted into the text by al-Ṭūsī.[4] These latter diagrams reverse the standard conventions — lines are drawn in black ink and letters labeling the points are written in red. Thus the two classes of diagrams are instantly distinguishable from one another — assuming that the copyist has correctly followed the conventions.[5]

The manuscript contains extensive marginalia in what appear to be either three different hands or written with three different pens.[6] Several of these marginalia have been damaged because the paper used in the codex is brittle and the margins have begun crumbling away during usage. There are also interlinear annotations, most of them brief. In addition to the brief interlinear comments, there are also interlinear references to earlier propositions, definitions, and axioms apparently inserted by or for a student as an aid in understanding the logic of the argumentation. These internal references to earlier propositions are given in *abjad* form, the first numeral referring to the proposition and the second to the book number (if different from the book in which the reference occurs). These references are usually placed below the word or phrase to which they belong and are frequently in red ink.

Some diagrams, both in the text and in the margins, also contain inserted material in the form of numerals. In books V and VII – IX someone has inserted specific numerical values, apparently as an aid to understanding or perhaps as a verification test of the relationship being demonstrated. It is often difficult to decide whether the numerical values were simply inserted from the source being used by the copyist, by the copyist himself, or by a later reader of the manuscript.[7] These numerical values would typically be written with red ink

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[3] A sample of diagrams from various medieval manuscript sources — Byzantine Greek, Arabic, Latin, Hebrew — can be found on the internet: `http://www.davidboeno.org/herodote/000EL.html`. Clicking on any diagram will redirect to a page with multiple examples of that diagram, many in full color. Clicking on any of these examples will open a new window showing the diagram in context on the codex page. This collection of diagrams, although far from complete, illustrates the unity and consistency of the historical transmission, as well as the ideosyncratic variations possible among individual copyists.

[4] Almost all of al-Ṭūsī's alternative demonstrations were taken from the *Kitāb fī ḥall shukūk Kitāb Uqlīdis* of Ibn al-Haytham, while the added cases come from a variety of Arabic or Greek sources [De Young, 2009]. Al-Ṭūsī has often simply adopted Ibn al-Haytham's diagrams for these alternative demonstrations.

[5] Use of color to distinguish Euclidean from extra-Euclidean diagrams is known in other manuscripts as well. It is difficult to estimate how wide-spread the use of two colors might have been, since it is often necessary to rely on black-and-white microfilm and many catalogers do not mention or carefully describe the use of color in diagrams among the characteristics of the manuscripts.

[6] Most of these margin notes bear no ascription, but a few are attributed to a variety of sources. These will be discussed in sections 3 and 4.

[7] We know from preliminary studies of the history of Euclidean diagrams in Arabic that errors once introduced into diagrams were sometimes repeated by a later copyist when the manuscript was recopied [De Young, 2012,

if the diagrams were prepared using two colors of ink. Such numerical values occur in a minority of the surviving manuscripts of al-Ṭūsī's treatise, but they occur often enough so that one hesitates to say that they are rare occurrences.

II.2 BSB arab. 2697

BSB arab. 2697 is a collection of mathematical tracts that appear to have been copied by the same person, who is not named in the text.[8] A colophon to one of the internal treatises in the codex (folio 194a) gives the date of copying as 1142 of the Islamic Hijra calendar or 1729 CE. Since the text appears to be copied in the same hand throughout, this colophon provides at least an approximate dating for this copy of al-Ṭūsī's treatise. The *Taḥrīr* occupies the first 145 of the 214 folios in the codex. The text has been copied with a neat, precise *naskhi* hand, 21 lines per page. Red ink has been used for rubrication and to mark textual features such as the *abjad* numeral that precedes each proposition and the references to earlier propositions. Red ink is also used to signal shifts from Euclidean text to comment — the stereotypical *aqūlu* ("I say") and similar transition words or phrases that introduce many of the inserted comments of al-Ṭūsī.

The diagrams have been constructed in two colors, using red and black inks. Euclidean diagrams — diagrams corresponding to the original Euclidean proposition (the propositions in al-Ṭūsī's main text) — have been rendered with red lines and black labels. Diagrams belonging to added cases or alternative demonstrations — cases or demonstrations which were not present in the original Euclidean text — are constructed with black lines and red letter labels and are generally placed in the margins. The diagrams also sometimes contain added annotations in the form of inserted numerals.[9] These numerals appear in the diagrams of books V and VII – IX as well as in some of the propositions of book X. The numerals are generally written in the same color as the lines of the diagram.

The diagrams of books V and VII – IX in al-Ṭūsī's text are usually constructed of line segments oriented vertically (unless there is only one line segment) and typically placed in rectangular openings or "windows" aligned on the margins of the text space, although they invade the margins on occasion. These diagrams are typically surrounded by adequate white space and rarely impinge on the surrounding text. A few examples show unusually large white spaces surrounding the diagram, suggesting that the diagrams were inserted into spaces left by the copyist only after the copying was completed. If the diagrams had been constructed during the copying process, we would expect the copyist to have continued the line of text to fill in the unneeded space. We also find a few diagrams are rotated ninety

32]. When two manuscripts repeatedly use the same numerical values in their diagrams, one begins to suspect that one may have been copied from the other or that both were copied from the same prototype.

[8] Available on-line http://daten.digitale-sammlungen.de/ db/0003/bsb00036526/images/index.html.

[9] Such annotations must have entered the tradition through a reader of the text, although once placed in the diagrams they were sometimes treated as an integral part of the diagram by subsequent copyists [De Young 2012, 32].

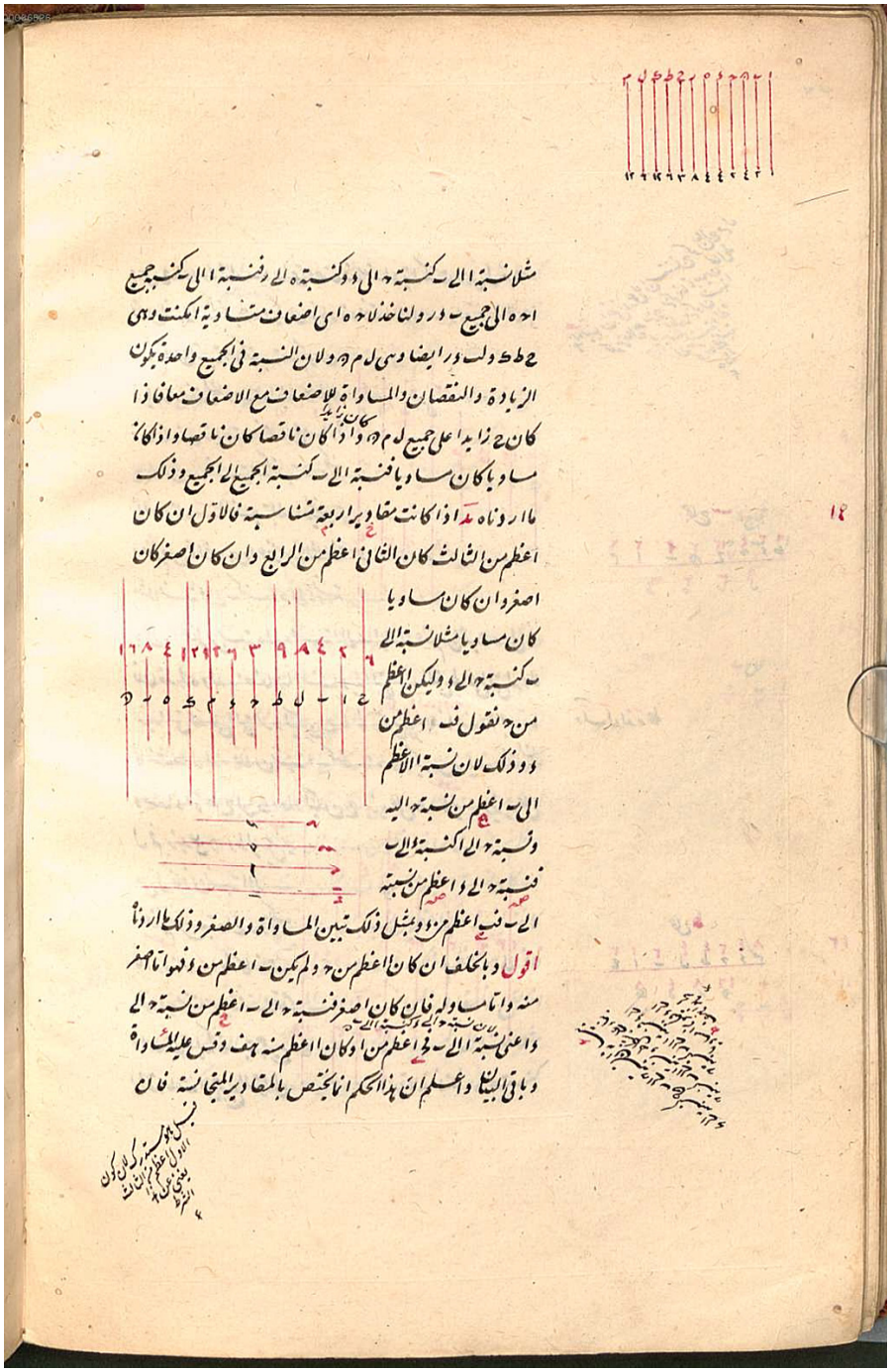

Plate 3: Sample page from Bayerische Staatsbibliothek, cod. arab. 2697, folio 57b. The diagram at the top of the page does not carry an ascription to al-Ḥajjāj nor a proposition number. But it contains complete diagram information. Cf. the same diagram in Plate 1.

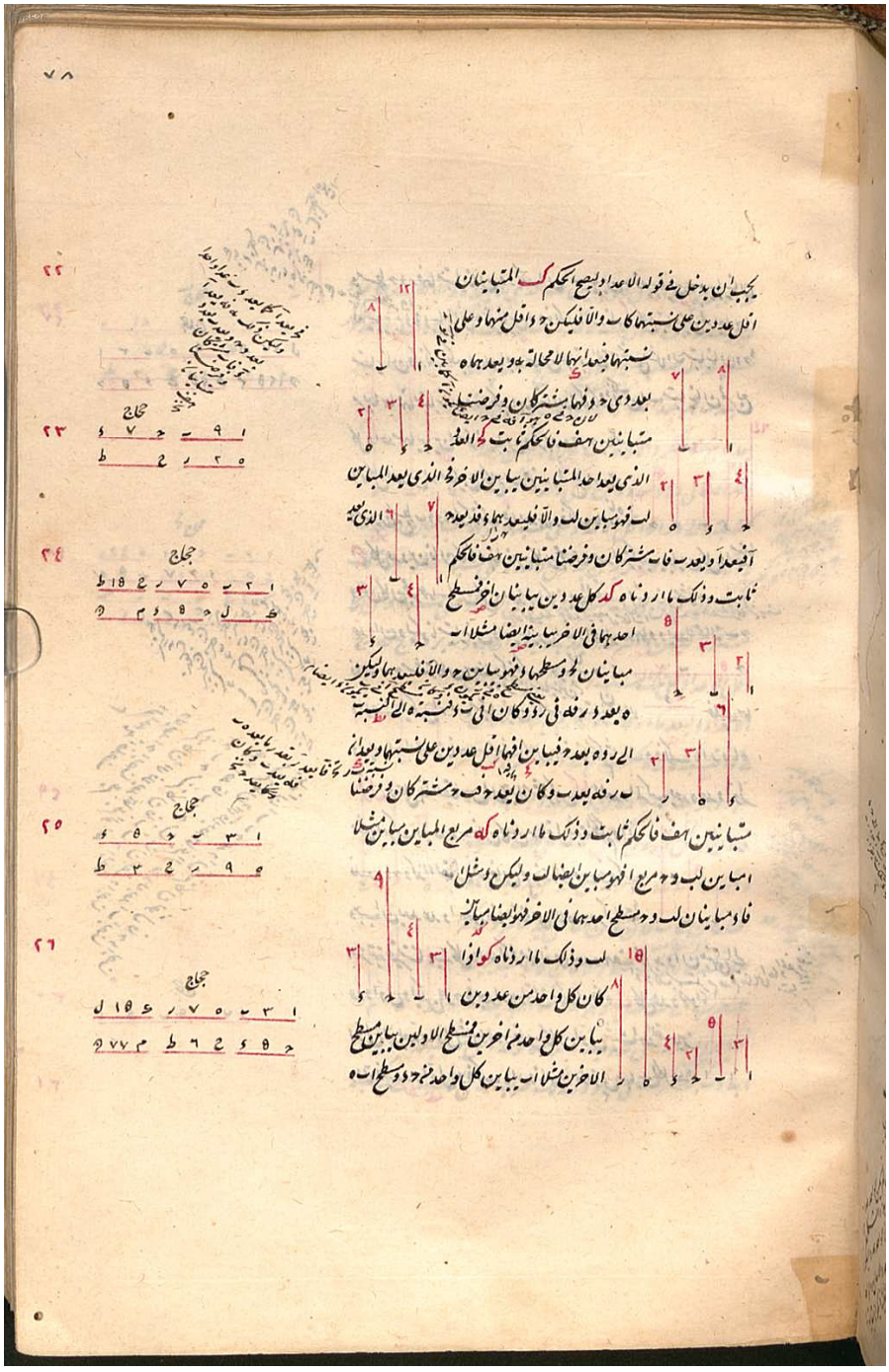

Plate 4: Sample page from Bayerische Staatsbibliothek, cod. arab. 2697, folio 78a, showing several alternative diagrams from book VII, each with clear ascription to Ḥajjāj.

degrees relative to the text. Such rotation often is indicative of insufficient space being left for the insertion of the diagram.

The collection of alternative diagrams that appears in books V and VII – X are almost always placed horizontally in the margins. Most are given a number referring to the proposition they should accompany. The style of this number can vary, sometimes *abjad* (alphanumeric) and sometimes numeral. The color of ink used for these numbers can also vary. Often it is red, but black can also be found. Perhaps this variation appears because the existence of a tradition of alternative diagrams was not common enough to enforce a specific color convention as was the case for diagram construction. Alternative diagrams accompanying propositions for which al-Ṭūsī indicates a difference in numbering between the transmission of al-Ḥajjāj and Isḥāq (such as proposition VIII, 25/27) typically also carry a dual numbering, one in red and one in black, following the color convention al-Ṭūsī uses to distinguish the proposition numbers of the two textual traditions: red for Thābit and black for al-Ḥajjāj.

The manuscript also contains marginalia, most of it appearing to be in the same hand as the text itself. The diagrams attributed to al-Ḥajjāj that are our focus in this paper constitute a subset of this material. In addition, there are short interlinear annotations, most if not all of them apparently in the hand of the copyist, and many acting as internal references (written in red ink and usually placed below the line), pointing the reader back to propositions or definitions that justify the assertions in the demonstrations. Other interlinear notes are added to explicate the correlation between the specific geometrical proof and the components of the general statement of the proposition.[10]

Although these verbal margin notes may have a random orientation on the page, each different from its neighbors, each marginal note or gloss has a consistent internal text base line. Some margin notes are verbally identical to marginalia in Yahuda 4848, but others occur only in this manuscript. Several of these unique margin notes refer to or are attributed to specific mathematical sources:[11]

- Qāḍīzāde (attribution of a note attached to *Elements* I, 25)
- Taqī al-Dīn (attribution of a note attached to the first of the eight propositions inserted by al-Ṭūsī following *Elements I, 29* as a demonstration of Euclid's parallel lines postulate)
- Ibn al-Haytham (quoted in a note attached to *Elements* V, 1)
- Abharī (attribution of a comment attached to *Elements* VI, 16)

Many glosses throughout the treatise are signed with a confusing array of abbreviations,

[10]For example, if the proposition involves dividing line AB into two unequal segments at point G, such that AG is larger than GB, then when the demonstration mentions AG, the glossator often reminds the readers that this segment is "the larger".

[11]None of these attributed comments appear in the marginalia of Yahuda 4848. Marginalia shared between the two manuscripts will be discussed in sections 3 and 4.

such as the Arabic letter م — perhaps signifying they are the work of the author (in Arabic, *mu'allif*). Others are signed with the Arabic letter ن or, less frequently, with the Arabic letter ظ. Many are also signed with what appears to be the name "Badr" — sometimes appearing remarkably like the Indo-Arabic numeral "12" — and a few are signed "Sayyid". I have been unable to identify the mathematicians to whom these ascriptions may refer. Margin glosses are sometimes keyed to specific points in the text through placement of a variety of symbols, usually in red ink, above the line. Often the gloss is placed beside the section to which it refers, but it is also possible to find a comment separated by a considerable distance from its textual referent if there are many glosses on the same section of text.

III Shared verbal marginalia

These two manuscripts not only share a nearly identical collection of alternative diagrams ascribed to al-Ḥajjāj, they also share a surprising amount of verbal marginalia. These shared marginalia discuss the same topics, are frequently word for word the same, and typically carry the same authorial ascriptions. For example, shared glosses ascribed to Thābit ibn Qurra and Nīsābūrī are immediately evident among the comments on the definitions of books II, V, VII, and XI. I translate here six examples from this shared collection of marginal glosses. The first three come from the glosses to book V and the remaining three from the glosses to book VII. I discuss the implications of these shared features for the textual history of the marginalia in section 4.

- "The *ex aequali* ratio occurs [when there are] a number of magnitudes and other magnitudes according to their number [such that] the ratio of the antecedent magnitude of the first to the last of the first magnitudes is as the ratio of the first to the last magnitude of the other magnitudes." The note is signed "Thābit". The last line and signature are either damaged or invisible in Yahuda 4848.
- "A perturbed proportion exists when there are three magnitudes and three other magnitudes such that the ratio of the first of the first magnitudes to the second of them is as the ratio of the second of the second magnitudes to the third of them and the ratio of the second of the first magnitudes to the third of them is as the ratio of the first of the other magnitudes to the second of them." The note is signed Nīsābūrī.[12]
- "If HB is a multiple of E and TD is not a multiple of Z according to its measure then let

[12]It is not known whether this is Yūsuf ibn Aḥmed al-Nīsābūrī who is thought to have been active in the 5th / 11th century and who wrote a well-known introduction to arithmetic, or Niẓām al-Dīn Ḥasan ibn Muḥammad ibn Ḥussayn Qumī al-Nīsābūrī (died 729 / 1329), author of the widely read *al-Risāla al-shamsiyya fī'l-ḥisāb*. Although his primary writing was on astronomy / cosmology, Niẓām al-Dīn is credited with a treatise on mathematics as well. Since both treatises are extant [Sezgin, 1975, 313; Rosenfeld & Ihsanoğlu, 2003, 238-239], the question of which Nīsābūrī is being quoted might eventually be resolved through careful examination of their contents.

the multiple taken according to this measure be TL. Thus in all the first, I mean AH, and the fifth, I mean HB, the multiples of the second, I mean E, are as [the multiples] in all the third, I mean ET, and the sixth, I mean TL, from the multiples of the fourth, I mean Z, according to proposition 2. Thus GD [and] GL, the whole and the part, are equal to one another because each one of them is a multiple of Z according to the number of what is in AB from E." This note, placed near the end of proposition V, 6, is unascribed. The note found in the margin of Yahuda 4848 is partially obscured by the tightness of the binding, although from what is visible, the note appears to be identical in content.

- "Thābit said: And the number which is called prime (*awwal*) is that which only the unit measures and all the numbers which are called mutually prime (*awwal ʿinda al-ākhar* [are those] which have no common measure except the unit." This note is apparently intended to explicate or expand a comment inserted into al-Ṭūsī's text following the definition of a prime number: "In the text of Thābit [he says that] a [number] prime to another number is that [for which] there does not measure the two of them together [a number] other than the unit." The word "together" (*maʿan*) seems to have caused some concern for both manuscripts also include an interlinear comment (very difficult to read in Yahuda 4848): "That is, not that it measures their sum but that it measures [each of] them individually."
- "The smaller number, if it measures the larger, is conventionally called a part of it and if it does not measure it [it is called] parts of it. For example, four in relation to ten is two fifths of it and by the same principle, three [in relation] to five is three fifths of five." The note is ascribed to the "*Commentary on Farāʾid* of al-Sayyid".[13]
- "The Sheikh said, in the Arithmetic [section] of the *Shifāʾ*, concerning the derivation of the perfect [number]:[14] there occurs a test for some people, namely that any even [number] multiplied into a prime number, whatever it may be, and additionally if there exists an even-times-even [number] greater than half that prime number by half of the unit, then their product is always a perfect number.[15] For example when two [is

[13] *ʿIlm al-farāʾid* is the branch of Islamic jurisprudence dealing with inheritance regulations. A legal scholar adjudicating inheritance distribution would routinely be required to deal with fractional parts. I have not been able to identify this commentary or its author. Perhaps this treatise is also the source of other notes ascribed to "Sayyid". In any case, the note suggests that the commentator who first penned these marginalia had a familiarity with this branch of legal literature and that he presumed his readers would have a similar interest.

[14] The Sheikh is a reference to Ibn Sīnā (c. 369 / 980 – 428 / 1037), compiler of the massive philosophical compendium, *Kitāb al-Shifāʾ*. The comment is referring to Euclid's discussion of perfect numbers in *Elements* IX, 36. The phrases in italic type appear only in Yahuda 4848.

[15] That is, a perfect number will always be the product of an even and an odd number, but not every such product is perfect. The product will be a perfect number only if there exists an even-times-even number which exceeds half the prime number by a half. The product of this even-times-even number and the prime number yields a perfect number. In more modern formulation, we can say that if, in the expression $1+2+4+8...+2^{k-1} = 2^k-1$,

multiplied] into three and four into seven *and sixteen into thirty one* then the result in the first example is six and in the second, twenty eight *and in the third, four hundred ninety six*."[16]

IV Are the two manuscripts independent?

As we contemplate these examples of shared marginalia, we might also ask whether BSB arab. 2697 could have been copied from Yahuda 4848. The available evidence initially appears to be ambiguous. Some diagrams contain identical internal features, not just identical mathematical content. For example, an unattributed diagram placed beside proposition V, 11 in BSB arab. 2697 has been constructed exactly like the Ḥajjāj diagrams for propositions V, 9 and 10. The values it contains, however are exactly those found in al-Ṭūsī's diagram for proposition V, 10. And in Yahuda 4848 we find an unattributed diagram in the margin beside the end of proposition V, 10 and beside the *abjad* number of proposition 11. It has been constructed exactly like the diagrams contained in al-Ṭūsī's text (that is, with neat careful vertical line segments, unlike the freehand sketches of the diagrams that are attributed to Ḥajjāj), suggesting that it may be a diagram mistakenly omitted by the copyist of Yahuda 4848 and supplied in the margin. Moreover, the numerical values included in this Yahuda diagram, although difficult to make out, appear to be consistent with the values found in the margin diagram of BSB arab. 2697. In any case, since the numerical values are not consistent with the text of proposition 11, it is difficult to account for this unattributed diagram labeled with the *abjad* label 11 in BSB arab. 2697 unless the copyist were looking at Yahuda 4848. Similarly, in the diagram for proposition VIII, 19, the alternative diagrams in both Yahuda 4848 and BSB arab. 2697 omit the same set of letters. It is somewhat difficult to imagine that these parallel omissions are completely independent.[17] Of course these features could also be explained by being copied from a single source which had omitted some diagram labels.

Despite these striking similarities, however, I believe that BSB arab. 2697 was not simply copied directly from Yahuda 4848. First, the numerals inserted into the diagrams of al-Ṭūsī's text are not the same in the two manuscripts. Thus if BSB arab. 2697 had been copied from Yahuda 4848, we must assume either that these numbers were added after the

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the value $2^k - 1$ is a prime number, then the product $2^{k-1}(2^k - 1)$ is a prime number when k is an integer greater than 1.

Nicomachus defined the even-times-even number as a number which can be divided into two equal parts, each of which can be divided into two equal parts, and so on until the division of the successive even parts reaches the unit. One can produce the even-times-even numbers by beginning from the unit and carrying out successive doublings to yield the sequence 1–2–4–8–16–32 . . . .

[16]The phrase in italic is found in Yahuda 4848 but not in BSB arab. 2967.

[17]Additional examples of an apparent close linkage between the margin diagrams of the two manuscripts are indicated in the Appendix.

copying process was completed or that the copyist of BSB arab. 2697 deliberately chose to insert different numbers into the diagrams while copying the text. Moreover, the alternative diagrams in BSB arab. 2697 usually appear more precise and sometimes are more complete than those of Yahuda 4848. It would require an astute and mathematically literate copyist to complete the missing or obscure elements of the alternative diagrams found in Yahuda 4848. Furthermore, we can see that in a few cases the marginal note in Yahuda 4848 contains more information than the corresponding note in BSB arab. 2697. One could only explain this observation by assuming that the copyist had deliberately chosen to ignore some statements found in the marginalia of Yahuda 4848 — a scenario that seems unlikely.

But even if BSB arab. 2697 was not copied directly from Yahuda 4848, could the two manuscripts have shared a common textual ancestor for their shared marginalia? Since these two manuscripts share both the collection of alternative diagrams ascribed to al-Ḥajjāj and considerable verbal marginalia, it seems reasonable to assume that these common marginalia may have been added to an early copy of al-Ṭūsī's *Taḥrīr* by some unknown scholiast . If this hypothesis is correct, the collection of shared marginalia must have been compiled some time after 646 AH / AD 1248, the date when al-Ṭūsī completed his *Taḥrīr*, but earlier than the date of copying of Yahuda 4848 in 736 AH / AD 1336. I consider it unlikely that the shared marginalia in the two manuscripts were copied from the same source. First, the two manuscripts do not contain exactly the same marginalia. At first glance, it appears that Yahuda 4848 has a more extensive collection of marginalia since its margins frequently appear more crowded. This appearance may be misleading, however. The margins of Yahuda 4848 are significantly smaller than those of BSB arab. 2697, so that the margins of the latter manuscript would naturally appear to be less crowded. Moreover, the number of diagrams attributed to al-Ḥajjāj in the margins of the two manuscripts are not exactly the same — there are more alternative diagrams in Yahuda 4848 than in BSB arab. 2697 (Table 1). If both sets of al-Ḥajjāj diagrams had been copied from the same source, we would expect identical collections of diagrams and identical attributions.

I believe the most likely scenario to be that the marginalia in our manuscripts derive from a common source transmitted through an unknown number of intermediaries. In the transmission process, some differences in diction and in diagram construction have been introduced. Thus the core collection of shared marginalia represent, in my view, two branches of a transmission that apparently originated in a fairly early copy of the *Taḥrīr*.

V Diagrams ascribed to al-Ḥajjāj

This collection of alternative diagrams appears in the margins of books V and VII–X of these two manuscripts. There are 58 alternative diagrams in the margins of Yahuda 4848 and 49 in BSB arab. 2697. Of these, 44 are explicitly ascribed to Ḥajjāj in Yahuda 4848 and 37 are ascribed to Ḥajjāj in BSB arab. 2697 (see Table 1). The collection of alternative diagrams extends to proposition X, 8 in Yahuda 4848, while the last diagram in the BSB arab. 2697 collection accompanies proposition X, 1. Most of these diagrams appear in both

treatises, usually with ascriptions to al-Ḥajjāj. A few margin diagrams in Yahuda 4848 carry the attribution to al-Ḥajjāj but are unattributed in BSB arab. 2697. There are only eight alternative diagrams that do not carry an attribution to Ḥajjāj in at least one of the two manuscripts (see Table 2 and Table 3 in section 10). Although we find additional diagrams in the margins of BSB arab. 2697, most appear to be text diagrams which the copyist apparently neglected to insert while copying the text of al-Ṭūsī. They can be distinguished from the collection of diagrams attributed to al-Ḥajjāj both by their vertical (rather than horizontal) orientation and by the color conventions used in the diagram.

	Yahuda 4848		BSB arab. 2697	
	Ḥajjāj	Total	Ḥajjāj	Total
Book V	7	9	7	8
Book VII	23	24	18	21
Book VIII	6	11	9	11
Book IX	6	12	2	8
Book X	2	3	1	1

Table 1: Number of diagrams ascribed to al-Ḥajjāj and total number of ascribed and unascribed margin diagrams for each manuscript.

The redaction of al-Ṭūsī is itself accompanied by diagrams, as one would expect in a treatise on Euclidean geometry. These diagrams typically follow the basic construction and labeling patterns of the diagrams found in the extant Arabic primary transmission manuscripts, all of which have descended from the transmission attributed to Isḥāq ibn Ḥunayn as revised by Thābit ibn Qurra. The primary transmission manuscripts still exstant have, it appears, all experienced, to one degree or another, contamination or influence from the earlier, now-submerged translation of al-Ḥajjāj. One indirect evidence of this contamination is the bifurcation in the textual transmission occurring primarily in books V [Engroff, 1980], and books VII–IX [De Young, 1981] and the beginning of book X . In addition, several independent lines of testimony purportedly quote from the Ḥajjāj transmission or report differences in ordering or propositions or definitions in that transmission. These testimonia show little overlap with one another, however, making it difficult to substantiate the various attributions.

Apart from the collection of alternative diagrams under study now, we have little explicit evidence about the diagrams that may have been used by al-Ḥajjāj. One report, referring to *Elements* X, 17 and 18, states that al-Ḥajjāj and Isḥāq differed only in the placement of the diagram letter labels but does not state what the different placement involved. There appears to be some textual evidence indicating an alternative lettering of the diagrams for these two propositions, but it comes from the Latin transmission, not from the Arabic [De Young 2002-2003, 160]. This report provides another tantalizing hint how the diagrams of al-Ḥajjāj may have differed from those of Isḥāq, but no alternative diagram in our collection corresponds to this report.

VI Comparing al-Ḥajjāj diagrams and al-Ṭūsī's diagrams

Since only some propositions have been endowed with these alternative diagrams, we assume that the diagrams of al-Ṭūsī for other propositions (following the basic conventions of the Isḥāq–Thābit transmission) did not differ substantially from those attributed to al-Ḥajjāj. The alternative diagrams attributed to al-Ḥajjāj differ from those in the al-Ṭūsī's *Taḥrīr* in several ways. I have already discussed these differences elsewhere [De Young, 2005], so I merely summarize and refine my conclusions here:

- In both Yahuda 4848 and BSB arab. 2697 we find numerical values inserted into the majority of the diagrams of books V and books VII–IX, apparently intended as examples to verify the principle being stated. Numerical values are also inserted into the alternative diagrams attributed to al-Ḥajjāj. These numerical values differ from those found in the text diagrams. Sometimes the numerical values are the only discernable difference between the two diagrams. And even when the same numerical values are used in both the text and the alternative diagram, the values are frequently arranged differently. For example, when illustrating numbers that are continuously proportional, the *Taḥrīr* typically begin with smaller and proceed to larger values (8–12–18–27), while the alternative diagram begin with larger and proceed to smaller values (27–18–12–8).
- Both Yahuda 4848 and BSB arab. 2697 represent magnitudes and numbers using line segments. Whenever possible these line segments are labeled using a single letter. This single letter is typically situated at the center of the line segment. The numerical values are placed either at the top of the line segment or more commonly slightly above or beside the letter label. The alternative diagrams in book VII attributed to al-Ḥajjāj, however, regularly use two letters, each labeling one of the endpoints of the segment, with the numerical value centered between them. It seems as though al-Ḥajjāj could never quite forget that these diagrams are only visual representations of a magnitude or number and not true line segments. In some of the alternative diagrams, the lines segments are completely omitted and one finds only pairs of letters, apparently representing the labels of lines, with the corresponding numerical values placed between them. Even when no line segments are physically present, al-Ḥajjāj apparently felt compelled to demarcate their implicit endpoints.
- The line segments representing magnitudes or numbers in Yahuda 4848 and BSB arab. 2697 are typically placed vertically in square or rectangular openings within the text — although when insufficient space was left by the copyist of BSB arab. 2697, the diagrams were sometimes rotated ninety degrees.[18] The typical orientation for the alternative al-Ḥajjāj diagrams in the margin of BSB arab. 2697 is horizontal. In Yahuda 4848, the alternative diagrams can assume different orientations but in many cases

[18]The diagram of proposition V, 9 is rotated only 45° anticlockwise, a rather rare occurrence.

abandoning the horizontal orientation appears to be constrained by space available in the margins. It is not known whether this apparent preference for a horizontal orientation was merely a visual technique adopted by the copyists to differentiate the diagrams attributed to al-Ḥajjāj from the diagrams of the *Taḥrīr* or was intended to mirror the actual orientation favored by al-Ḥajjāj.

Behind differences between the text diagrams of al-Ṭūsī and the alternative diagrams ascribed to al-Ḥajjāj, there is a distinct difference between the alternative diagrams for book VII and those for books VIII and IX. The alternative diagrams for book VII emphasize the use of double letter labeling for each line segment, as noted above.[19] In books VIII and IX, however, we find an entirely different technique is employed. No more double-letter labeling appears, but now the diagrams consist only of letters (the line segments have apparently been completely suppressed) with numberical values written below each letter. In these two books, the major difference between the textual diagrams and the alternative diagrams appears to be the numerical values employed rather than anything inherent in the diagram architecture itself. It is at present impossible to guess why this shift of technique should occur at this point in the treatise. But a parallel shift from double letter labeling to single letter labeling is observed to occur between books VII and VIII in manuscripts belonging to the Group B component of Arabic primary and secondary transmission. Since it mirrors changes that appear in the primary Arabic transmission, this stylistic change observed in these alternative diagrams, whatever its rationale, seems unlikely to represent merely the ideosyncratic technique of an individual copyist or annotator.

VII The attribution to al-Ḥajjāj

The two collections of diagrams in the margins of Yahuda 4848 and BSB arab. 2697 are united through the explicit attribution of a large majority of them to al-Ḥajjāj. We do not know what the annotators intended when they inserted these alternative diagrams in the margins of the *Taḥrīr*. Because of the confused state of textual evidence found in the manuscripts of the primary Arabic transmission, we should be cautious about accepting attributions to al-Ḥajjāj at face value. No manuscripts containing the original translation by al-Ḥajjāj are extant, so the characteristics of the diagrams they may have contained remain unknown. The diagrams here attributed to al-Ḥajjāj were added some time after the redaction of al-Ṭūsī was completed and thus they must date from some centuries after al-Ḥajjāj completed his translation. Until we can find independent evidence that these alternative diagrams do

[19]But we should note that not all examples of single-letter labelling are replaced by double-lettering. Propositions 3, 4, 12, 13, 14, 17, 18, 23, 24, 25, 26, 27, 32, 33, 37, 38, 39 have alternative diagrams that include the double-letter labeling typical of al-Ḥajjāj. Propositions 19, 21, 22, 29, on the other hand, are not provided with alternative diagrams although they also would seem to be eligible for the double-letter labeling. At present this apparent inconsistency cannot be explained.

indeed represent the diagrammatic features of the earliest Arabic translation, we should be cautious in our interpretation. But whatever their original source may be, and whatever the intent of the annotators may have been when they inserted this collection of alternative diagrams, the readers of these manuscripts would at least have received the message that these diagrams were associated with the transmission of the *Elements* into Arabic by al-Ḥajjāj ibn Yūsuf ibn Maṭar. And these readers would at least have been cognizant of the al-Ḥajjāj transmission from the editorial notes of al-Ṭūsī in his widely read *Taḥrīr kitāb Uqlīdis* [De Young, 2003, 134–138]. We also know that certain features of these alternative diagrams, such as the use of double-letter labeling in book VII, are shared with the Group B branch of the Arabic primary transmission manuscripts — the branch that often appears to have a greater influence from the work of al-Ḥajjāj.

VIII Evidence about the Ḥajjāj transmission

Since the primary Arabic transmission associated with al-Ḥajjāj is no longer extant, our main sources of information are reports or purported quotations in secondary and tertiary sources.[20] This evidence is fragmentary and sometimes difficult to interpret. Most frustrating, perhaps, is the fact that the various bits of information claiming to represent the work of al-Ḥajjāj rarely coincide with one another. Since studies of these pieces of information are scattered in articles, some of which are not easily accessible, I summarize the main lines of evidence here.

- The tenth century biobibliographer, Ibn al-Nadīm, is usually the first source of evidence to be cited. In his *Fihrist* he reports that al-Ḥajjāj first translated the *Elements* under the aegis of Caliph Ḥārūn al-Rashīd (ruled 169 / 786 – 193 / 809). He made a new version or new translation during the caliphate of al-Ma'mūn (ruled 197 / 813 – 217 / 833) in a bid for patronage from the state. Ibn al-Nadīm goes on to mention the translation by Isḥāq ibn Ḥunayn and a possible (partial?) translation by Abū ʿUthmān al-Dimishqī. Later biobibliographers mention additional names (Ḥunayn ibn Isḥāq, Thābit ibn Qurra) as translators. The possible relationships between these purported translations and the work of al-Ḥajjāj remains a matter of debate. Djebbar [1996, 92-95] provides a succinct survey of this biobibliographical evidence.
- The commentary on the *Elements* by al-Nayrīzī (c. 251 / 865 – c. 309 / 922), is extant in at least two incomplete manuscripts (Leiden University 399.1 and Qūm, Kitābkhān-i ʻUmūmī 6265). The Leiden manuscript contains an anonymous preface which explicitly links the text to the second version of al-Ḥajjāj. But Engroff [1980] showed that

[20]It is precisely for this reason that Brentjes [1996] called for historians to take a wider view of the surviving literature instead of focusing too narrowly on the surviving primary documents of the Euclidean transmission. It is, she argued, essential to include evidence from secondary and tertiary sources if one wishes to understand the complex transmission of the *Elements* during the medieval period.

the quotations from earlier authors included by al-Nayrīzī in his commentary had been extensively edited or paraphrased to match the diction of the main text. Given the extent of the editing evidenced in these identifiable quotations, Engroff suggested that it is impossible to be certain whether any of the verbal elements in the text accurately reflect the the second version of al-Ḥajjāj. Perhaps one could attribute structural features of the treatise such as the order of definitions and propositions to al-Ḥajjāj.[21] The actual text of al-Ḥajjāj, however, is now apparently obscured beneath the editing of al-Nayrīzī.

- Manuscripts of the "Andalusian" family within the "Group A" Arabic primary transmission documents (Escurial, ms. arabe 907; Rabat, al-Khizānah al-Mālikiyyah 1101; and Rabat, al-Khizānah al-Mālikiyyah 53) contain quotations explicitly attributed to al-Ḥajjāj of alternative formulations for the enunciations of the first nine propositions of book II. They differ from those enunciations found in the Isḥāq–Thābit text in technical vocabulary (using arithmetical rather than geometrical terminology) but retain the same phrasing as in the other manuscripts of the primary transmission.[22] The verbal features of these alternative enunciations do not reflect what we read in the commentary of al-Nayrīzī, probably because he has edited the diction to suit his own purposes. Another group of interpolations ascribed to al-Ḥajjāj introduces additional cases for several propositions of books III and IV. Curiously, only one of these insertions of added cases is mentioned by al-Ṭūsī as attributable to al-Ḥajjāj. These "Andalusian" manuscripts also contain alternative demonstrations for propositions VIII, 20 and 21 and three condensed alternate demonstrations following proposition X, 67 [De Young, 1991]. Since the commentary of al-Nayrīzī as known today breaks off after a few lines of book VII, we cannot compare these alternative demonstrations to his purported quotations from the second version of al-Ḥajjāj.
- An appendix to book X in Oxford, Bodleian Library, Thurston 11 (folios 156a–157a) reports that the ending of book X in the translation of al-Ḥajjāj is different from that of Thābit. The passage is of interest because it employs technical vocabulary different from that found in the Isḥā–Thābit version. The explicit citation of al-Ḥajjāj in this passage suggests that the alternative terminology may reflect the diction used in his translation [Djebbar, 1996, 100–104.]
- Abū'l-Qāsim 'Alī b. Aḥmad al-Anṭākī, in his comments on the definitions of book

[21] The omission of *Elements* I, 45 and III, 36 from the translation of al-Ḥajjāj, for example, are mentioned by al-Ṭūsī [De Young 2003, 135]. These propositions are also omitted from the commentary of al-Nayrīzī.

[22] In the Hebrew transmission of the *Elements*, Munich, hebreu 36 also has, in the margins of book II, alternative formulations for these propositions ascribed to al-Ḥajjāj [Lévy, 2005]. This formulation does not only change technical vocabulary, but states the enunciation in terms of numbers rather than line segments. So we now have two different alternative formulations for these book II enunciations each of which is attributed to al-Ḥajjāj. It is possible that each represents a different version or edition of the translation of al-Ḥajjāj. Without additional information, though, it is impossible to decide the question.

V, quotes an alternate form of the third definitions of book V, which he attributes to al-Ḥajjāj. His statement is: "A ratio is some relation (*ayyiyyah qadr*) of two magnitudes of the same species" (Oxford, Bodleian Library, Huntington 70, folio 3a). The use of the term *ayyiyyah* has been associated with the translation of al-Ḥajjāj [Djebbar, 1996, 98–100]. Nearly the same quotation, also attributed to al-Ḥajjāj, is found in an anonymous Arabic commentary in Hyderabad, India [De Young 2002-2003, 144]. This anonymous commentary has been shown to include substantial quotations from the commentary of al-Anṭākī, including quotations from the now-lost section on books I–IV [De Young, 2008a], so these two quotations cannot be considered truly independent of one another.

- Three Arabic manuscripts (Copenhagen, Kongelige Bibliothek, Mehren LXXXI; Istanbul, Fatih 3439/1; and Tehran, Majlis Shūrā, 2060/1) contain a note at the beginning of book XI stating that the remainder of the text is from the second version of al-Ḥajjāj. Klamroth [1884] compared the text of these manuscripts with that of other manuscripts and found the differences to be insignificant. He concluded that Isḥāq had not translated this section but simply took over the translation of al-Ḥajjāj. Kunitzsch [1985] examined the question again and reached the conclusion that this block of material represents a transmission different from that of Isḥāq. It is now widely accepted that the attribution to al-Ḥajjāj should be treated with caution. Whether Klamroth's hypothesis concerning Isḥāq's translation technique is correct or whether the extensive similarity of all the surviving manuscripts results from an intermingling of the two transmissions remains a subject of investigation.
- The *Taḥrīr* of al-Ṭūsī contains, among its many mathematical notes, several reports of differences in the ordering of definitions and propositions between the Ḥajjāj and Thābit versions [De Young, 2003, 136-138]. In addition to explicitly inserting commentary notes to indicate where the ordering of al-Ḥajjāj differs from that of Isḥāq, he tells his readers in his introduction that whenever there is a difference in ordering he will indicate the proposition number of Thābit using red and that of Ḥajjāj using black ink. These differences in ordering occur only in book V and books VII – IX.
- The *Taḥrīr* of the *Elements* prepared by the Pseudo-Ṭūsī — so-called because the printed edition of his treatise published in Rome by the Medici Press in 1594 incorrectly names Naṣīr al-Dīn al-Ṭūsī as the author of the redaction — also contains a number of notes on differences between the version of Ḥajjāj and that of Thābit. Several of these statements parallel reports found already in al-Ṭūsī's *Taḥrīr*.

IX Edition of the "Hajjāj" diagrams

I now present an edition of the collection of alternative diagrams ascribed to al-Ḥajjāj based on the evidence in these two manuscripts. In most cases, the ascription occurs in both manuscripts, but in a few cases the ascription appears in only one manuscript although the alternative diagram may appear in both. A few diagrams are unascribed in both manuscripts

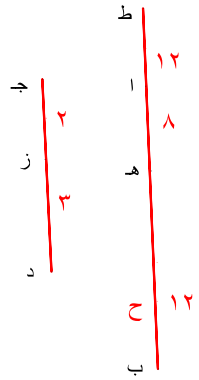
but seem so similar in construction and content that they are probably part of the collection as well. A few diagrams appearing in the margins of BSB arab. 2697 are not really alternative diagrams but represent diagrams omitted from the text. These latter diagrams are not included in the edition.

Editing is never a passive or mechanical activity. The editor is continually called upon to make decisions. This initial excursion into diagram editing is no different. The decisions one must make are global (general policies) and local (regarding individual features of specific diagrams). Although it is difficult if not impossible to discuss every local decision I have made during the editing process, the salient features of individual diagrams are discussed in the notes following the edition of each diagram. I have tried here to indicate the basic global editorial decisions governing my edition of these diagrams.

- I have edited the diagrams from BSB arab. 2697 because they are typically clearer to read and easier to edit. And I did not find the mathematical content of the diagrams in Yahuda 4848 to differ in any material way from those in BSB arab. 2697. But when the alternative diagram appears only in Yahuda 4848, I have used its diagrams.
- Since BSB arab. 2697 is available in full-color scan, I have retained the use of red in my edited diagrams as well. (The DRaFT software allows this kind of flexibility in editing.) Although Princeton University Library has informed me that both red and black ink are used throughout Yahuda 4848, I have only available the black and white microfilm supplied by Princeton University Library. Therefore, I have typically edited its diagrams only in black. But in cases where the evidence is overwhelming (such as the appearance of lines or numerals in much lighter color than normal) I have used red in editing its diagrams as well. But since that use of red is at least somewhat conjectural, I have specifically indicated its use in the notes to the diagrams.
- Whenever I edit a Ḥajjāj diagram from Yahuda 4848, I edit the corresponding diagram of al-Ṭūsī from the same manuscript unless the diagram is illegible or damaged. One should bear in mind that although the diagrams of Yahuda 4848 have the same general architecture, those of BSB arab. 2697 tend to have a higher degree of over-specification and the two manuscripts do not necessarily insert identical numerals in their diagrams.
- I have, in all cases, retained the orientation of the diagram as it exists in the manuscript.
- I have placed the diagram labels as close as possible to their position in the diagram, but I have not tried to retain the original size variations for labels within individual diagrams. Many letter labels are placed directly on the points to which they belong. In the interest of legibility, I have sometimes moved these labels to a position beside the line segment. In these cases, I try as much as possible to retain the position of the label relative to the line segment.
- When editing the diagrams, I regularly replace the independent form of the letter *hā* in the diagram labels, which closely resembles the numeral 5 in Arabic, with the initial form of the letter, which is more easily distinguished from the numeral. Although the

context within the diagram should nearly always make clear whether we have a letter or a numeral, I wish to avoid any possible confusion.

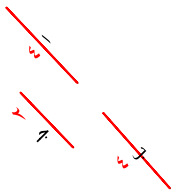
Ḥajjāj, V, 4
BSB arab. 2697, folio 55a



al-Ṭūsī, V, 5

Although the diagram attributed to al-Ḥajjāj is labeled as number four, its structure corresponds to the diagram for proposition five in al-Ṭūsī's *Taḥrīr*. There is no indication in the Arabic transmission (either primary or secondary) of a re-ordering of propositions at this point. The letter *ḥā* in al-Ṭūsī's diagram is written in red because it is part of an alternative demonstration but not part of the primary demonstration.

Ḥajjāj, V, 9
BSB arab. 2697, folio 56b

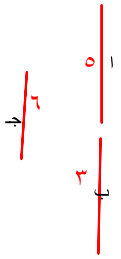


al-Ṭūsī, V, 9

There are three Ḥajjāj diagrams in the top margin of Yahuda 4848, folio 26b (see Plate 1). Two of them have no number now visible. It is impossible to determine which was intended for proposition 9 or for proposition 10. I have arbitrarily identified the diagram at the right as belonging to proposition 9 only because the two diagrams have exactly the same format in both manuscripts. No numerals are visible in Yahuda 4848. It is unusual to find a diagram rotated $45°$ as in this copy of al-Ṭūsī's *Taḥrīr*. Diagrams are rotated primarily because there is insufficient space available to place the diagram in a vertical position. When diagrams are rotated, it is almost always by $90°$.

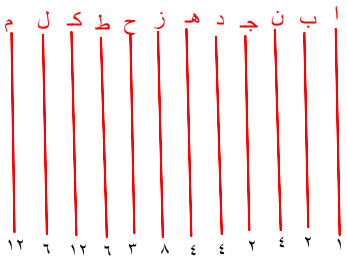
Ḥajjāj, V, 10
BSB arab. 2697, folio 56b

Ḥajjāj V, 10
Yahuda 4848, folio 26b

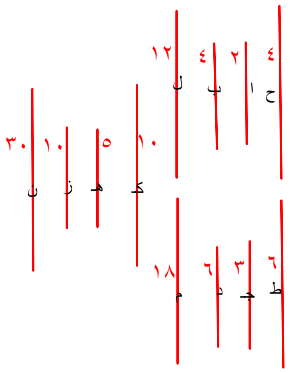


al-Ṭūsī, V, 10

Since I have arbitrarily assigned the diagram at the top right of Yahuda 4848, folio 26b (see Plate 1) to represent the diagram for proposition 9, the other unnumbered diagram should represent proposition 10. But in this case, the two Ḥajjāj diagrams are quite different in format. Hence I have edited both versions here.

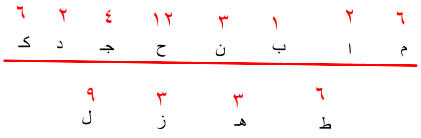


Ḥajjāj, V, 11(?)
BSB arab. 2697, folio 57b

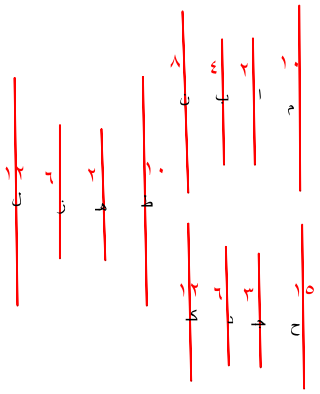


al-Ṭūsī, V, 11

The ascription to Ḥajjāj appears only in Yahuda 4848. It is placed here because in Yahuda 4848 the diagram is given the *abjad* number 11. In BSB arab. 2697, there is no proposition number given, although the diagram is placed on folio 57b, following the page containing the Ḥajjāj diagrams labeled 12 and 11, on the page bearing the diagram of propositions 13 and 14 in al-Ṭūsī's text. Although the number of letters is consistent with any of al-Ṭūsī's propositions 11, 12, 13, the numerical values appear to be consistent only with proposition 11.

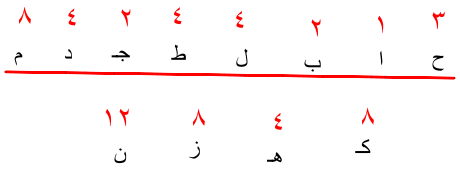


Ḥajjāj, V, 12
BSB arab. 2697, folio 57a

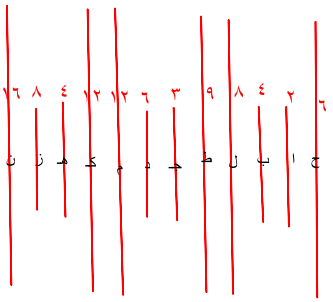


al-Ṭūsī, V, 12

In BSB arab. 2697, this diagrams is labeled as proposition 12 but is placed beside the diagram of al-Ṭūsī's proposition 11. The order of the letters corresponds to the order of letters found in the diagram of al-Ṭūsī's proposition 12. This ordering suggests that in the work of Ḥajjāj the order of al-Ṭūsī's propositions 11 and 12 was inverted. The diagram in Yahuda 4848 (folio 26b) now lacks letter *mīm* due to crumbling manuscript edges.

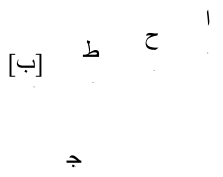


Ḥajjāj, V, 13



al-Ṭūsī, V, 11
BSB arab. 2697, folio 57a

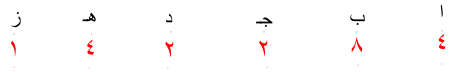
In BSB arab. 2697, this diagram is labeled as proposition 11 but it is placed beside the diagram of al-Ṭūsī's proposition 12. The order of the letters corresponds to te order of letters found in the diagram of al-Ṭūsī's proposition 11. This ordering suggests that in the work of Ḥajjāj the order of al-Ṭūsī's propositions 11 and 12 was inverted. The diagram in Yahuda 4848 (folio 26b) lacks letters *ḥā* and *alif*, lost to the crumbling of the margins.



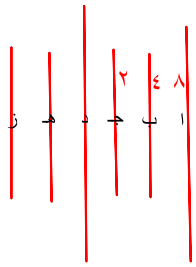
Ḥajjāj, V, 15 (?)
Yahuda 4848, folio 29a

al-Ṭūsī, V, 15

The alternative diagram, which appears only in Yahuda 4848, seems to be a fragment from the diagram of al-Ṭūsī's proposition 15.

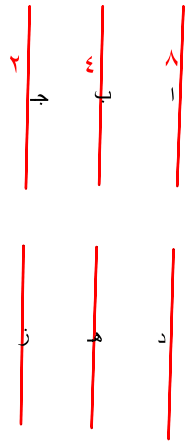


Ḥajjāj, V, 20
BSB arab. 2697, folio 59b

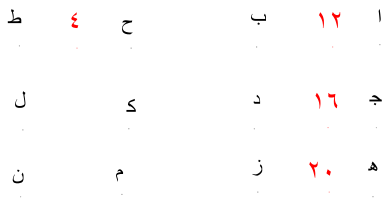


al-Ṭūsī, V, 20

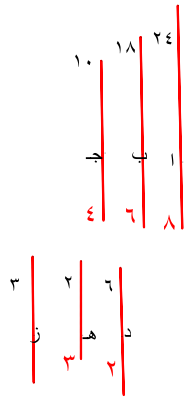
Ḥajjāj, V, 21
BSB arab. 2697, folio 60a



al-Ṭūsī, V, 21

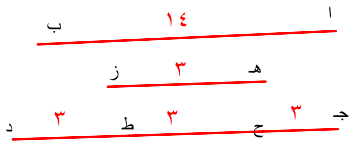


Ḥajjāj, VII, 3
BSB arab. 2697, folio 74b

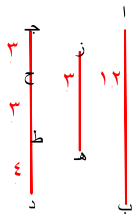


al-Ṭūsī, VII, 3

Although the Ḥajjāj diagram in BSB arab. 2697 does not contain lines, the comparable diagram in Yahuda 4848 (folio 37a) does contain lines. The diagram of al-Ṭūsī contains two sets of numerals, one in red ink and the other in black ink. The explanation seems to be that the proposition corresponds to Euclid's proposition VII, 3, in which two different scenarios are discussed. The red set of numerals represents the first (and simpler) case in which *dāl* represents the greatest common measure of numbers *alif*, *bā*, and *jīm* while the black numerals represent the second case in which *dāl* does not represent the greatest common measure of the three given numbers. The comparable diagram of al-Ṭūsī in Yahuda 4848 does not have a set of numerical double values. In fact, it has numerals only on lines *alif*, *bā*, *jīm*, *dāl*, which correspond to the simpler Euclidean case and to the four line segments with values in the diagram attributed to Ḥajjāj.

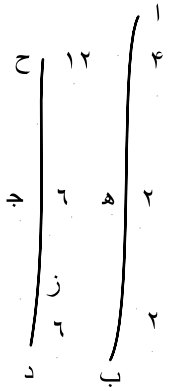


Ḥajjāj, VII, 4
BSB arab. 2697, folio 74b

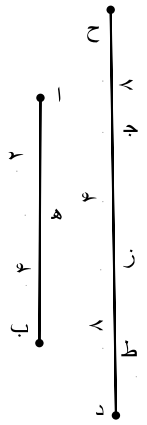


al-Ṭūsī, VII, 4

The alternative diagram in BSB arab. 2697 has been placed beside the text of al-Ṭūsī's proposition 4. But because the text of the proposition ends in the last line of the page, the copyist has placed al-Ṭūsī's diagram on the following page. This separates the alternative diagram both spatially and visually from al-Ṭūsī's diagram.

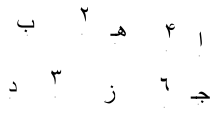


Ḥajjāj, VII, 8
Yahuda 4848, folio 37b



al-Ṭūsī, VII, 8

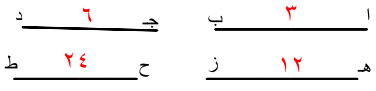
This unattributed diagram appears only in Yahuda 4848. Unlike most of the alternative diagrams, it is placed in the inner margin. There is no proposition number given, but the diagram is placed beside al-Ṭūsī's diagram for proposition 8. It differs from al-Ṭūsī's diagram in omitting the letter *ṭā* and in its numerical values.



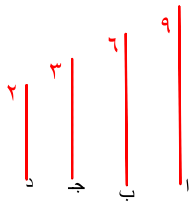
Ḥajjāj, VII, 11
Yahuda 4848, folio 38a

al-Ṭūsī, VII, 11

This diagram appears only in Yahuda 4848. Although no proposition number is assigned, it is placed immediately beside the diagram for al-Ṭūsī's proposition 11. It has the same essential architecture as al-Ṭūsī's diagram for proposition 11, differing only in terms of its numerical values.

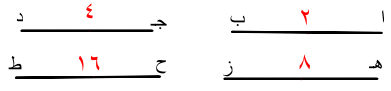


Ḥajjāj, VII, 12
BSB arab. 2697, folio 76a

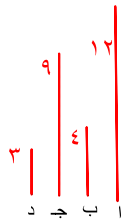


al-Ṭūsī, VII, 12

The alternative diagrams for propositions 12 and 13 are placed as a unit in the upper left corner of the page, although it appears that it would have been possible to place them closer to the propositions to which they refer. They are also placed in the upper left corner of Yahuda 4848 (folio 38a). But in Yahuda 4848, the placement of the marginal notes prevents insertion of the Ḥajjāj diagrams closer to the text of the propositions.

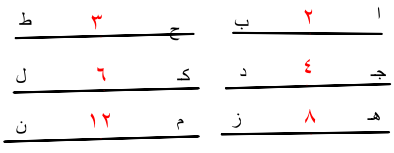


Ḥajjāj, VII, 13
BSB arab. 2697, folio 76a

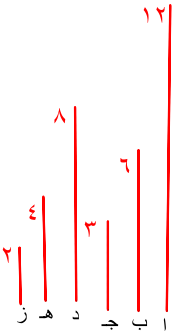


al-Ṭūsī, VII, 13

Although al-Ṭūsī's proposition begins on folio 76a, its conclusion is on the next page, so al-Ṭūsī's diagram is on folio 76b. Thus the alternative diagram is both spatially and visually separated from the diagram of the *Taḥrīr* in BSB arab. 2697.

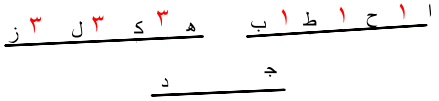


Ḥajjāj, VII, 14
BSB arab. 2697, folio 76b

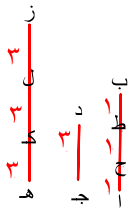


al-Ṭūsī, VII, 14

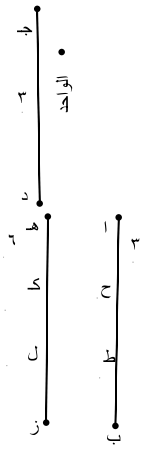
The diagram in BSB arab. 2697 is placed in the upper right corner of the page. Two extended glosses occupy the margin and prevent it from being placed beside the text of al-Ṭūsī's proposition 14. Because these glosses are placed differently in the margins of Yahuda 4848, the Ḥajjāj diagram there appears directly beside al-Ṭūsī's diagram (folio 38a).



Ḥajjāj, VII, 15
BSB arab. 2697, folio 76b

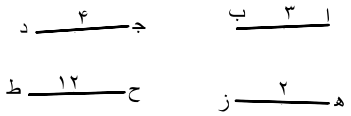


al-Ṭūsī, VII, 15
BSB arab. 2697,
folio 77a

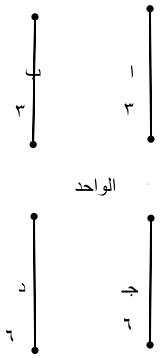


al-Ṭūsī, VII, 15
Yahuda 4848,
folio 38b

Only the diagram in Yahuda 4848 (folio 38b) is ascribed to Ḥajjāj. The values in the alternative diagram are the same as those in the diagram of al-Ṭūsī in BSB arab. 2697. Al-Ṭūsī's diagram has been placed in the margin of BSB arab. 2697 (folio 77a), although it retains the usual archetectural features of al-Ṭūsī's diagrams. The copyist has, however, forgotten to include the unit in al-Ṭūsī's diagram. I have shown also al-Ṭūsī's diagram from Yahuda 4848 (folio 38b) for comparision.

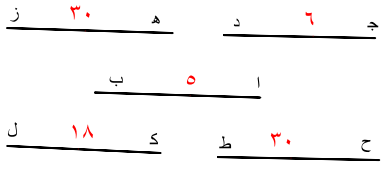


Ḥajjāj, VII, 16
Yahuda 4848, folio 38b

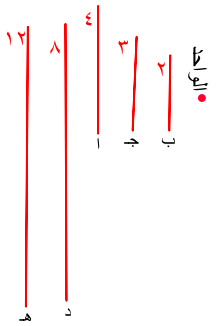


al-Ṭūsī, VII, 16

This alternative diagram, ascribed to Ḥajjāj, is found only in Yahuda 4848. The alternative diagram does not include the unit although it is included in the diagram of al-Ṭūsī.

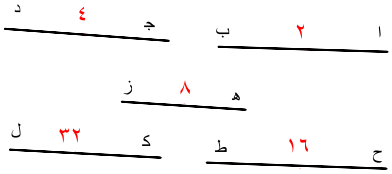


Ḥajjāj, VII, 17
BSB arab. 2697, folio 77a

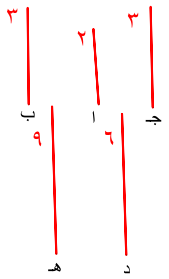


al-Ṭūsī, VII, 17

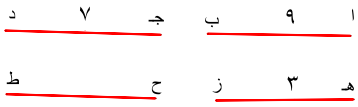
In BSB arab. 2697, the lines of the alternative diagrams for propositions VII, 17 and VII, 18 are drawn with black rather than the more usual red ink.



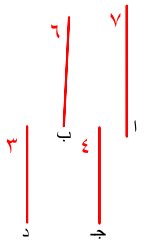
Ḥajjāj, VII, 18
BSB arab. 2697, folio 77a



al-Ṭūsī, VII, 18

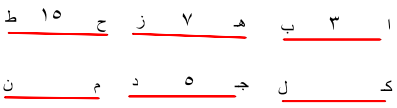


Ḥajjāj, VII, 23
BSB arab. 2697, folio 78a

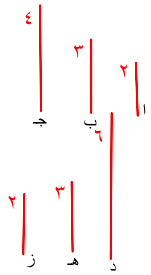


al-Ṭūsī, VII, 23

For the remainder of book VII, the copyist of BSB arab. 2697 has written the numerals with black ink. Until this point, they have been consistently written with red ink. There is no obvious reason for this change in color of ink.

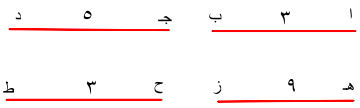


Ḥajjāj, VII, 24
BSB arab. 2697, folio 78a

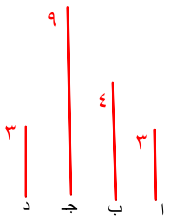


al-Ṭūsī, VII, 24

The value attached to *alif–bā* in the alternative diagram in BSB arab. 2697 is ambiguous — it could be read either 2 or 3. But the value is clearly 3 in the alternative diagram in Yahuda 4848. This value is also consistent with the mathematical content of the proposition.

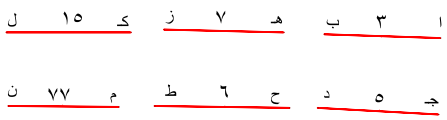


Ḥajjāj, VII, 25
BSB arab. 2697, folio 78a

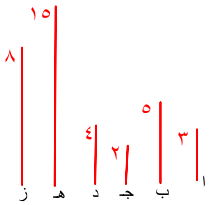


al-Ṭūsī, VII, 25

In Yahuda 4848, this diagram is placed in the upper margin without a proposition number. It can be identified by comparison with the diagrams in BSB arab. 2697 (where the diagram is placed immediately beside the proposition) or, with less certainty, by comparison with the diagrams of al-Ṭūsī.

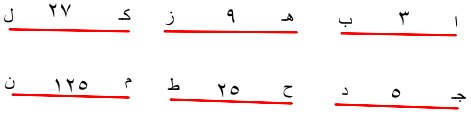


Ḥajjāj, VII, 26
BSB arab 2697, folio 78a

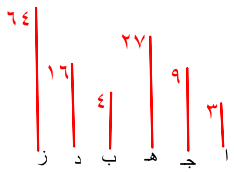


al-Ṭūsī, VII, 26

In Yahuda 4848, this diagram is placed in the upper margin without a proposition number. It can be identified by comparison with the diagrams in BSB arab. 2697 (where the diagram is placed immediately beside the proposition) or, with less certainty, by comparison with the diagrams of al-Ṭūsī.

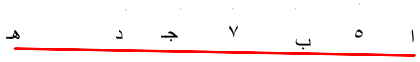


Ḥajjāj, VII, 27
BSB arab. 2697, folio 78b

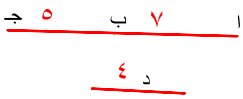


al-Ṭūsī, VII, 27

The label *jīm* has been lost from the diagram in Yahuda 4848 (folio 39b) due to the crumbling manuscript edges.



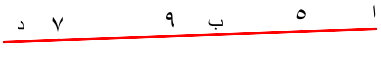
Ḥajjāj, VII, 28
BSB arab. 2697, folio 78b



al-Ṭūsī, VII, 28

The ascription to Ḥajjāj appears only in Yahuda 4848 (folio 39b). The alternative diagram is now incomplete — the *alif* and part of the numeral five are now lost to the crumbling of the manuscript edges. The copyist of BSB arab. 2697 has drawn the alternative diagram as one connected line. This appears to be an error. Based on the mathematical content of the proposition, the portion of the line labeled *dāl–hā* should be an independent line segment, as is the case in the alternative diagram in Yahuda 4848.

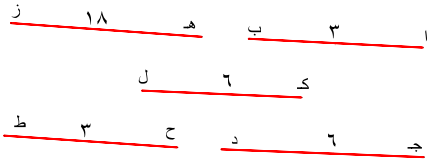
.



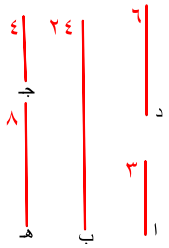
Ḥajjāj, VII, 30
BSB arab. 2697, folio 78b

al-Ṭūsī, VII, 30

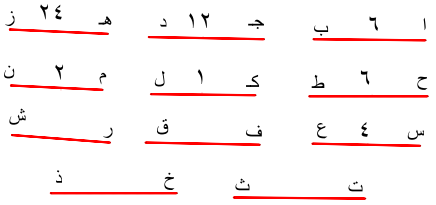
In the diagram for proposition 30, the copyist of BSB arab. 2697 has again drawn the alternative diagram as a single line. The current diagram is also unusual because it contains two numbers between *bā* and *dāl* and has no label *jīm*. Perhaps the numeral 9 was once letter *jīm* and the diagram originally contained two line segments? The letter *jīm* replaces the value 9 in the Ḥajjāj diagram in Yahuda 4848. The copyist of BSB arab. 2697 has also omitted the diagram that should appear in al-Ṭūsī's text for proposition 31, so it has been placed in the margin at the top of folio 79a. It is difficult to correlate al-Ṭūsī's diagram with the alternative diagram in the margin.



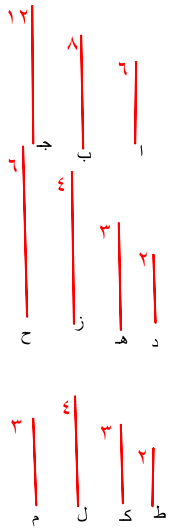
Ḥajjāj, VII, 32
BSB arab. 2697, folio 79a



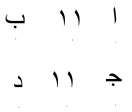
al-Ṭūsī, VII, 32



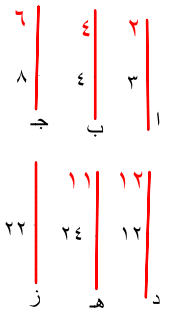
Ḥajjāj, VII, 33
BSB arab. 2697, folio 79a



al-Ṭūsī, VII, 33

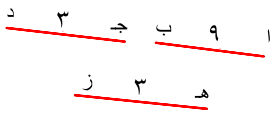


Ḥajjāj, VII, 36
BSB arab. 2697, folio 79b

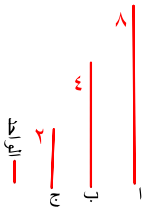


al-Ṭūsī, VII, 36

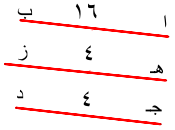
Although there are no lines in the diagram from BSB arab. 2697, there are lines visible in Yahuda 4848 (folio 39bis a). It is difficult to understand how this alternative diagram can be understood in relation to the text of the proposition. It corresponds to proposition 37 in the Greek text. The diagram in the BSB arab. 2697 copy of al-Ṭūsī's text contains two sets of numerals. (There are no numerals discernable in the text diagram in Yahuda 4848.) The red numerals apparently represent the first case discussed in Euclid's demonstration, in which the third given number measures the fourth number. The black numerals represent the second case, in which the third given number does not measure the fourth.



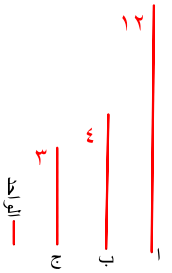
Ḥajjāj, VII, 37
BSB arab. 2697, folio 80a



al-Ṭūsī, VII, 37

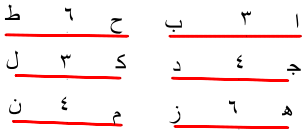


Ḥajjāj, VII, 38
BSB arab. 2697, folio 80a

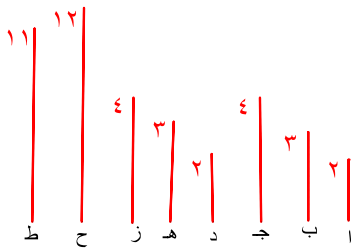


al-Ṭūsī, VII, 38

The letters *alif*, *hā*, and *jīm* have all been lost from the diagram in Yahuda 4848 (folio 39bis b) in the crumbling of the manuscript edges.

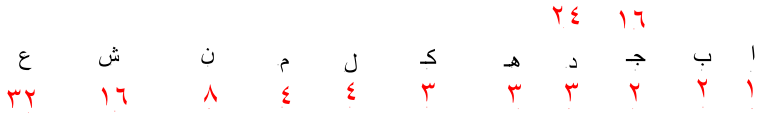


Ḥajjāj, VII, 39
BSB arab 2697, folio 80a

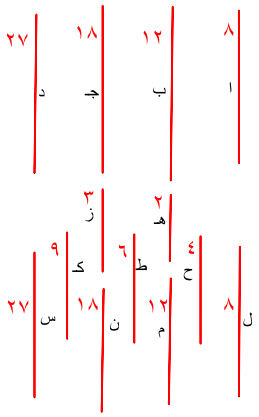


al-Ṭūsī, VII, 39

The ascription to Ḥajjāj appears only in Yahuda 4848 (folio 39bis b). There are no numbers visible in the Yahuda 4848 diagram of al-Ṭūsī.



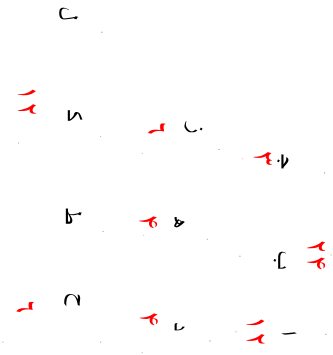
Ḥajjāj, VIII, 4
BSB arab. 2697, folio 81a



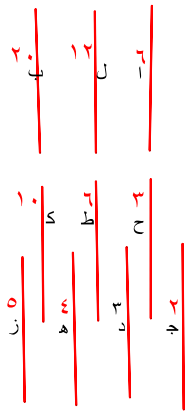
al-Ṭūsī, VIII, 3

In BSB arab 2697, the Ḥajjāj diagram is given the *abjad* number *dāl*. There is no proposition number visible in the microfilm of Yahuda 4848. In both manuscripts, the diagram is placed in the top margin, rather than beside the text of al-Ṭūsī. Three traditional letters are missing from each diagram: *zay*, *ḥā*, and *ṭā*. The Ḥajjāj diagram also contains non-traditional letters (*shīn* and *ʿayn*) following the letter *nūn*. Moreover, in each Ḥajjāj diagram, there are double values attached to letters *jīm* and *dāl*. The Ḥajjāj diagram does not directly correspond to any diagram in al-Ṭūsī's text, although in terms of number of elements, it is somewhat closer to al-Ṭūsī's proposition 3, which is edited here for comparison.

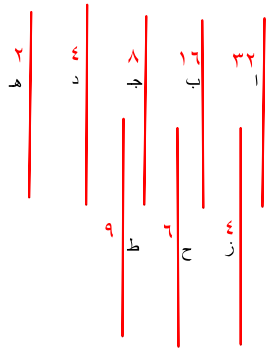
Ḥajjāj, VIII, 6 (?)
BSB arab. 2697, folio 81b



Ḥajjāj, VIII, 6
Yahuda 4848, folio 40a



al-Ṭūsī, VIII, 5

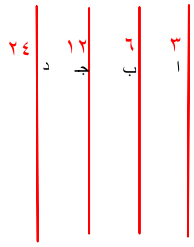


al-Ṭūsī, VIII, 6

The two Ḥajjāj diagrams differ slightly, so I have included both in my edition. The numerals in Yahuda 4848 appear much lighter than the letters, implying that they have been written with red ink, so I have inserted them in color in my edition as well. The numerals for *ṭā* and *lām* are missing or unreadable. In BSB arab. 2697, the Ḥajjāj diagram has no number but is placed beside proposition 6. In Yahuda 4848, it has the *abjad* label *wāw* (6) and is placed opposite the beginning of proposition 5. (The location may be accidental because the diagram has several verbal notes crowded around it.)

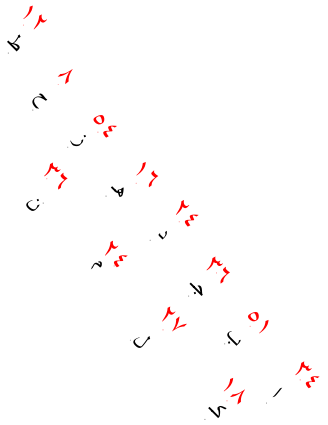
The Ḥajjāj diagram in BSB arab. 2697 does not match the diagram for al-Ṭūsī's proposition 6 since the latter lacks the letter *kāf*. Nor does it match al-Ṭūsī's diagram for proposition 5 because it lacks the letter *lām*. In the Ḥajjāj diagram of Yahuda 4848, however, the letter *lām* is present. Thus the diagram apparently corresponds to al-Ṭūsī's fifth, not his sixth, proposition. This reordering of propositions implicitly attributed to Ḥajjāj does not reflect any known ordering in the extant Arabic manuscripts.

Ḥajjāj, VIII, 7
BSB arab. 2697, folio 82a

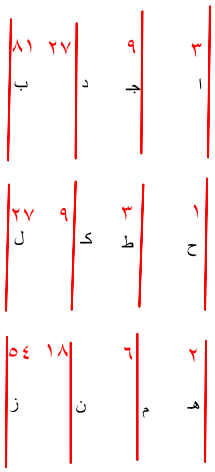


al-Ṭūsī, VIII, 7

This Ḥajjāj diagram is labeled *zay* (7) also in Yahuda 4848 (folio 40a), where the last part of the diagram (the letter *ṭā*) has been lost due to the crumbling manuscript edges. The diagram does not correspond to the content of al-Ṭūsī's seventh diagram. In structure it appears to correspond to the diagram of al-Ṭūsī's proposition 6 — see the previous diagram, but the numerical values seem incorrect for this proposition. There is no report of a reordering in the primary or secondary Arabic transmission.

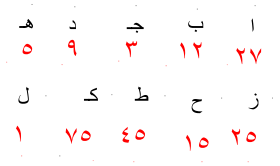


Ḥajjāj, VIII, 8
BSB arab. 2697, folio 82a

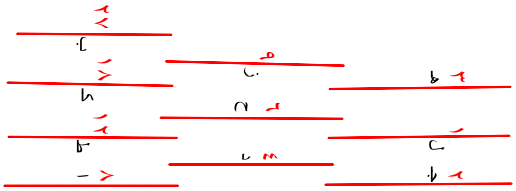


al-Ṭūsī, VIII, 8

The diagram in Yahuda 4848 (folio 40b) lacks the attribution to Ḥajjāj. And the letter *nūn*, along with the values for letters *lām* and *nūn* have been lost due to the crumbling manuscript edges.

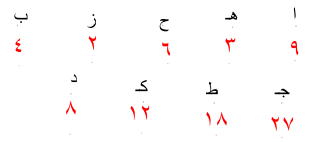


Ḥajjāj, VIII, 11
BSB arab. 2697, folio 82b

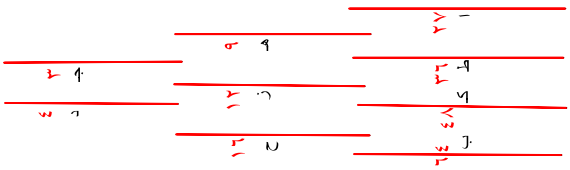


al-Ṭūsī, VIII, 10

The Ḥajjāj diagram has no proposition number attached but it is placed opposite the beginning of al-Ṭūsī's proposition 11 in BSB arab. 2697, but in terms of its content it must represent proposition 10. This placement is somewhat puzzling because there is ample space to insert the diagram beside the text of proposition 10. It is also placed beside the text of proposition 10 in Yahuda 4848 (folio 40b), but appears to be given the proposition number 11. The alternative diagram in Yahuda 4848 also lacks the ascription to Ḥajjāj. The rotation of al-Ṭūsī's diagram is a local adjustment in order to fit the diagram on the page. (The diagram is not rotated in Yahuda 4848.) Another solution might have been to decrease the length of the line segments, but apparently the copyist chose not to do so.

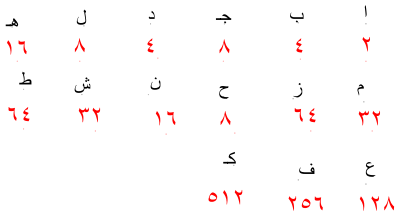


Ḥajjāj, VIII, 12
BSB arab. 2697, folio 83a

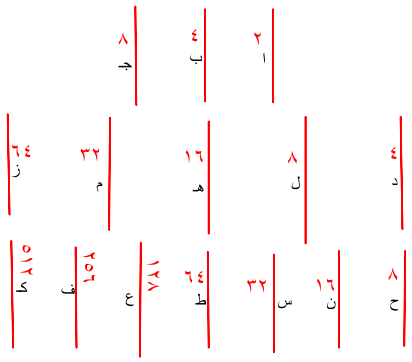


al-Ṭūsī, VIII, 12

The alternative diagram lacks the attribution to Ḥajjāj in Yahuda 4848 (folio 41a). Al-Ṭūsī's diagram in BSB arab. 2697 has again been rotated by the copyist in order to save space on the page since the text of the proposition does not require extensive space. Why the copyist chose to rotate this diagram in the opposite direction to that used in proposition 10 is not known. Perhaps an unexpressed aesthetic demands that the widest portion of the diagram (its "base") be oriented to the inner margin of the page?



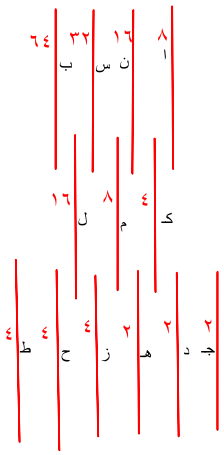
Ḥajjāj, VIII, 13
BSB arab. 2697, folio 83a



al-Ṭūsī, VIII, 13

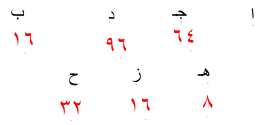
The first three letters of the diagram along with their numerical values as well as any ascription to Ḥajjāj are now missing from Yahuda 4848 (folio 41a) due to the crumbling manuscript edges. Since the alternative diagram contains the same numerical values as found in al-Ṭūsī's diagram, the Ḥajjāj alternative seems to offer no additional or different information than already contained in the diagram of al-Ṭūsī. It is not known why it has been inserted at this point.

Ḥajjāj, VIII, 17
BSB arab. 2697, folio 84a

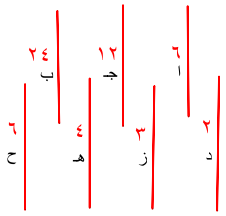


al-Ṭūsī, VIII, 17

The alternative diagram lacks the ascription to Ḥajjāj in Yahuda 4848 (folio 41b) and the diagram has been damaged due to crumbling manuscript edges — the values for letters *ḥā* and *ṭā* are now missing.

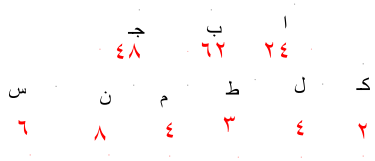


Ḥajjāj, VIII, 18
BSB arab. 2697, folio 84a

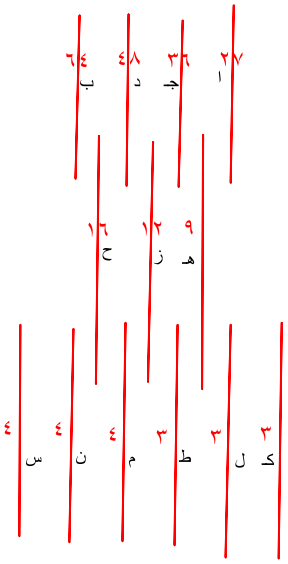


al-Ṭūsī, VIII, 18

Only the alternative diagram in Yahuda 4848 (folio 41b) carries an ascription to Ḥajjāj. That diagram also lacks the numerical value for the letter *alif*.

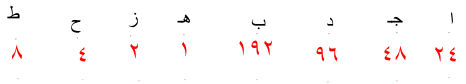


Ḥajjāj, VIII, 19
BSB arab. 2697, folio 84b

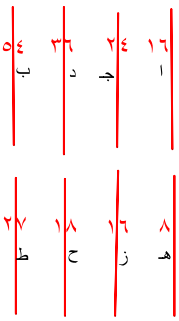


al-Ṭūsī, VIII, 19

Only the alternative diagram in the margin of Yahuda 4848 (folio 42a) carries the ascription to Ḥajjāj. Most of the numerical values in this diagram are illegible or invisible in the microfilm. The letter *dāl* is missing from the diagram. And, like BSB arab. 2697, the copyist has also omitted letters (*hā*, *zay*, and *ḥā*) from the diagram.

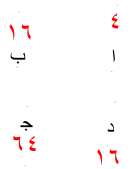


Ḥajjāj, VIII, 25/27
BSB arab. 2697, folio 85b

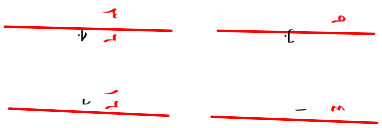


al-Ṭūsī, VIII, 27

The alternative diagram carries a dual set of *abjad* proposition numbers — 25 and 27. This observation can be readily explained. Al-Ṭūsī reports in the incipit of book VIII that the text of Thābit exceeded that of al-Ḥajjāj by two propositions, namely 24 and 25 (BSB arab. 2697, folio 80a) and at the end of proposition 25, he includes a brief note that "these two propositions [referring to propositions 24 and 25] are not in the text of al-Ḥajjāj" (BSB arab. 2697, folio 85b). Similarly, the *Taḥrīr* of the Pseudo-Ṭūsī notes, following proposition 23, that what he has treated as porisms in this and the previous proposition Thābit ibn Qurra had made into propositions 24 and 25 [De Young, 2002-2003, 151-152]. The important point of this diagram for the annotator, though, is not the numbering of the proposition but the numerical values it includes. The numerical values included in Yahuda 4848 (folio 42b) are impossible to decipher from the microfilm.

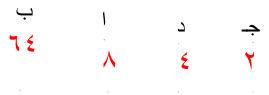


Ḥajjāj (?), IX, 1
BSB arab. 2697, folio 85b

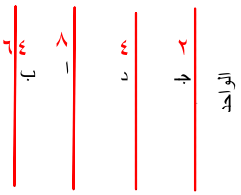


al-Ṭūsī, IX, 1

This alternative diagram is unattributed in both manuscripts. Its similarity to other attributed diagrams suggests that it probably belongs to the collection of Ḥajjāj diagrams.



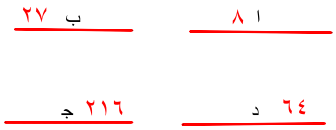
Ḥajjāj, IX, 3
BSB arab. 2697, folio 86a



al-Ṭūsī, IX, 3

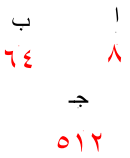
The numerical values in the diagram ascribed to Ḥajjāj do not differ from those in the diagram of al-Ṭūsī, except that Ḥajjāj does not include the unit. No numerical values can be read in the alternative diagram in Yahuda 4848 (folio 42b).

Ḥajjāj (?), IX, 5
BSB arab. 2697, folio 86a

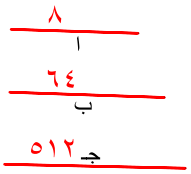


al-Ṭūsī, IX, 5

This alternative diagram is unattributed in both manuscripts. Its similarity to other attributed diagrams suggests that it probably belongs to the collection of Ḥajjāj diagrams. The value for letter *jīm* is either missing or illegible in Yahuda 4848 (folio 43a).



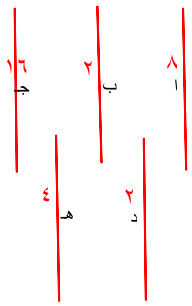
Ḥajjāj (?), IX, 6
BSB arab. 2697, folio 86b



al-Ṭūsī, IX, 6

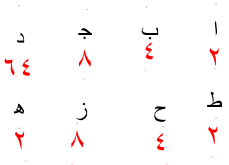
This alternative diagram is unattributed in both manuscripts. Its similarity to other attributed diagrams suggests that it probably belongs to the collection of Ḥajjāj diagrams. Since its values and comosition are identical to those in that found in the diagram of al-Ṭūsī, the reason for its inclusion here is uncertain.

Ḥajjāj (?), IX, 7
BSB arab. 2697, folio 86b

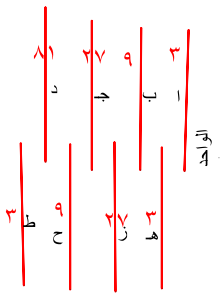


al-Ṭūsī, IX, 7

This alternative diagram is unattributed in both manuscripts. Its similarity to other attributed diagrams suggests that it probably belongs to the collection of Ḥajjāj diagrams.

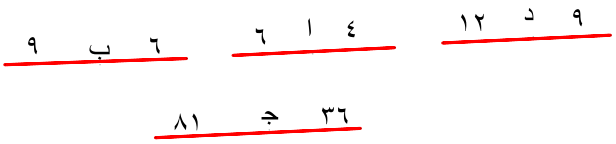


Ḥajjāj , IX, 11
BSB arab. 2697, folio 87a

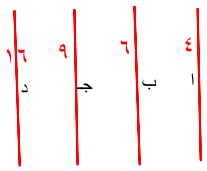


al-Ṭūsī, IX, 12

This alternative diagram is attributed to Ḥajjāj only in Yahuda 4848 (folio 43b) but most of the diagram has been lost due to the crumbling margins of the manuscript. In Yahuda 4848 it is given proposition number 11, while in BSB arab. 2697 it is not given a number but is placed beside proposition 12.

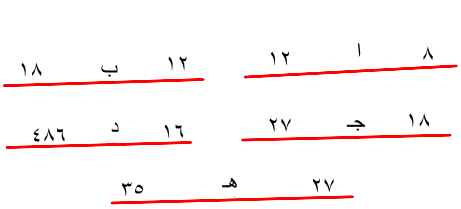


Ḥajjāj, IX, 18
BSB arab. 2697, folio 88b

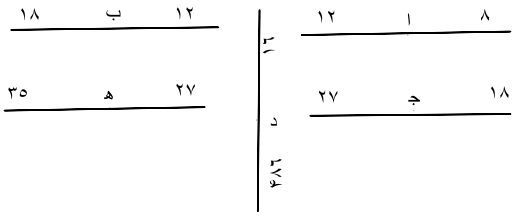


al-Ṭūsī, IX, 19

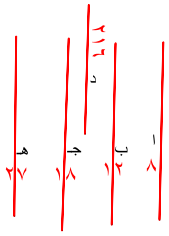
This diagram is ascribed to Ḥajjāj only in Yahuda 4848. The number at the left side of *jīm* has been lost due to crumbling manuscript edges (folio 44a). The proposition corresponds to Euclid's proposition 18, in which Euclid explores two possible scenarios — either A measures or does not measure D. The numbers on the right side of the line represent the first case and those on the left represent the second case.



Ḥajjāj, IX, 19
BSB arab. 2697, folio 88b

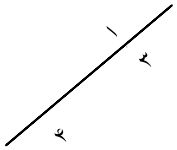


Ḥajjāj, IX, 19
Yahuda 4848, folio 44b

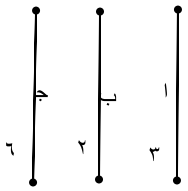


al-Ṭūsī, IX, 20

Since the alternative diagrams have such different architecture (although they agree in numerical values), I have included both in my edition. The proposition corresponds to Euclid's proposition 19. Euclid considers two scenarios — either A measures or does not measure D. The numbers on the right side of the line represent the first case and the numbers on the left represent the second case.



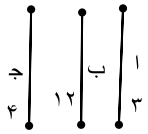
Ḥajjāj, IX, 29 (?)
Yahuda 4848, folio 44b



al-Ṭūsī, IX, 29

The alternative diagram appears only in Yahuda 4848. It is not given an identification number and it has lost letters *bā* and *jīm*. Since the diagram is placed in the inner margin beside propositions 29 and 30, the missing letters may be obscured by tight binding. I make the hypothesis that it is associated with proposition 29 because the legible values correspond to the values included in al-Ṭūsī's diagram for proposition 29.

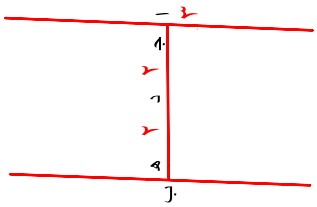
Ḥajjāj, IX, 31
Yahuda 4848, folio 45a



al-Ṭūsī, IX, 31

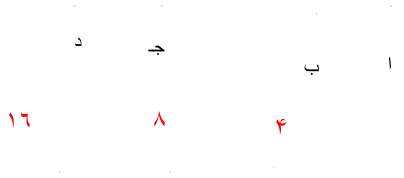
The diagram, ascribed to Ḥajjāj, appears only in Yahuda 4848. The letter *jīm* appears to be lost due to the crumbling margin. No numerical values are visible in the microfilm.

Ḥajjāj, IX, 33
Yahuda 4848, folio 45a

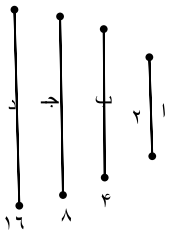


al-Ṭūsī, IX, 33

The alternative diagram appears only in Yahuda 4848. Because the numerals in the diagram appear much lighter than the letters, I believe they were probably written in red so I have edited them in red. The corresponding diagram in the text of al-Ṭūsī is damaged or obscured in Yahuda 4848, so I have edited it from BSB arab. 2697. It is difficult to correlate al-Ṭūsī's diagram with that ascribed to al-Ḥajjāj.

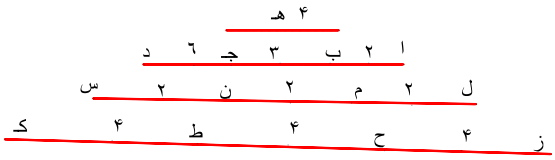


Ḥajjāj, IX, 34
Yahuda 4848, folio 45a

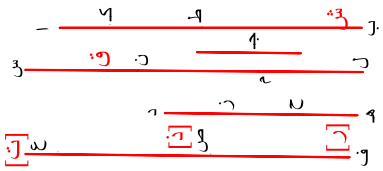


al-Ṭūsī, IX, 34

There is no apparent difference between the alternative diagram and that of al-Ṭūsī, except that the diagram attributed to al-Ḥajjāj is now lacking the numerical value attached to the line segment *alif*.

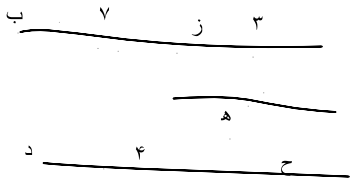


Ḥajjāj, X, 1
BSB arab. 2697, folio 91a

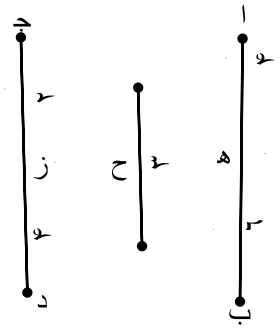


al-Ṭūsī, X, 1

The diagram attributed to Ḥajjāj has suffered significant damage in Yahuda 4848 (folio 46a) — nearly a third of the diagram is missing. The diagram of al-Ṭūsī has been rotated so as to gain more space for the diagram. The three letters in square brackets do not appear in the diagram found in Yahuda 4848.

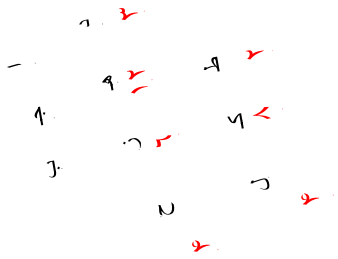


Ḥajjāj (?), X, 3
Yahuda 4848, folio 46b

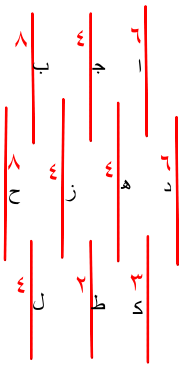


al-Ṭūsī, X, 3

The alternative diagram appears only in Yahuda 4848. It has suffered significant damage — it has now lost letters *alif* and *jīm* and the value of the central magnitude due to the crumbling manuscript edges.



Ḥajjāj, X, 8
Yahuda 4848, folio 47a



al-Ṭūsī, X, 10

The diagram ascribed to Ḥajjāj is found only in Yahuda 4848. I have placed the numerals in red because in the microfilm they appear much fainter than the letters of the diagram — often an indication that red ink is being used. The diagram of al-Ṭūsī has the same basic architecture in both manuscripts but the diagramin BSB arab. 2697 contains numerals, while the diagram in Yahuda 4848 does not.

X Tabulated Data

These tables summarize the key observations concerning these alternative diagrams. The diagrams are listed by proposition number to which they appear to be attached, key differences between the alternative diagrams and those used by al-Ṭūsī (key follows Table II), location of the diagram in each manuscript and whether attributed to Ḥajjāj or unattributed, and comparable proposition in Heiberg's Greek edition or translation based on Heiberg's edition.

Number	Differences	Yahuda 4848	BSB arab. 2697	Heiberg
V, 4	L, N, S	folio 28a (H)	folio 55a (H)	V, 5
V, 9	N	folio 28b (H)	folio 56b (H)	V, 9
V, 10	N	folio 28b (H)	folio 56b (H)	V, 10
V, 11	N	folio 28b (H)	folio 57b (U)	V, 11
V, 12	N, S	folio 28b (U)	folio 57a (H)	V, 13
V, 13	N	folio 28b (H)	folio 57a (H)	V, 12
V, 15		folio 29a		V, 15
V, 20	N	folio 29b (H)	folio 59b (H)	V, 20
V, 21	N	folio 29b (H)	folio 60a (H)	V, 21
VII, 3	D, N	folio 37a (H)	folio 74b (H)	VII, 3
VII, 4	N	folio 37a (H)	folio 74b (H)	VII, 4
VII, 8	N	folio 37b (U)		VII, 8
VII, 11	N	folio 38a (H)		VII, 11
VII, 12	D, N	folio 38a (H)	folio 76a (H)	VII, 12
VII, 13	D, N	folio 38a (H)	folio 76a (H)	VII, 13
VII, 14	D, N	folio 38a (H)	folio 76b (H)	VII, 14
VII, 15	N	folio 38b (H)	folio 76b (U)	VII, 15
VII, 16	D, N, S	folio 38b (H)		VII, 16
VII, 17	D, N, S	folio 38b (H)	folio 77a (H)	VII, 17
VII, 18	D, N	folio 38b (H)	folio 77a (H)	VII, 18
VII, 23	D, N	folio 39a (H)	folio 78a (H)	VII, 23
VII, 24	D, N	folio 39a (H)	folio 78a (H)	VII, 24
VII, 25	D, N	folio 39b (H)	folio 78a (H)	VII, 25
VII, 26	D, N	folio 39b (H)	folio 78a (H)	VII, 26
VII, 27	D, N	folio 39b (H)	folio 78b (H)	VII, 27
VII, 27	N	folio 39b (H)	folio 78b (U)	VII, 28
VII, 30	N	folio 39b (H)	folio 78b (H)	VII, 32
VII, 32	D, N	folio 39b (H)	folio 79a (H)	VII, 30
VII, 33	D, N	folio 39bis a (H)	folio 79a (H)	VII, 33
VII, 36	N(?), S	folio 39bis a (H)	folio 79b (H)	VII, 36
VII, 37	D, N	folio 39bis b (H)	folio 80a (H)	VII, 37
VII, 38	D, N	folio 39bis b (H)	folio 80a (H)	VII, 38
VII, 39	D, N	folio 39bis b (H)	folio 80a (U)	VII, 39

Table I: Alternative diagrams in *Elements*, book V and book VII.

Number	Differences	Yahuda 4848	BSB arab. 2697	Heiberg
VIII, 4	N, S(?)	folio 40a (H)	folio 81a (H)	VIII, 4 (?)
VIII, 6	N	folio 40a (H)	folio 81b (H)	VIII, 5 (?)
VIII, 7	N	folio 40a (H)	folio 82a (H)	VIII, 6 (?)
VIII, 8	N (?)	folio 40b (U)	folio 82a (H)	VIII, 8 (?)
VIII, 10	N	folio 40b (U)	folio 82b (H)	VIII, 10
VIII, 12	N	folio 41a (U)	folio 83a (H)	VIII, 12
VIII, 13	E	folio 41a (U)	folio 83a (H)	VIII, 13
VIII, 17	N	folio 41b (U)	folio 84a (H)	VIII, 19
VIII, 18	N	folio 41b (H)	folio 84a (U)	VIII, 20
VIII, 19	N	folio 42a (H)	folio 84a (U)	VIII, 21
VIII, 27	N	folio 42b (H)	folio 85b (H)	VIII, 27
IX, 1	N	folio 42b (U)	folio 85b (U)	IX, 1
IX, 3	S	folio 42b (H)	folio 86a (H)	IX, 3
IX, 5	N	folio 43a (U)	folio 86a (U)	IX, 5
IX, 6	E	folio 43a (U)	folio 86b (U)	IX, 6
IX, 7	N	folio 43a (U)	folio 86b (U)	IX, 7
IX, 11		folio 43b (H)	folio 87a (U)	IX, 12
IX, 18	N	folio 44a (H)	folio 88b (U)	IX, 18
IX, 19	N	folio 44b (H)	folio 88b (H)	IX, 19
IX, 29	N	folio 44b (H)		IX, 28
IX, 31		folio 45a (H)		IX, 29 (porism 2)
IX, 33		folio 45a (U)		IX, 31
IX, 34	E	folio 45a (U)		IX, 32
X, 1	N, S	folio 46a (H)	folio 91a (H)	X, 1
X, 3	N, L	folioi 46b (U)		X, 3
X, 8	N	folio 47a (H)		X, 8

Table II: Alternative diagrams in *Elements*, books VIII, IX, and X.

Key for diagram differences:

- D = Use of double letter labels in place of single letter labels
- E = Equivalent diagram; no discernable differences between the alternative diagram and that of al-Ṭūsī
- L = Different conventions of assigning labels to points in the diagrams
- N = Different numerical values used in the alternative diagrams and those of al-Ṭūsī
- S = Diagrams have different architecture or style of composition

In a few cases, the alternative diagrams are in such poor condition that they cannot be adequately compared to those of the main text. In such cases, the differences column is left empty.

XI Concluding thoughts

XI.1 What we have learned

This study represents a first excursion into the editing of mathematical diagrams found in manuscript sources. It has demonstrated that such editing is possible. The software tool, DRaFT, provides an easy yet powerful instrument for use in editing mathematical diagrams. At the same time, my experience has shown that the editing of manuscript diagrams still lags behind the scholarly study of textual materials. Several technical questions concerning best practices for editing manuscript diagrams remain to be discussed within the scholarly community. For example, to what extent is it important to preserve the actual orientation of the diagrams or the diagram labels? When diagrams are prepared in two colors, does the failure of the copyist to use the correct color constitute a variation that needs reproduction of the diagram? Should incomplete diagrams be reproduced as they appear now in the manuscript? Or if reading the text indicates the actual form of the diagram should we edit only the reconstructed diagram? I present some personal and preliminary suggestions in the appendix, but such questions deserve to be more thoroughly discussed by the wider community of historians as well.

The actual mathematical content of the collection of diagrams forming the core of this study differs from the diagrams found in the mainstream of the Arabic transmission of Euclid's *Elements*. The designation "al-Ḥajjāj" attached to them suggests that they are in some way associated with his now lost Arabic translation of the *Elements*. The attribution to al-Ḥajjāj is difficult to assess since the few fragmentary sources of information about the Ḥajjāj transmission focus on verbal elements or on differences in ordering of definitions and propositions. Although it has so far proved impossible to establish a definitive connection to the transmission of al-Ḥajjāj, we can at least say that these alternative diagrams existed as a coherent collection and that the collection was transmitted in association with the name of al-Ḥajjāj.

In relation to the broader historical issues raised by this specific collection of diagrams, we have also been able to make some advances. Most importantly, we have can now look beyond verbal elements to include also the content of diagrams when studying the impact of the tradition of al-Ḥajjāj. Editing this collection of diagrams has allowed us to shine a small beam of light into the past, revealing some additional aspects about his work. Unfortunately, most of the beams of light that have so far been directed to this historical question have been more like laser beams than like sunlight. They do not fully illuminate the broad picture but only tiny areas of it. We have seen, for example, that inclusion of specific numerical examples are somehow characteristic of the diagrams associated with the Ḥajjāj tradition. But how this information can be used to gain a more coherent and nuanced picture of al-Ḥajjāj and his translation work remains remains unresolved.

XI.2 Looking to the future

Nearly four centuries separate the dates of the copying of these two manuscripts, during which time this collection of Ḥajjāj diagrams continued to exist in a coherent form. Because the tradition of these diagrams lasted so long, there is a possibility that additional copies may yet be found, perhaps in association with other copies of the *Taḥrīr* of al-Ṭūsī. There are hundreds of copies of his treatise still in existence all around the world, and manuscript catalogers rarely provide a detailed analysis of any marginalia and may do little more than mention the existence of marginalia. If additional copies of these diagrams should be found, they may help to fill in the gaps and uncertainties that still exist in our edition. But even if other versions of the diagram collection can be located, we will be unable to definitively confirm the attribution to al-Ḥajjāj until additional sources of information about this now-lost transmission are discovered — sources which offer independent confirmation to support the variety of reports that currently exist in the secondary and tertiary literature.

Since diagrams are more readily identified in manuscripts than are specific verbal elements and since they are more likely to be preserved in any transmission across linguistic boundaries, they offer a quick and convenient way to assess the influence of the Ḥajjāj translation. For example, on the basis of some of the reported structural differences existing between the transmissions of Ḥajjāj and Thābit, it would appear that the epitome of Euclidean geometry included in the philosophical compendium of Ibn Sina, *Kitāb al-Shifā'*, was based on a version of al-Ḥajjāj. Yet its diagrams in book VII do not show the distinctive double lettering that seems to be so prominent in our collection of Ḥajjāj diagrams. On the other hand, the medieval Latin versions ascribed to Hermann of Carinthia [Busard, 1977] and to Adelard of Bath [Busard, 1983] both reveal the distinctive double lettering pattern in book VII. This observation, coupled with the presence of several structural features that have also been reported to be typical of Ḥajjāj, suggest that these Latin versions, at least, were considerably influenced by the tradition that also gave rise to our collection of diagrams. Since these translations were made early in the twelfth century they provide additoinal evidence that the tradition of Ḥajjāj, even though not strongly evident in many of our best-known surviving Arabic sources, played a significant role in the transmission of Euclid to medieval Europe.

XII Acknowledgements

I am grateful to Princeton University Library for providing a microfilm of Yahuda 4848 and to the staff of the Rare Books and Special Collections who took time to answer queries about the color conventions of the diagrams in the manuscript. I am also very grateful to the Bayerische Staatsbibliothek for making so many manuscripts freely available for study on their website.

All diagrams in this study have been edited using DRaFT, a free software tool designed to capture the core geometric information contained in mathematical diagrams and preserve

it in the form of EPS files. The software may be downloaded from Professor Saito's website: `http://www.greekmath.org`. Dr. Saito personally introduced me to the software, and I have benefitted much from his sage advice over the years.

I thank also Dr. Michio Yano and the anonymous reviewer who offered both constructive criticisms and helpful suggestions for making the paper better. I have tried to follow their useful pointers. Those errors that still remain — and hopefully their number has been reduced thanks to their efforts — are my own.

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Appendix A: Technical issues in editing mathematical diagrams

Mathematical diagrams, reduced to their essence, consist of points which are connected to one another by line segments, which may be either rectilinear or curved. Some points may have letter labels attached to facilitate discussion.[23] The basic task of editing diagrams centers around things that go wrong with the reproduction of diagram lines and points as diagrams are copied from one manuscript to another. Just as when editing a verbal text, the principles of editing include the recording as clearly as possible of all significant variations within the tradition and indicating transparently any editorial interference in the text. In this section, I will discuss some editing issues already raised by others and suggest the potential role of DRaFT software in the editing process.

When we begin to think about editing diagrams, we must ask ourselves what kinds of things can go wrong when diagrams are being copied? A number of possibilities may come readily to mind — at the most basic level, one or more point labels might be missing — either omitted or lost due to manuscript damage — or incorrectly placed when considered in relation to the mathematical content of the text. Missing labels are one of the easiest variants to indicate. One can identify the insertion of a missing element by placing the diagram label in square brackets in analogy to the way an editor inserts a missing word or phrase into an edited text by enclosing the supplied word or phrase in square brackets.

Similarly, one or more lines might be missing or incorrectly constructed. And a closely related form of missing information occurs when a manuscript has suffered physical damage during its lifetime, resulting in parts of the diagram lines that have disappeared along with the actual paper on which the diagram appeared. Occasionally, diagram lines are erased or, if not erased, stricken out by the copyist (or perhaps a later reader). Or a line might be inserted into an already existing diagram — recognizable by use of different ink or different pen or different drawing technique (for example, a line drawn freehand within a diagram constructed with traditional drawing aids) or even from the fact that the line is not present in other copies of the diagram). Electronic tools such as DRaFT allow us to indicate editorial actions by altering the weight of solid lines or by using various forms of dashed lines.[24] Since there is as yet no strong consensus on how to indicate various forms of editor interference within diagrams, there is the potential for confusing or contradictory indicators to develop. One of the key problems facing editors of mathematical texts containing diagrams is the need to develop widely recognized editorial conventions. Variants such as those just described I call substantive variants in distinction from what might be called cosmetic variants —

[23] Not every point in a diagram requires a letter label. And not every label is a letter — sometimes we find numerals and sometimes words are used to identify geometrical entitites.

[24] For example, Saito [2013, 70] and Sidoi & Li [2013, 3] have recently used a double weight solid line to indicate the occasional insertion of lines drawn by hand. Similarly, one can use various forms of dashed lines to indicate lines inserted by the editor to replace lines missing from the diagram, or to indicate lines that have been erased or struck out by the copyist.

including such factors as orientation on the page and the different metrics used by copyists — which result from idioyncratic factors and do not alter the mathematical meaning of the diagram.[25]

Beyond the question of how to represent diagram variants, there are two additional questions that the editor must consider. First, when editing diagrams in texts written in alphabets other than Roman or perhaps Greek, should the editor retain the original alphabet used for lettering the diagrams? At first glance, the answer seems simple — one should remain true to the characteristics of the diagram in the manuscript and retain the original alphabet. But doing so raises the distinct possibility that users might not be able to read the diagram or compare it with diagrams from other traditions without additional assistance. A possible solution might be a transliteration table in the introduction to the edition so that users can transfer the diagram labels into more widely recognizable alphabets. One possible confusing factor could be that diagrams constructed in Arabic or Hebrew treatises, for example, will typically be mirror images of those found in Latin or Greek manuscripts because the diagrams are constructed to be read from right to left in the same direction as the verbal text. I would not consider such mirror image diagrams to represent substantive variants.

A second important question is whether the editor should alter the diagram to fit modern conceptions of what a particular diagram should look like. Diagrams constructed within the conventions of medieval perspective may be virtually unrecognizable to modern readers.[26] And a closely related question revolves around the "over-specification" so often encountered in medieval diagrams. Should one retain the over-specified form of the diagram or should one redraw it in accord with contemporary understanding of mathematical generality. In such cases, there is a growing concensus that pre-modern diagrams must be edited within their historical context. One should interfere with the original diagram as little as possible, and one should indicate as clearly as possible the precise nature of any interference that is made. Thus Jardine & Jardine [2010, 410] argue strongly in favor of preserving the essential features of the originial diagram, making it the task of the editor to explain the "foreign" features of the diagram, educating the modern reader to understand its characteristics and its mathematical messages. If the text containing the diagrams is to be translated, one has the option to retain the original diagram within the edition of the text itself, while presenting a more modern version of the diagram in the translation section.

The principles for editing manuscript texts are now well-developed. But what principles should be used in our editing of manuscript diagrams? A number of questions will require consideration and answers. In the following paragraphs, I discuss some of these questions and suggest possible answers.

When a line or arc is incomplete because the ink in some places has faded or flaked

[25] Even these idiosyncratic characteristics might be of importance if one wishes to investigate traditions of scribal techniques and traditions of diagram construction [Saito, 2012].

[26] Among numerous examples, one could conisider the diagrams accompanying an anonymous study of semi-regular solids and how they differ from modern representations of the same structures [De Young, 2008].

away from the paper, but the enough remains to show the original placement of the element, should the editor restore the line to its original completeness in his edition? And what if the entire line or arc is completely missing except for its endpoints? I believe that the best editing practice is to preserve the maximum information from the manuscript and when restoration can be done without ambiguity it is preferable to restore it rather than to leave the diagram incomplete. Since DRaFT and other modern electronic editing tools offer options to specify a variety line styles, it is possible to indicate clearly the portion of the diagram that is restoration and the portion that is original. For example, one could use a dashed line to indicate restoration of a line when its endpoints are clearly visible, while a dotted line could be used to restore a portion of a line when one of its endpoints is not visible — perhaps lost due to physical damage such as a torn page — but its position can be inferred from surviving diagram elements or from the text. Similarly, if diagram labels are missing, either through scribal error or physical damage to the manuscript, the editor should replace them when possible while clearly indicating the editorial action. One easy solution is to place them within square brackets, as is done when replacing a missing word or phrase in the text.

When a diagram contains a mistake — perhaps an incorrectly drawn line which was either erased or merely crossed out — I believe that the editor should include the line and indicate its character in the same way that an editor will indicate in the critical apparatus that a copyist incorrectly copied a word which he then crossed out and rewrote correctly. It is again a relatively simple matter to indicate visually such diagram corrections using the options available in modern electronic sofware such as DRaFT. If the editor chooses to omit the line, that editorial action should at least be noted in the diagram caption or in a note within the critical apparatus. And certainly if the manuscript copyist did not notice his error, it is necessary to report it in our editing of the diagram. An incorrectly placed diagram label should similarly be reported in the critical apparatus.

Hand-drawn diagrams, even when made with drawing aids, often display minor imperfections — lines do not meet as precisely as they should, circles may be distorted, diagrams are not always drawn to the correct proportion as described in the text, etc. To what degree should one reproduce these minor flaws when editing the diagram? Should every minor flaw be preserved or noted in the editing of the diagram? Probably this is not necessary unless the actual drawing techniques of the copyist form an important focus of the discussion. Lines which are clearly intended to meet (as indicated both visually and textually) can usually be edited or repositioned to do so without doing injustice either to the diagram content or the text. On the other hand, if the diagram copyist has clearly ignored the proportions of lines implicit in the text, one might wish to redraw the basic diagram to make it conform to text.[27]

[27]This situation is different from the case of over-specification (which is very common in manuscripts of the Euclidean tradition), in which the copyist produces a diagram that conforms to the requirements of the text but draws it in a form more specific than is required by the mathematics of the text, such as when a copyist uses an isosceles triangle instead of a more general scalene triangle to represent a given triangle. And since over-specification is so common, what of the occasional copyist who does not reproduce the diagram in its

As is the case when editing a text, the most basic editing principle is that for every substantive change introduced by the editor into the manuscript diagram, it is essential to leave a transparent record of what has been done. To make substantive changes without informing the reader is to perform a historical disservice and imply a diagram that was not present in the manuscript tradition. Whenever there is any possibility of ambiguity, it is preferable to provide both the original diagram and the amended diagram, explaining the correction(s) or alteration(s) in a caption or in the critical apparatus.

In many cases, diagrams in different manuscripts of the same text will have identical mathematical content and geometric structure but will differ from one another only in terms of internal metrical features. Either the diagrams as entire entities vary in size or individual lines may be longer or shorter or perhaps each diagram will use a somewhat different value for the ratio of the altitude to the base of a triangle. In such cases, I suggest that there is no necessity to indicate individual differences between diagrams. In the critical introduction, one can discuss general trends or characteristics of each copyist's diagram style — how strongly each chooses to emphasize diagram overspecification, for example.

But what about cases where one of the copyists uses a mirror image of diagram? The basic geometrical parts are still the same and the labeling is still the same but only the layout on the page is different. If only one copyist or one family of manuscripts routinely adopts a mirror image arrangement, this characteristic could probably be discussed in the critical introduction. Sometimes a copyist will alter the orientation of a diagram — usually by a 90° degree rotation. Usually this rotation includes both the geometrical elements and the letter labels. The tactic of rotating a diagram usually appears to have been adopted when the copyist of the text has left too little space for the insertion of the diagram within an opening or "window" in the text. If there is no other significant change introduced, there seems no necessity to reproduce the rotated diagram as a variant. Its existence can simply be noted in the caption of the diagram. Similarly, if it is only a few scattered diagrams that are flipped left for right, for example, then we might better indicate that fact for each flipped diagram, either through a diagram caption or in the critical apparatus. Similarly, in the case of reorientation of diagrams from vertical to horizontal arrangement, for example. If the difference in orientation is a regular practice of an individual copyist or a family of manuscripts, this characteristic is better discussed in the critical introduction. But if it occurs only sporadically in the text, then it may be best to note the phenomenon in a diagram caption or in the critical apparatus.

When diagrams exhibit a perspective different from that with which modern readers feel comfortable, there may be the temptation to reconstruct the diagram using more modern perspective. Arguments against this approach have been discussed by Jardine & Jardine [2010], who suggest that part of the editorial task is to educate the modern reader to understand the pre-modern conventions. I suggest that these differences in perspective can be

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over-specified form? Such general characteristic of mathematical manuscripts might be better addressed in the critical introduction to the manuscripts, rather than treated as variants.

resolved when one retains the original perspective in the text edition but uses the more modern perspective when preparing the translation of the text — for an example, see De Young [2008b]. In any case, an editor should respect the diagrams in the manuscript and should assist modern readers to understand them as those who read the original manuscript would have understood them.

Do we need a separate critical apparatus for diagrams whenever we edit a pre-modern mathematical treatise? The answer one gives will depend in large measure on the character of the text and on the character of the diagram transmission. If there are many significant errors or variants in the diagrams, it may be desirable to edit them in a separate appendix where one can display the nature of the variants. If there are only a few significant variants, one might be satisfied with discussing them in the critical introduction or in the diagram captions. The essential considerations must be (a) that the editor accurately reflects the state of the diagrams and their variants during the editing process and (b) that any reader will be able to understand clearly what changes the editor has made in the diagram and why these changes have been made.

Appendix B: Some recent contributions to diagram studies

Although diagrams were for a long while accorded little serious study, in the past two decades a number of scholars have turned attention specifically to diagrams and their historical development. As yet there has been little effort to consolidate these different initiatives. In part this neglect stems from a perception that diagrams in ancient and medieval mathematical texts are little more than a technical detail and from the fact that scholars who focus on diagrams are often concerned only with one linguistic tradition and may not be conversant with texts and diagrams in other languages. Moreover, discussions of diagrams have not usually been the main focus of the manuscript studies and so might not be picked out for notice by catalogers, indexers, or abstractors, thus limiting access to the published studies.

Some of the earliest discussions of mathematical diagrams occurred in the context of manuscript illustrations in general. Hall, although surveying a field much broader than mathematical diagrams, pointed to an important clue concerning diagram construction techniques. Discussing a 12th century Latin astronomical manuscript, he noted [1996, 11] that when the text crowds up against and surrounds the diagram, adapting itself to the shape of the diagram, the figure must have been drawn before the text surrounding it was written. The observation, once stated, seems simple and obvious. But it provides a test to decide whether illustrations or mathematical diagrams were inserted by the copyist during the copying process or whether they might have been inserted later, possibly by another draughtsman.

Building on the observation of Hall, I suggest that when diagrams are presented in square or oblong blank areas on the margin of blocks of text, it is still sometimes possible to determine whether the diagrams were constructed at the time of copying or inserted later. When the diagram is improperly proportioned, so that the top sections must be deformed in order to fit in the allotted space or when the top portions intrude upon the text, we can be quite sure that the diagrams were inserted after the copying of the text was complete.[28] Similarly, when diagrams must be rotated in order to be accommodated within the allowed space or when diagrams impinge on the margins in order to be fit into the allowed area, we can be quite confident that they have been inserted later.[29] On the other hand, when diagrams fit within the openings left for them or are placed in the margins beside the text, one cannot be absolutely sure when they were inserted.

Netz [1998; 1999] was among the first to focus scholarly attention on diagrams, arguing that diagrams preceded text in the development of Greek mathematical discourse. Netz was primarily interested in the role of diagrams in the development of mathematical logic. In the course of his analysis he offered some useful insights into several features of diagram

[28] The fact that the deformation of the diagram is always at the top and it is usually at the top that the diagram may impinge on the text implies that the draughtsman typically began construction of the diagram from a baseline at the level of the lowest line of text rather than any other part of the diagram.

[29] This principle does not, however, allow us to decide whether the diagram was inserted by the original copyist of the text or by another draughtsman.

construction, including the typical order or patterns in which letters were assigned to diagram points. Kunitzsch [1991] also explored similarities and differences in labeling conventions across linguistic boundaries (Greek – Arabic – Latin).

Crozet [1999] was one of the first to discuss diagrams specifically in relation to Arabic mathematical texts and the problems that diagrams can raise for editors of mathematical manuscripts. In an introductory section he discusses an important issue for the philosophy of mathematics — the distinction between the Arabic terms *shakl* and *ṣūrah* and the role of diagrams within mathematical texts. In the second part of his paper, Crozet, based on his studies of the mathematical oeuvre of al-Sijzī, discusses in some detail examples of copyist errors in the construction of diagrams and offers suggestions for how editors might deal with diagrams containing errors. His suggestions have become more easily implemented after the development of DRaFT software for editing mathematical diagrams.

Saito developed the software tool, DRaFT, specifically for use in editing manuscript diagrams.[30] In one sense, the software project was Saito's response to the changing historiography of mathematics which no longer tries to reconstruct the mathematics of an autograph text [Saito, 1998] but rather asks what was the mathematical praxis of a given era as revealed in surviving texts. Although the tradition of mathematical diagrams extends at least as far back as the earliest extant fragments of Greek mathematics, many editors have until quite recently felt no qualms about redrawing manuscript diagrams to conform to what they thought the author of the diagrams should have intended.[31] Such reconstructions of diagrams have ignored and sometimes masked important features in the historical diagrams, such as the persistent use of "over-specification".

An early study by Decorps-Foulquier [1999] considered the construction problems in the *Conica* of Apollonius and the implications of these constructivist activities for the production of textual diagrams. More recently, Sidoli & Saito [2009] have explored similar themes in the case of the *Sphaerica* of Theodosius. Their focus is not on the mechanical production of diagram lines but on the relation of the diagram to the text.

Keller [2005] was one of the first to study diagrams in a non-Western mathematical tra-

[30] The software is designed to extract the geometric data from mathematical diagrams. It allows users to register the significant points in a diagram and connect these points with straight or curved lines. The most recent releases have increased the flexibility of the sofware, making it highly customizable, capable of running on multiple platforms, and able to accept unicode input so that diagram labels can now be assigned in any language for which there exists a unicode font. The program produces as output an EPS file that can be inserted into many kinds of documents, including LaTeX and XeLaTeX. All diagrams used in this paper have been edited using DRaFT software. Interested readers may refer to Saito's web site for more information concerning the software or to download a gratis copy of the software: `http://www.greekmath.org`.

[31] To be fair, not all editors have completely ignored the evidence of ancient and medieval mathematical diagrams. Perhaps the most glaring example of an editor ignoring diagram evidence is Heiberg's edition of the *Elements*, in which he simply copied the form of the diagrams in August's earlier Greek edition [Saito & Sidoli, 2012, 136–139].

dition. She showed that diagrams are not merely ancillary devises subordinate to textual mathematics but performed several interlocking functions: they could be a schematic representation of reality, they might explicate complex mathematical relations, they might serve to convince readers of mathematical principles.

Chemla [2005; 2010] also explored diagrams in non-Western mathematics, focusing especially on traditional Chinese mathematics. She showed convincingly that diagrams are not isolated from the mathematical tradition but change along with that tradition. And perhaps more importantly, diagrams from one tradition which may appear on the surface to be related to those of another mathematical tradition sometimes fulfill different mathematical functions and may represent a completely different mathematical reality. It is essential to understand diagrams within the context of the mathematical tradition that gave rise to them. Volkov [2007] has also described the function of diagrams in Chinese traditional mathmatics.

Saito [2006; 2012], based primarily on studies of Greek manuscripts, has also added to our analytical tool kit the concept of "over-specification" or the tendency of diagrams to present a more limited case than is required by the text of the proposition.[32] Over-specification is very common in ancient and medieval manuscripts of the Euclidean tradition, although it is difficult to determine its origins. This over-specification is one of the first features to be lost when modern editors redraw the diagrams for their editions since modern mathematics tends to focus on the most abstract and most general presentation.

Malpangotto [2010] has studied the diagram transmission in manuscript copies of the *Spherica* of Theodosius and their transformation in Renaissance printed editions. Although her concern is not directly with problems of editing diagrams, her study reveals both the complexity of diagram conventions and the fact that these conventions change over time. Renaissance editors who prepared printed editions of the treatise adopted a variety of different techniques to deal with these diagramatic conventions in the manuscripts. I have briefly discussed some of the same issues in relation to the Arabic transmission of Euclid's *Elements* [De Young, 2012].

Jardine & Jardine [2010] tackle the same questions about using and editing pre-modern diagrams in the context of editing astronomical diagrams in early modern manuscripts. The issues they raise and their suggested principles for editing pre-modern diagrams are applicable to many other branches of mathematical sciences as well. They provide an important and thoughtful discussion of editorial aims, suggesting that rather than "domesticate" medieval diagrams so that they appear in forms familiar to modern readers, the editor might better aim to explicate the "foreign" features and conventions of diagram placement in relation to text

[32] A classic example of "over-specification" appears in the diagram for *Elements* I, 47. In ancient and medieval manuscripts the right triangle is almost always drawn as isosceles, although the enunciation of the proposition does not require it. In this case, the squares on the two sides of the right angle must be equal to one another. That this over-specification was intentional and not merely an artistic convention, is indicated by numbers inserted into the diagram in several Byzantine Greek manuscripts.

as well as conventions of presentation and perspective so as to educate the modern reader. Like Saito [1998], they question whether it is possible to recapture the actual intent of an ancient or medieval author when studying the diagrams associated with his work. Their remarks on the use of electronic media and software suggest a vision of the future for editing and studying mathematical diagrams.

In a little-noticed study, Barany [2012] examined the representation of the point in early modern printed editions of Euclid. It is fascinating that such representations rarely appear in medieval manuscripts — at least not in manuscripts that have come to my notice. Barany points out that these tiny diagrams or representations of the mathematical point are not just interesting pictures but imply a whole mindset and a way of reading mathematical treatises and illuminating the meaning of mathematical diagrams.

Saito & Sidoli [2012] have analyzed the diagrams in several modern printed editions of ancient and medieval mathematical texts. Their work shows that some editors attempted to be true to the diagrams in the original manuscripts while others were less scrupulously devoted to preserving the information in the medieval diagrams. Takahashi [2008], while introducing his study of the collection of diagrams found in Euclid's *Optica*, includes a careful discussion of their characteristics and their relation to the text, as well as remarks on the later history of the diagrams as treated by modern editors. Although he does not introduce new intellectual tools for the study of diagrams, his work provides a possible model for how an editor might use a critical introduction to discuss mathematical diagrams when editing a mathematical manuscript.

In their study of the diagrams in al-Harawī's Arabic version of the *Sphaerica* of Menelaus, Sidoli and Li [2013] introduced a distinction between mathematical and visual elements in the diagrams. Mathematical elements include labeling and positioning of points, positioning of lines, and differentiation into cases. Visual elements include such features as orientation of the diagrams, direction of curvature of lines, the presence of additional or non-essential lines, etc. Their distinction is especially helpful when examining scribal traditions of diagram construction as they evolve over time.

A recent study of the history of diagrams in East Asian mathematics by Ying [2013] focuses on the role of diagrams in the mathematical tradition in Korea during the Choson period. Specific conventions used in Korean diagrams are compared with the Chinese mathematical diagrams that have already been studied. Diagrams were used in Korea to show how to use instruments in solving mathematical problems, to clarify or explain mathematical identities, or to represent geometrical objects.

In a significant step forward, Raynaud [2014] has argued that it is possible to construct a stemma for diagrams and that it is possible, under certain conditions, to construct a stemma for mathematical manuscripts based on the diagrams alone. His approach to diagram studies relies on the application of the phylogenetic techniques that have been successfully employed to study manuscript documents in several genres. The process proposed is similar to that applied to textual studies, except that one focuses on errors in the diagrams rather than errors in the text. Since diagrams usually contain fewer elements than does a text,

the process of identifying and cumulating specific errors in diagrams is significantly less time consuming than the same process applied to the textual elements. Thus constructing a stemma codicum on the basis of the diagrams can be less labor intensive than traditional methods which rely on patterns of textual errors. In an initial test case he has demonstrated that this phylogenetic approach yields the same stemma codicum that is produced from traditional textual study.

Joyce van Leeuwen's discussion of the diagrams in the Greek manuscripts of Aristotle's *Mechanics* [2014] has shown that the diagrams of this treatise are often essential for understanding and editing the verbal text. And like Raynaud, she has shown that a study of the diagrams can confirm the stemma codicum derived from the text.

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