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Determining the Sine of One Degree in the *Sarvasiddhāntarāja* of Nityānanda

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Abstract

The *Sarvasiddhāntarāja* (1639) of Nityānanda is a monumental treatise that provides a comprehensive treatment of various aspects of astronomy. Since trigonometry was vital for professional astronomers, Nityānanda gives a systematic and detailed account of this topic in over sixty-five verses in one of the early chapters in this treatise. As is the case with much of the *Sarvasiddhāntarāja*, Nityānanda introduces certain novel features which are not found in prior treatments, some of which are his own insights, others of which were inspired by Arabic sources. We present a critically edited text of part of the trigonometry section of this treatise on the basis of six manuscripts, as well as a translation, and a commentary of the technical content.

I Introduction

The *Sarvasiddhāntarāja* ("King of all *siddhāntas*") is a monumental Sanskrit astronomical treatise that provides insight into the trends and practice of seventeenth century astronomy in the Indian subcontinent. Composed in 1639 by Nityānanda, astronomer at the Mughal court of Shāh Jahān in Delhi, the *Sarvasiddhāntarāja* is prodigious in both length and scope and contains verses covering topics in astronomy from calendrics to instrumentation in twelve chapters (a brief overview of the topics can been seen in Table 1). Preliminary investigations of the work have revealed that Nityānanda infused his traditional *siddhānta* with some Islamic concepts and parameters. His familiarity with this tradition of astronomy no doubt stemmed from the translation of a massive set of astronomical tables of Ulugh Beg from Arabic, via the Persian intermediary, the *Zīj-i Shāh Jahānī* which he completed almost a decade earlier. Given this influence on Nityānanda, the *Sarvasiddhāntarāja* remains one of the most important sources in which to trace the introduction of the Arabic astronomical and mathematical sciences into the Sanskrit *siddhāntic* tradition in the seventeenth century.

One topic which is emblematic of this introduction is trigonometry. Nityānanda's exposition on this appears early on in the work, as is typical with most *siddhāntas*. He devotes sixty-seven verses (I 3, 19–85) to trigonometry near the beginning of chapter three which concerns the determination of the true positions of the planets. It is thus

Part	Names of the different chapters	
I *gaṇita*	1. *mīmāṃsā*	(Philosophical Rationales)
	2. *madhyama*	(Mean Positions)
	3. *spaṣṭa*	(True Positions)
	4. *tripraśna*	(Three Questions: Direction, Place, Time)
	5. *candragrahaṇa*	(Lunar Eclipses)
	6. *sūryagrahaṇa*	(Solar Eclipses)
	7. *śṛṅgonnati*	(Elevation of the Lunar Cusps)
	8. *bhagrahayuti*	(Conjunction of the Stars and Planets)
	9. *bhagrahāṇām unnatāṃśāḥ*	(Altitudes, in Degrees, of the Stars and Planets)
II *gola*	1. *bhuvanakośa*	(Cosmography)
	2. *golabandha*	(Armillary Sphere)
	3. *yantra*	(Astronomical Instruments)

Table 1: The contents of the *Sarvasiddhāntarāja*.

one of the rare texts that presents an extended discussion on trigonometry. Since the *Sarvasiddhāntarāja* is primarily an astronomical treatise, Nityānanda could have skimmed over the topic, and simply presented a table of sines. However, he has chosen to deal with it in a very systematic and detailed manner. This is perhaps because of Nityānanda's conviction that a clear understanding of the theoretical framework is important not only as a practitioner of science, but also to establish himself as a true scholar, a sentiment he himself conveys at the outset (*Sarvasiddhāntarāja* 3, 19):

> आचार्यवर्या गणकास्त एव जानन्ति ये ज्यानयनोपपत्तिम् ।
> ततोऽधिगन्तुं पदवीं च तेषां महाजडो वाञ्छति मादृशोऽपि ॥ १९ ॥
>
> *ācāryavaryā gaṇakās ta eva jānanti ye jyānayanopapattim |*
> *tato 'dhigantuṃ padavīṃ ca teṣāṃ mahājaḍo vāñchati mādṛśo 'pi* ||19||
>
> Oh revered teachers! Only those are considered mathematicians who know the rationale behind the computation of Sines. Therefore, [by venturing to explain that,] even a dullard like me wishes to accomplish their status.

In a recent paper (Montelle, Ramasubramanian, and Dhammaloka 2016) we examined the first part of this exposition (verses 19–59), in which Nityānanda, over the course of five "sections" (*prakāra*), provides rules and relations for determining the Sines, Cosines, and Versines of various arcs, the double and half-angle formulae, as well as the Sines of sums and differences (see Table 2).[1] With these rules, one can produce Sine values of smaller arcs including an arc of 3° but not 1°.[2]

Verses	Contents	
19–23		Preamble and definitions
24–30	Section 1	The Sines of ninety, thirty, eighteen degrees
31–36	Section 2	The Sine of half the arc
37–40	Section 3	The Sine of double the arc
41-48	Section 4	The Sine of the sum of two arcs
49–54	Section 5	The Sine of the difference of two arcs
55–59		Demonstration of equivalences by geometrical construction (*śilpa*)

Table 2: The structure and contents of the first five sections on trigonometry.

Nityānanda continues with a sixth section (verses 60–85) focused on determining the Sine of 1°. He alludes to this in the preamble (*Sarvasiddhāntarāja* 3, 23):

> अथोच्यते ज्यानयनोपपत्तिः पञ्चप्रकारैर्गणितस्य तावत् ।
> त्रित्र्यंशकोत्थानि परिस्फुटानि त्रिंशज्ज्यकार्धानि यतो भवन्ति ॥ २३ ॥
>
> *athocyate jyānayanopapattiḥ pañcaprakārair gaṇitasya tāvat |*
> *tritryaṃśakotthāni parisphuṭāni triṃśajjyakārdhāni yato bhavanti || 23 ||*
>
> Now, the rationale behind the computation of Sines is indeed described through five sections. Since there are only 30 Sines (*tritryaṃśa*), accurate values [of Sines of any angle] are obtained from [the Sine] of one third of three degrees (i.e., Sin 1°).

Finding the Sine of 1° was of great importance to Nityānanda as he includes a table of Sines whose argument ranges from 1° to 90° (see, for instance, the table included in one of the manuscripts of the *Sarvasiddhāntarāja*, in Appendix IV, Figure 9). In addition to this, his table-text work, the *Siddhāntasindhu*, contains the values of Sines down to single minutes of arc! Clearly there was some technique at Nityānanda's disposal for computing the Sine of 1° (and from that the Sine of $0;01°$, which

[1] To reflect the difference between the modern sine, with Radius equal to one, and the Indian trigonometric equivalents with non-unity Radii, we represent the latter with a capitalised letter, i.e., Sin and Cos, such that $R \sin\theta = \text{Sin}\,\theta$. Here Nityānanda uses $R = 60$.

[2] This can be achieved simply by using a combination of the half-angle relation, and the sines of sums or differences. For instance, from the sine of 72°, one can determine 36°, then 18°, then 9°, then 4.5°. Similarly, from the sine of 30°, one can determine 15°, then 7.5°. Taking the difference of the sines of 7.5° and 3.5° produces the sine of 3° exactly.

we discuss below), which he could use to compute all the other values. Various Sine tables, as they appear in manuscripts, are given in Appendix IV.

It is a well known fact that the Sine of 1° can not be determined geometrically. Hence, Nityānanda presents an algorithm which is derived from a rearrangement of the multiple-angle formula—a computation which effectively reduces determining the Sine of 1° to solving a cubic equation. Nityānanda does not stop at simply presenting the algorithm. Besides giving three approaches to the solving the cubic, he also provides a detailed justification of the algorithm, introducing this account as an *upapatti* (demonstration), and sets out a geometrical diagram as well as detailed algebraic reasoning. In addition, he includes a "worked example" (*udāharaṇa*) with a quasi-symbolic style of reasoning to derive the formula. He concludes the discussion with a very short account of how to determine the Sine of 1;01°, which is, to our knowledge, a previously undocumented technique in Sanskrit sources. This multifaceted approach to the determination of the Sine of 1° gives rich insight into the modes of reasoning, clever geometrical manipulation, nascent symbolic styles of algebra, inventive approximation techniques, and diagrammatic analysis, which Nityānanda had at his command.

The techniques that Nityānanda presents clearly indicate connections to earlier expositions which deal with determining the Sine of 1°. The earliest of these that we know of is an Arabic work ascribed to the Iranian astronomer al-Kashi (*ca.* 1380–1429), although the only sources we have describing his insights appear to have been composed after his death in a treatise entitled *Risāla fī stikhrāj jayb daraja wāḥida* ("Treatise on the Determination of the Sine of One Degree"), which has been edited and translated by Rosenfeld and Hogendijk (2003). This treatise contains the description of two geometrical theorems which underscore the cubic equation (pp. 33–39), and then two (equivalent) worked methods for finding the Sine of 1°, one by the unnamed author of the treatise (pp. 39–43), and the other associated with al-Kashi (pp. 43–50). The former essentially works through the iterative procedure using Chords, and the latter, using Sines.

The second text of relevance is one written in Sanskrit prose which contains a compilation of procedures, and can be traced back to Islamic astronomer Ulug Beg (1394–1449). This currently unpublished text, known as the *Jyācāpa-utpatti* (SRCMI 269.I.13.i.3),[3] includes several variants of the iterative techniques for finding the Sine of 1°, at least two of which are attributed directly to Ulug Beg.

Though the common threads through all of these texts point to the transmission of techniques and procedures from the Arabic through to the Sanskrit sources, there are key distinctions between the expositions. A careful comparison of the individual methods between all three treatises must be made in order to understand exactly the extent of transmission, transformation, and innovation in the later texts. The current

[3] This text is currently under preparation by Montelle, Plofker, and Van Brummelen.

study offers a critically edited text (see section II) based on six available manuscripts, a literal English translation, and a technical analysis of the contents. By presenting a thorough exposition of the work of Nityānanda, this paper will provide a basis from which to better understand the transmission of these techniques from Islamic sources to Sanskrit texts.

II Critical Edition of the Verses

II.1 A Description of the Sources and Conventions Adopted

The six manuscripts of the *Sarvasiddhāntarāja*[4] that were used to produce the critical edition are listed in Table 3:

Siglum	Description
B_1	Sarasvati Bhavan Library, Benares, 35741
B_2	Sarasvati Bhavan Library, Benares, 37079
B_o	Bhandarakar Oriental Research Institute 206 of A, 1883–1884
N	National Archives Nepal, Kathmandu, Microfilm Reel No. B $\frac{354}{15}$
R	Rajasthan Oriental Research Institute (Alwar) 2619
W	Wellcome MS Y550

Table 3: Manuscripts of the *Sarvasiddhāntarāja* and their sigla.

In preparing our critical edition, we have adopted the following conventions:

- Common orthographical variants have been emended silently and have not been recorded in the critical apparatus. These include misplaced *daṇḍas*, omitted *visarga, virāma*, or *avagraha*, *anusvāra* for a conjoined nasal, doubled consonants such as *ācāryya* or *arddha*, frequent confusion over *ṅa* and *ḍa*, *ṭha* and *ṭa*, and *va* in place of *ba*.
- We have not critically edited the diagrams, leaving that to a future dedicated study. We do however reproduce them in our technical analysis.
- All manuscripts had trivial variants in the numeration of verses. We have not recorded them.
- In the *udāharaṇa* section, the scribes have represented the equations as per the convenience of the space available to insert the equation. Sometimes they have even broken a single word to insert an equation (For example: RORI f.14r, "san–dhi" line -1). Such layout discrepancies have been ignored by us when noting variants in the apparatus criticus.
- The small dot which is used above the number to represent that the coefficient is

[4] Many more copies exist: see Pingree (1970–1994, A3 173–4, A4 141, A5 184) which lists 16.

negative is found to be missing in some of the manuscripts in certain instances. We have not noted these in the apparatus, due to their frequency.

- While representing the algebraic equations, some of the manuscripts, for instance B_0, leave out terms in the denominator. This also has not been noted by us in the apparatus criticus.
- While representing the quantity that appears in the denominator, some manuscripts place it in the center, sometimes to the left, and sometimes to the right. We have not noted this as a variant. However, in our critical edition, we have consistently placed the numeral in the center.

II.2 Critical Edition

अथ षष्ठप्रकारे चापज्याज्ञाने सति चापतृतीयांशज्याज्ञानम् ।

त्रिभागजीवां द्विगुणां विधाय तां त्रिभिर्भजेल्लब्धलवान् पृथङ्न्यसेत् ।
त्रिभज्यकावर्गहृतं तु तद्घनं लवावशेषेण युतं च कारयेत् ॥ ६० ॥

पुनस्त्रिभिस्तं विभजेत् फलं कला न्यसेच्च पङ्क्तौ घनतोऽपि पङ्क्तिजात् ।
अनन्तरोक्तं पतितं घनं पुनः त्रिभज्यकावर्गहृतं च कारयेत् ॥ ६१ ॥

अनेन शेषेण कलावशेषकं प्रयुज्य भूयो विभजेत् त्रिभिः फलम् ।
विलिप्तिकाः पङ्क्तिपटे निवेशयेत् इतीह कुर्यात् पुनरेव शेषतः ॥ ६२ ॥

पङ्क्तेर्दलं स्यात् प्रथमांशजीवा तां च प्रकारान्तरतः प्रवक्ष्ये ।
त्र्यंशज्यका स्वत्रिलवेन हीना पृथग्घनोस्यास्त्रिहृतः फलेन ॥ ६३ ॥

त्रिभज्यकावर्गहृतेन युक्ता मुहुस्तदर्धं प्रथमांशजीवा ।
त्रिभागजीवात्रिलवः पृथक्स्थः स्वत्र्यंशयुक्तेन घनेन तस्य ॥ ६४ ॥

त्रिभज्यकावर्गहृतेन युक्तः मुहुर्मुहुर्वा प्रथमांशजीवा ।
एभिः प्रकारैर्धनुषस्त्रिभागज्यकां स्फुटां वा गणकः प्रकुर्यात् ॥ ६५ ॥

रामानागाजिनादेवागोक्षावारिधिवह्नयः ।
पिण्डाः शक्रावियद्बाणा भागाद्या त्र्यंशसिञ्जिनी ॥ ६६ ॥

अत्रोपपत्तिः ।

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1 षष्ठप्रकारे] ष्टप्रकारे B_0; तृतीयप्रकारे B_1
1 चाप-] चप- B_0
1 -यांशज्या-] -यांसज्या- B_2, R
2 जीवां द्विगुणां विधाय] जीवा द्विगुणा विधायं B_2
2 -जेल्लब्धलवान् पृथङ्न्यसेत्] -जेल्लब्द- W; -वानथङ्न्यसेत् B_1
3 -वर्गहृतं] -वार्गहृतं W; -वहृतं B_1
4–5 पुनस्त्रिभिस्तं...कारयेत् ॥ ६१ ॥] *verse om.* B_1
5 वर्गहृतं] वार्गहृतं R
6 अनेन शेषेण] लवादिलब्धेन W
6 त्रिभिः फलम्] त्रिभिफलम् B_2, R
9 पृथग्घनोस्यास्त्रि-] पृथग्घनास्यास्त्रि- B_1; पृथग्घनोस्यस्त्रि R
10–11 त्रिभज्यकावर्गहृतेन...तस्य ॥ ६४ ॥] *verse om.* B_1
10 वर्गहृतेन] वर्गघ्नहृतेन W
11 तस्य] तस्या B_1
13 प्रकारैर्धनुष *dup. in* B_1
13 -ज्यकां] -ज्यको B_0
14 -वह्नयः] -बह्नयः R; -वह्नय B_1

त्रिज्याप्रमाणेन विधाय वृत्तं झकेन्द्रकं तत्र तु चापमिष्टम् ।
तुल्यत्रिभागं कखगाघचिह्नं तच्चापकर्णं विलिखेत् कघञ्च ॥ ६७ ॥

कखं खगं गाघमिति क्रमेण त्र्यंशास्तदा तच्छ्रुतयोऽपि लेख्याः ।
कगं खघं कर्णमिती क्रमेण खछं गचं लम्बमिती च तद्वत् ॥ ६८ ॥

भूमिः कघं वक्त्रमिहास्ति खागं काखं गघं द्वौ क्रमशो भुजाख्यौ ।
एवं समानास्यभुजं महाक्ष्मं चतुर्भुजं क्षेत्रमिहोहनीयम् ॥ ६९ ॥

ततस्तदन्तर्गतमन्यदेतत् कखागसंज्ञं त्रिभुजं प्रकल्प्यम् ।
कागं क्ष्मा तत्र भुजौ क्रमेण कखं खगं लम्बमितिस्तु खाजम् ॥ ७० ॥

अथात्र वृत्ते त्रिभमौर्विकास्ति कझं खझं वा कजमेव दोर्ज्या ।
कोटिज्यका जाझमिता शरोऽपि खजं सदा काखशरासनस्य ॥ ७१ ॥

यावत्तावद्दोःप्रमाणं प्रकल्प्यं तद्धीनक्ष्मा या दलं स्याच्च सन्धिः ।
सन्ध्यूना भूः पीठसंज्ञं च बाहोः वर्गः साध्यः सन्धिवर्गेण हीनः ॥ ७२ ॥

उत्पन्नोऽयं लम्बवर्गोऽथ तेन युक्तः कार्यः पीठवर्गोऽपि तज्ज्ञैः ।
एवं जातो भूमिवक्त्रावघातः दोर्वर्गाढ्यः कर्णवर्गस्स एव ॥ ७३ ॥

शरेण हीना त्रिभसिञ्जिनी चेत् तदा धनुःकोटिलवज्यका स्यात् ।
त्रिज्याकृतिस्तत्कृतिवर्जिता चेत् तदा भुजज्याकृतिरेनयाढ्यः ॥ ७४ ॥

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17 चापमिष्टम्] चापिमिष्टम् B_2,
 B_1
18 कखगा-] खगा- B_1
18 तच्चापकर्णं] तच्चापचिह्नं B_1
19 -शास्तदा] -शास्ततः B_1
20 कर्णमिती] कर्णमिति B_2, R
21 -मिहास्ति] -मिहोस्ति R
21 द्वौ क्रमशो] दोः क्रमतो N
22 भूमिः ... महा] *om.* B_2
22 महाक्ष्मं] समाक्ष्मं B_1
23 -मन्यदेतत्] -मेतदन्यत् B_1
23 त्रिभुजं] त्रेमुहाजां B_2
24 कागं] कामं W
24 खाजम्] खीजम् B_0
25 कझं खझं वा] कडं खडं वा B_1; कझं ख वा N
26 जाझमिता] ज्याप्रमिता B_1; झमिति B_2; जाझमिति R
26 -सनस्य] -स⁻स्य N
27 प्रकल्प्यं] प्रकखल्प्यं B_2
27 दलं स्या-] द स्या- B_1
28 बाहोः वर्गः] बाहोवर्गः B_2, R
29 उत्पन्नोऽयं लम्ब-] उत्पन्नो लम्ब- B_2, R
29 तज्ज्ञैः] तज्ज्ञै B_1
30 भूमिव-] भू⁻व- B_1
30 दोर्वर्गाढ्यः] दोवर्गाढ्यः B_2, R
30 कर्णवर्गस्स] कर्णवर्गास्स B_2, R
31 शरेण] शरेन B_1; त्रिभेण W
32 -याढ्यः] -याढ्या N; -याढ्याः B_2

बाणस्य वर्गो भुजवर्गकः स्यात् शराहतव्याससमो भवेत् सः ।
ततो भजेत्तद्भुजवर्गमानं व्यासेन बाणो भवतीह लब्धिः ॥ ७५ ॥

तयोनितं व्यासदलं च कोटिज्या जायते तत्कृतिवर्जिता च ।
त्रिज्याकृतिः स्याद्भुजमौर्विका या वर्गश्चतुर्घ्नः स तु कर्णवर्गः ॥ ७६ ॥

एवं प्रकारद्वयतोऽपि कुर्यात् कर्णस्य वर्गौ गणकप्रवीणः ।
बीजक्रियातः पुनरत्र पक्षौ तुल्यौ विदध्यात् कथितप्रकारैः ॥ ७७ ॥

यावत्तावन्मानकेनापवर्त्यौ यावत्तावन्मानजो यो घनोऽत्र ।
सोऽपि व्यक्ते पक्ष एवानुवेश्यो व्यक्तं कार्यं गूढमानं ततोऽपि ॥ ७८ ॥

यावत्तावद्धनो योऽत्र त्रिज्यावर्गविभाजितः ।
तत्त्र्यंशयुक्तं भूत्र्यंशं बाहुमानं स्फुटं जगुः ॥ ७९ ॥

सामान्यतो भूमितृतीयभागो व्यक्तस्य मानं परिकल्पनीयम् ।
कृत्वा घनं तस्य तु तत्त्रिभागं त्रिभज्यकावर्गहृतं फलं यत् ॥ ८० ॥

भूमित्रिभागेन युतं प्रकुर्यात् मुहुर्मुहुः स्पष्टतरो भवेत् सः ।
प्रत्येकपंक्तिस्थितलब्धिजातं घनं त्रिमौर्व्याः कृतिभाजितं तम् ॥ ८१ ॥

तथा पुरोक्तेन घनेन हीनं क्षिपेत् त्रिभक्तोर्वरिते मुहुश्च।
एवं स्वबुद्ध्या गणको विदध्यादव्यक्तमानस्य परिस्फुटत्वम्॥ ८२ ॥

अत्रोदाहरणम् । भुजमानं यावत्तावत् या १ । तेनोनिता भूः [या १̇ भू १] ।

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33 शरा-] छरा- B_2, R
33 -राहतव्यासस-] -राप्तसस- B_1
34 लब्धिः] लब्दिः W
36 -श्चतुर्घ्नः स तु] -श्चतुः स तु B_1
36 कर्ण-] कतंर्ण- B_1
37 वर्गौ] वर्गो B_2
37 -प्रवीणः] -प्रवीणीः B_1
38 -यातः पुनरत्र] -यातपुनरतंत्रं B_2
38 पक्षौ तुल्यौ] पतुल्यौ B_1
38 कथितप्रकारैः] कथितः प्रकारैः B_2, R; कथितप्रकारै B_1
39 -त्तावन्मानजो] -त्तावन्तावन्मानजो B_1
39 घनोऽत्र] घनो ३ B_2
41 -जितः] जित B_2
43 -तीयभा-] -तीयेभा- B_2
43 व्यक्तस्य मानं] व्यक्तसमानं B_2
43 -मानं परि-] -मानं पपरि- B_1
44 तस्य तु] तस्य च N
44 तु तत्त्रिभागं] तु तृभणं B_1
44 यत्] स्यात् B_1
46 -लब्धिजातं] -लब्धजातं B_1
46 कृतिभा-] कृतभा- N
47 -क्तो वरि-] -तोर्वरि- B_0
49 तेनोनिता] तेरोनिता W; तेनोना B_1

अस्या दलं [या १ भू १] । जातोऽयं सन्धिः । सन्ध्यूना भूः पीठोऽयं [या १ भू १] ।
२ २
बाहुवर्गः [याव १] । सन्धिवर्ग एषः [याव १ याभू २ भूव १] । अनयोः समच्छेदयोरन्तरं
४
जातो लम्बवर्गोऽयं [याव ३ याभू २ भूव १] । अथ पीठवर्गः [याव १ याभू २ भुव १] ।
४ ४
लम्बवर्गपीठवर्गयोर्योगोऽयमेव कर्णवर्गः [याव १ याभू १] । अयमेव भूमिवक्त्रावघातो भुजवर्गयुक्तः सम्पन्नः ।

अथ प्रकारान्तरेण कर्णवर्गः साध्यते । भुजवर्गमानं [याव १] व्यासहृतं जातो बाणः [याव १]
व्या १
। अनेन हीनं व्यासार्धम [याव २ व्याव १] । इयं कोटिज्या । अस्याः कृतिरियं जाता
व्या २
। न्यासः [यावव ४ यावव्याव ४ व्यावव १] । अनया वर्जितो व्यासार्धवर्गो जातश्चतुर्गुणोऽयं
व्याव ४
कर्णवर्गः [यावव ४ यावव्याव ४] ।
व्याव १

प्रथमानीतकर्णवर्गेण समः कार्यः । तेन समौ पक्षौ [यावव ० यावव्याव १ याभूव्याव १] [यावव ४ यावव्याव ४ या ००००] एतौ यावत्तावता अपवर्त्यौ । जातौ -- [याघ ० याव्याव १ भूव्याव १] [याघ ४ याव्याव ४ भू ००००] अत्र यावत्तावद्धनं रूपेष्वेव निवेश्य परस्परशोधनाल्लब्धं यावत्तावान्मानम् [याघ ४ भूव्याव १] [याव्याव ३] । भाज्यभाजकौ चतुर्भिरपवर्तितौ [याघ १ भूत्रिव १] [यात्रिव ३] ।

भूत्रिभागं सामान्यतो भुजमानमङ्गीकृत्य तस्य घनं कृत्वा त्रिभिर्विभज्य पुनस्त्रिज्यावर्गेण भक्त्वा भुजमाने निक्षिप्य तदेव भुजमानं पुनः स्वीकृत्य तस्य घनं कृत्वा त्रिभिर्विभज्य त्रिज्यावर्गेण च भक्त्वा भूत्रिभागे क्षिप्त्वा मुहुः स्फुटं भुजमानयेत् प्रकारान्तरेण च ।

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50 सन्ध्यूना भूः] *om.*B_2, R

51 अनयोः] अननयोः R

51 -योः ... जातो] *om.*B_2

51–52 समच्छेद ... वार्गोऽयं] समच्छेदोहतयोरन्तरं जातीयं लम्बवर्गः B_1

53 लम्बवर्ग-] लंवर्ग- B_1

53 -र्योगो अय-] -र्योगाय- B_1

53 -वक्त्रावघा-] वक्त्रीवघा- W

55 -रान्तरेण] -रातरेण B_2

55 साध्यते । भुज-] साध्यः । तत्र भुज- N

55 भुजवर्गमानं] भुजवगन B_1

55 [याव १] व्यासहृतं] [याव १ व्यायाव १ व्या] सहृतं R

56 अनेन हीनं] अनेहीनं B_1

56 जाता] *om.* B_1

58 कर्णवर्गः] कार्णवर्गः W

59 -कर्ण ... समः] -कर्णवर्गसमः N

59 कार्यः] कार्य R

61 अत्र] *om.* N

64 -कृत्य तस्य] कृत्यस्य R

64 -वर्गेण] -वर्गेन B_1, B_2

65 भक्त्वा] *om.* B_0, R

65 स्वीकृत्य तस्य] स्वीकृत्यस्य B_1

66 -भागे क्षिप्त्वा] -भागेप्त्वा B_2, R

66 भुजमान-] भुमान- B_1

66 च] *om* B_1

अथ प्रत्येकांशज्याज्ञाने सति प्रत्येककलाज्याज्ञानम् ।

चापार्धचापत्रिलवोत्थजीवासंसाधनार्थाभिहितप्रकारैः ।
संसाधयेत् पञ्चकाल्पोत्थजीवामाद्यांशजीवात इह प्रवीणैः ॥ ८३ ॥

अथ तिथिकलिकानां सिञ्जिनीतो विदध्यात्
विनृपलवकलायाः सिञ्जिनीमुक्तरित्या ।
पुनरपि लवमौर्व्या भागभूपांशजीवां
गगनरसविभक्तां साधयेत् तन्त्रविज्ञः ॥ ८४ ॥

एवं हि चापद्वयजातमौर्व्या या जायते चापगुणैक्यजीवा ।
सैवाद्यलिप्तोद्भवसिञ्जिनी स्यात् ततोऽन्यलिप्ताप्रभवा गुणाः स्युः ॥ ८५ ॥

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68 -संसा-] -ससा- B_1

69 -जीवामाद्यां-] -जिवाद्यां- B_0

69 -द्यांशजी-] -द्यांसजी- N

70 सिञ्जिनीतो] सिजिनीतो B_2; सिञ्जिनीमुक्तरित्या *in next line N*

71 सिञ्जिनीमु-] सिजिनीतोमु- B_2

72 -र्व्या ...जीवां] *om.* B_2

73 गगनर-] गगर- W

74 -मौर्व्या या] -मौर्या यो B_0; -मौर्व्यार्यो B_2, R

75 -सिञ्जिनी] -सिजिनी B_2

III Text, Translation, and Technical Analysis

In the following, we present the critically edited text in *Devanāgarī*, a transliteration, and translation of verses 60 to 85 (see Table 4 for the contents of the verses). We also give a detailed technical analysis of the contents to aid the modern reader in following the text. Original diagrams, as they were executed by the scribe, have been included along with a modern reproduction where appropriate. Square brackets [ ] represent an addition to the translation for better readability. Round brackets ( ) indicate the corresponding text in an editorial gloss to clarify the text.

Verses	Contents
60–65	Description of the iterative procedures
66	Numerical value of the Sine of three degrees
67–71	Construction of a cyclic quadrilateral with three equal sides and conception of various triangles and their associated elements inside the cyclic quadrilateral
72–73	Expression for the square of the diagonal in a cyclic quadrilateral
74–77	Alternative expression for the diagonal of the cyclic quadrilateral
78–79	Expression for the measure of the chord
80–82	Iterative procedure for obtaining the chord
prose	Worked example to derive the recursive algorithm
83–85	Determining the Sine of one minute from the Sine of one degree

Table 4: The structure and contents of the sixth section dealing with the Sine of 1°.

In particular, given the graphical nature of the symbolic style of reasoning in the "worked example" (*udāharaṇa*) section (preceding verse 83), we have included small cropped images from one of the more legible and error free manuscripts, so that the reader can directly appreciate the way in which the "algebraic" reasoning was expressed in this context nearly four centuries ago.

III.1 Determination of the Sine of 1°

Text and Translation

अथ षष्ठप्रकारे चापज्याज्ञाने सति चापतृतीयांशज्याज्ञानम् ।

atha ṣaṣṭhaprakāre cāpajyājñāne sati cāpatṛtīyāṃśajyājñānam |

Now, in the sixth unit, with the knowledge of the Sine of the arc, the knowledge of the

Sine of one third of the arc [is described].

त्रिभागजीवां द्विगुणां विधाय तां
त्रिभिर्भजेल्लब्धलवान् पृथङ्न्यसेत् ।
त्रिभज्यकावर्गहृतं तु तद्घनं
लवावशेषेण युतं च कारयेत् ॥ ६० ॥ ॥ वंशस्थम् ॥
पुनस्त्रिभिस्तं विभजेत् फलं कला
न्यसेच्च पङ्क्तौ घनतोऽपि पङ्क्तिजात् ।
अनन्तरोक्तं पतितं घनं पुनः
त्रिभज्यकावर्गहृतं च कारयेत् ॥ ६१ ॥ ॥ वंशस्थम् ॥
अनेन शेषेण कलावशेषकं
प्रयुज्य भूयो विभजेत् त्रिभिः फलम् ।
विलिप्तिकाः पङ्क्तिपटे निवेशयेत्
इतीह कुर्यात् पुनरेव शेषतः ॥ ६२ ॥ ॥ वंशस्थम् ॥
पङ्क्तेर्दलं स्यात् प्रथमांशजीवा
तां च प्रकारान्तरतः प्रवक्ष्ये ।

tribhāgajīvāṃ dviguṇāṃ vidhāya tāṃ
tribhir bhajel labdhalavān pṛthaṅnyaset |
tribhajyakāvargahṛtaṃ tu tadghanaṃ
lavāvaśeṣeṇa yutaṃ ca kārayet || 60 || || *vaṃśastham* ||
punas tribhis taṃ vibhajet phalaṃ kalā
nyasec ca paṅktau ghanato'pi paṅktijāt |
anantaroktaṃ patitaṃ ghanaṃ punaḥ
tribhajyakāvargahṛtaṃ ca kārayet || 61 || || *vaṃśastham* ||
anena śeṣeṇa kalāvaśeṣakaṃ
prayujya bhūyo vibhajet tribhiḥ phalam |
viliptikāḥ paṅktipaṭe niveśayet
itīha kuryāt punar eva śeṣataḥ || 62 || || *vaṃśastham* ||
paṅkter dalaṃ syāt prathamāṃśajīvā
tāṃ ca prakārāntarataḥ pravakṣye |

Having multiplied the Sine of 3 degrees by two, one should divide that by three. [Then] one should set down separately the degrees [of the quotient] thus obtained. Now, one should add the cube of that (i.e., the *labdhalava*, the quantity kept separately) divided by the square of the Radius, to the remaining part obtained by dropping the degrees (i.e., the minutes, seconds, and so on). Again, may one divide that result by three. The quotient [in the form of] minutes obtained may be put down in the result-line (*paṅkti*[5]). Subtracting the cube of the previous result from the cube of the number currently in the result-line, again may one divide by the square of the Radius.

[5] That is, the line in which the degrees were earlier noted down, and into which the successive digits of the result obtained in the successive steps of iteration have to be written down.

With this remainder [obtained by the previous operation), having added the remainder in minutes and so on (*kalāvaśeṣaka*), again, may one divide the result by three. May one set down the seconds [obtained thus] on the cloth (*paṭa*) [which contains] the result-line. With the remainder [obtained at this stage], again the same process has to be repeated here. [After iterating to achieve the desired accuracy], half the result-line produces the Sine of 1°. And I will explain that by another method (*prakāra*).

त्र्यंशज्यका स्वत्रिलवेन हीना
पृथग्घनोस्यास्त्रिहृतः फलेन ॥ ६३ ॥ ॥ बाला उपजातिका ॥
त्रिभज्यकावर्गहृतेन युक्ता
मुहुस्तदर्धं प्रथमांशजीवा ।

tryaṃśajyakā svatrilavena hīnā
pṛthag ghano'syās trihṛtaḥ phalena || 63 || *|| bālā upajātikā ||*
tribhajyakāvargahṛtena yuktā
muhustadardhaṃ prathamāṃśajīvā |

[Considering] the Sine of three degrees [and] removing one third of it, [may one put it down] separately. The cube of this is divided by three. When this result [obtained thus] is divided by the square of the Radius [and] added to [the initial result that was stored], [and the process] is repeated, half of it (i.e., the iterated value) is the Sine of one degree.

त्रिभागजीवात्रिलवः पृथक्स्थः
स्वत्र्यंशयुक्तेन घनेन तस्य ॥ ६४ ॥ ॥ जाया उपजातिका ॥
त्रिभज्यकावर्गहृतेन युक्तः
मुहुर्मुहुर्वा प्रथमांशजीवा ।

tribhāgajīvātrilavaḥ pṛthaksthaḥ
svatryaṃśayuktena ghanena tasya || 64 || *|| jāyā upajātikā ||*
tribhajyakāvargahṛtena yuktaḥ
muhurmuhur vā prathamāṃśajīvā |

A third part of the Sine of three degrees [is obtained and] is put down separately. Then, one third of the cube of this is divided by the square of the Radius and this is added to [the value which was put down separately, i.e., Sine of three degrees divided by 3]. [This process when carried out] iteratively [directly gives] the Sine of one degree.

एभिः प्रकारैर्धनुषस्त्रिभाग-
ज्यकां स्फुटां वा गणकः प्रकुर्यात् ॥ ६५ ॥ ॥ प्रेमा उपजातिका ॥

ebhiḥ prakārair dhanuṣas tribhāga-
jyakāṃ sphuṭāṃ vā gaṇakaḥ prakuryāt || 65 || *|| premā upajātikā ||*

By [any of] these methods, may the mathematician compute an accurate value (*sphuṭa*) of the Sine of one third of an arc.

रामा नागा जिना देवा गोऽक्षा वारिधिवह्नयः ।
पिण्डाः शक्रा वियद्बाणा भागाद्या त्र्यंशसिञ्जिनी ॥ ६६ ॥ ॥ पथ्यावक्त्र ॥

rāmā nāgā jinā devā go 'kṣā vāridhivahnayaḥ |
piṇḍāḥ śakrā viyadbāṇā bhāgādyā tryaṃśasiñjinī || 66 || || *pathyāvaktra* ||

The Sine of three degrees is: three (*rāma*), eight (*nāga*), twenty-four (*jina*), thirty-three (*deva*), fifty-nine (*go-akṣa*), thirty-four (*vāridhi-vahni*), twenty-eight (*piṇḍa*), fourteen (*śakra*), fifty (*viyat-bāṇa*) in degrees (*bhāga*) and so on.

Technical Analysis

The verses above present three distinct iterative algorithms for determining the Sine of 1°. The first involves solving the cubic equation by means of an iterative procedure which uses division and establishes the solution digit by digit. The other two algorithms which employ direct fixed point iteration also have their basis in the same cubic equation. While the relation of these three methods with earlier techniques needs to be more fully explored, overcoming the problem of interdependency by means of a simple iterative procedure (often referred to as *aviśiṣṭa-karma*) was well known to Indian mathematicians.[6]

Before proceeding to describe these three algorithms, we will present the trigonometric identity on which these algorithms are based upon.

III.1.1 Trigonometric Identity Forming the Basis for the Cubic Equation

The multiple-angle formula giving the key relation between the sine of an arc and the sine of three times that arc which forms the basis of all the three algorithms is

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta. \tag{1}$$

Given that the modern sine and the non-unit-radius Sine are related by the expression $R\sin\theta = \operatorname{Sin}\theta$, the above formula expressed in Sines reduces to

$$\operatorname{Sin} 3\theta = 3\operatorname{Sin}\theta - \frac{4\operatorname{Sin}^3\theta}{R^2}. \tag{2}$$

Though the above relation is valid for any angle θ, in this section Nityānanda is concerned with determining the Sine of 1°. Therefore, substituting $\theta = 1°$ and denoting $\operatorname{Sin} 1° = y$, Equation (2) becomes

$$\operatorname{Sin} 3° = 3y - \frac{4y^3}{R^2},$$

$$\text{or,}\qquad y = \frac{\operatorname{Sin} 3°}{3} + \frac{4y^3}{3R^2}, \tag{3}$$

where $\operatorname{Sin} 3°$, the Sine of the smallest integer number of degrees that can be computed by geometric methods, is known.

6 See the recent studies by Plofker (2002) and Sriram, Ramasubramanian, and Pai (2012).

Nityānanda presents three methods to solve the above equation, which we describe one after the other in the following sections.

III.1.2 Method One: Verses 60–63 (first half)

Nityānanda's first method commences straight away with the prescription of the iterative procedure for solving Equation (3). This recursive procedure, as it appears in Arabic sources, has been explained in previous publications.[7] This is the first time it has been explored in detail in a Sanskrit source.

Before analysing the Sanskrit text, we will outline the way in which this procedure works using modern symbolism. Since Arabic sources used chords to carry out this procedure, Nityānanda appears to be emulating this by doubling y in Equation (3),[8] so that there results

$$2y = \frac{2 \cdot \text{Sin}\, 3}{3} + \frac{(2y)^3}{3R^2}. \tag{4}$$

The left hand side $2y$ can be considered to be made up of successive sexagesimal digits, denoted by p_0, p_1, p_2, and so on. The text calls each of these components: degrees (*lava*), minutes (*liptā*), seconds (*viliptā*), and so on, to reflect the fact that they are successive sexagesimal parts.

$$2y = p_0, p_1, p_2, ...$$

By a careful analysis of the structure of Equation (4), the algorithm establishes each of these components one-by-one, by an iterative process. In other words, $2y$ is updated with each iteration by an additional sexagesimal place. At any stage of the process, an approximate value of the Sine of 1° can be obtained by dividing the current value of $2y$ by 2.

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7 See Van Brummelen (2009, 147–149), Rosenfeld and Hogendijk (2003), Aaboe (1954).

8 To justify this step, note that

$$2 \sin\theta = \frac{2 \cdot \sin 3\theta + (2\sin\theta)^3}{3}$$

is equivalent to (using the identity $2\sin\theta = \text{crd}2\theta$)

$$\text{crd}2\theta = \frac{\text{crd}6\theta + (\text{crd}2\theta)^3}{3},$$

the original Arabic relation.

III.1.3 Method One: Explanation of the Algorithm

Since y is small, neglecting y^3, as a first approximation, Equation (4) can be written as

$$2y \approx \frac{2 \cdot \mathrm{Sin}\, 3}{3}.$$

This division gives a quotient (p_0) and a remainder (r_0)

$$2y \approx \frac{2 \cdot \mathrm{Sin}\, 3}{3} = p_0 + \frac{r_0}{3},$$

where p_0 is the first sexagesimal digit of the resulting division (called *labhdalava* (quotient-degrees) in the text), and r_0 is the remainder (called *lavāvaśeṣa*) of this division. Then, p_0 is to be placed in the "result-line" (*paṅkti*).

Now, p_0 is used to generate the next segasesimal digit p_1

$$\frac{r_0 + \dfrac{(p_0)^3}{R^2}}{3} = p_1 + \frac{r_1}{3},$$

The quotient p_1 is added to the "result line" after p_0. Next, the third significant sexagesimal digit p_2 is to be found using the relation

$$\frac{r_1 + \dfrac{(p_0, p_1)^3 - (p_0)^3}{R^2}}{3} = p_2 + \frac{r_2}{3}.$$

Therefore, after the second iteration, the *paṅkti* has three entries

$$2y \approx \underline{p_0, p_1, p_2}.$$

Again, with r_2 and the elements in the *paṅkti*, the fourth significant sexagesimal digit p_3 is found using the relation

$$\frac{r_2 + \dfrac{(p_0, p_1, p_2)^3 - (p_0, p_1)^3}{R^2}}{3} = p_3 + \frac{r_3}{3}.$$

The procedure outlined above can be carried out as many times as one desires, depending on the sought accuracy. In the n^{th} iteration, one produces p_n using the equation

$$\frac{r_n + \dfrac{(p_0, ..., p_{n-1})^3 - (p_0, ..., p_{n-2})^3}{R^2}}{3} = p_n + \frac{r_n}{3}.$$

Thus after n iterations we have the following *paṅkti*:

$$2y \approx \underline{p_0, p_1, p_2, ..., p_n}.$$

Division by two produces the desired value for the Sine of 1°.

Method One: Numerical Evaluation of the Successive Sexagesimal Digits

Since Nityānanda simply describes the algorithm and does not specify the numerical value of the Sine of 1° in the verses presented in this section, for the sake of completeness, we carry out a few iterations of the procedure to demonstrate how the numbers might have been generated. The one and only numerical parameter Nityānanda does give is the Sine of 3° (verse 66). Its value, recorded using the object-numeral system is

$$\mathrm{Sin}\,3° = 3;08,24,33,59,34,28,14,50 \tag{5}$$

Interestingly, this value is not the same as the one given by al-Kashi, who gives 29 and not 50 in the eighth sexagesimal place (Rosenfeld and Hogendijk 2003, 45, n. 57). However, the value given in (5) is found in the earlier part of this text (Rosenfeld and Hogendijk 2003, 39). It is also found in the *Jyācāpa* manuscript referred to in the introduction.

The first few digits can be recomputed below as follows.

To find p_0: In the zeroth-order approximation, p_0 is simply taken to be one-third of the chord value corresponding to 6° (crd6 = 2 sin 3). That is,

$$\begin{aligned}\frac{2\cdot \mathrm{Sin}\,3}{3} &= \frac{2\times 3;08,24,33,59,\ldots}{3},\\ &= 2;05,36,22,39,42,\ldots,\\ &= p_0+\frac{r_0}{3}.\end{aligned}$$

Here, $p_0 = 2$ is in degrees and the remainder $\frac{r_0}{3} = 5,36,22,39,42,\ldots$ is in minutes, seconds, and so on, and thus $r_0 = 16,49,07,59,08,56,\ldots$ minutes and so on.

At this stage, the "result-line" (*paṅkti*) simply consists of 2. That is,

$$2y \approx p_0 = \underline{2}.$$

To find p_1:

$$\begin{aligned}\frac{r_0+\dfrac{(p_0)^3}{R^2}}{3} &= \frac{0;16,49,07,59,8,56,\ldots+\dfrac{2^3}{60^2}}{3},\\ &= \frac{0;16,49,07,59,08,56,\ldots+0;00,08}{3},\\ &= \frac{0;16,57,07,59,08,56,\ldots}{3},\\ &= 0;05,39,02,39,42,\ldots,\\ &= p_1+\frac{r_1}{3}.\end{aligned}$$

Here, $p_1 = 5$ is in minutes and the remainder $\frac{r_1}{3} = 39{,}02{,}39{,}42{,}\ldots$ is in seconds, thirds, and so on, and thus $r_1 = 1{,}57{,}07{,}59{,}08{,}\ldots$ minutes and so on.

The *paṅkti* at this stage is

$$2y \approx p_0, p_1 = \underline{2{,}5}.$$

To find p_2:

$$\begin{aligned}\frac{r_1 + \dfrac{(p_0,p_1)^3 - (p_0)^3}{R^2}}{3} &= \frac{0;01{,}57{,}07{,}59{,}8{,}56\ldots + \dfrac{(2;05)^3 - 2^3}{60^2}}{3},\\ &= \frac{0;01{,}58{,}10{,}31{,}13{,}56{,}\ldots}{3},\\ &= 0;00{,}39{,}02{,}23{,}30{,}24{,}38{,}\ldots,\\ &= p_2 + \frac{r_2}{3}.\end{aligned}$$

Here, $p_2 = 39$ is in seconds and the remainder $\frac{r_2}{3} = 2{,}23{,}30{,}24{,}38{,}\ldots$ is in thirds, fourths, and so on.

Now, the *paṅkti* is

$$2y \approx p_0, p_1, p_2 = \underline{2{,}05{,}39}.$$

This procedure can be continued until one wishes to stop. Dividing this by two produces

$$y = \mathrm{Sin}\, 1^\circ = 1;02{,}49{,}\ldots$$

The most precise figure for the Sine of 1° explicitly recorded by Nityānanda is included in his table of Sines in the *Sarvasiddhāntarāja* (see Figure 7 which gives the first page of the 18 page table of Sines). This numerical value is recorded as (second column, first row):

$$1;02{,}49{,}43{,}11$$

However, given the precision to eight places Nityānanda gives to the Sine of 3°, we assume that this is so that more places could be easily produced if needed.

III.1.4 Method Two: Verses 63 (second half)–64 (first half)

The second method Nityānanda describes to solve the cubic is again an iterative procedure, but instead of establishing the successive sexagesimal digits of the function one-by-one, it aims to find a better approximation of the functional value itself at each iteration.[9] The prescription of Nityānanda here is

[9] This method appears to be equivalent to one of the two given in *Jyācāpa*.

to again begin with the initial approximation: the Sine of 3° less a third part, which is equivalent to multiplying it by 2 and dividing by 3 as per the basic Equation (4). That is,

$$y_0 = \frac{2 \cdot \mathrm{Sin}\, 3}{3}. \tag{6}$$

This is to be increased by its own cube and divided by the product of 3 and the square of the radius to get the next iterate:

$$y_1 = y_0 + \frac{y_0^3}{3R^2}.$$

Now we use y_1 to get the next iterate. That is,

$$y_2 = y_0 + \frac{y_1^3}{3R^2}.$$

The subsequent iterations are

$$y_n = y_0 + \frac{(y_{n-1})^3}{3R^2}. \tag{7}$$

Since the values converge rapidly, the iterative process is terminated when one has the desired accuracy, namely

$$y_{n+1} \approx y_n.$$

The Sine of 1° is then found by taking half of this.

Method Two: Numerical Evaluation

In this particular case, we begin the numerical reconstructions with Equations (5) and (6). Thus, the initial approximation of the function is

$$\begin{aligned} y_0 = \frac{2 \cdot \mathrm{Sin}\, 3}{3} &= \frac{2}{3} \times 3;08,24,33,59,34,28,14,50, \\ &= 2;05,36,22,39,42,58,49,53,20. \end{aligned}$$

The values of successive approximations obtained using Equation (7) are displayed in Table 5. As may be noted from the third column of Table 5, the successive differences keep rapidly decreasing. Today we know that fixed point iterative methods converge rapidly provided one starts with a good first estimate. Thus, method two may be considered as a fixed point iteration as applied to a cubic equation.

III.1.5 Method Three: Verses 64 (second half)–65

From a mathematical viewpoint, this method is identical to method two, except that multiplying and dividing by 2 before and after the iterations is absent.[10] If

[10] This method appears to be equivalent to one of the two given in *Jyācāpa*.

Iteration number (n)	Sin 1° = $y_n/2$	difference = $\frac{y_n - y_{n-1}}{2}$
0	1;02,48,11,19,51,29,24,56,40	
1	1;02,49,43,04,32,00,53,37,37	0;00,01,31,44,40,31,28,40,57
2	1;02,49,43,11,14,14,50,04,39	0;00,00,00,06,42,13,56,27,02
3	1;02,49,43,11,14,44,14,17,10	0;00,00,00,00,00,29,24,12,31
4	1;02,49,43,11,14,44,16,26,08	0;00,00,00,00,00,00,02,08,58
5	1;02,49,43,11,14,44,16,26,17	0;00,00,00,00,00,00,00,00,09
6	1;02,49 43,11,14,44,16,26,17	0;00,00,00,00,00,00,00,00,00
7	1;02,49,43,11,14,44,16,26,17	0;00,00,00,00,00,00,00,00,00

Table 5: Successive approximations to the Sine of 1° obtained by method two.

so, why does Nityānanda present this as an alternative approach to determining the Sine of 1°? As indicated earlier, the scale factor of 2 is no doubt a remnant of the Arabic preference for solving this problem for chords. Indian trigonometry, on the other hand, right from its inception, embraced the Sine and this scale factor is thus not essential. Therefore, the initial approximation to the Sine of 1° is taken to be

$$y_0 = \frac{\text{Sin}\, 3}{3}.$$

The subsequent iterations are

$$y_n = y_0 + \frac{(y_{n-1})^3}{3R^2}.$$

Here, the successive iterates produce approximations of the Sine of 1° directly. Since the numerical values obtained by this method will not be different from those listed in Table 5, we have not tabulated them separately.

III.2 Geometrical Demonstration

III.2.1 Construction of a Cyclic Quadrilateral with Three Equal Sides

Text and Translation

अत्रोपपत्तिः ।

atropapattiḥ |

Now, the demonstration:

त्रिज्याप्रमाणेन विधाय वृत्तं
झकेन्द्रकं तत्र तु चापमिष्टम् ।
तुल्यत्रिभागं कखगाघचिह्नं
तच्चापकर्णं विलिखेत् कघज्ञ ॥ ६७ ॥ ॥ वाणी उपजातिका ॥

trijyāpramāṇena vidhāya vṛttaṃ
jhakendrakaṃ tatra tu cāpam iṣṭam |

tulyaprabhāgaṃ kakhagāghacihnaṃ
taccāpakarṇaṃ vilikhet kaghañ ca || 67 || || *vāṇī upajātikā* ||

Having drawn a circle with measure [equal to] the Radius, whose centre is *jha*, there may one mark an arc of desired [length. Then] divide the arc into [three] equal parts, with markings *ka*, *kha*, *ga*, *gha*, and may one draw the arc-hypotenuse (*cāpakarṇa*),[11] *ka-gha*, corresponding to that arc.

कखं खगं गाघमिति क्रमेण
त्र्यंशास्तदा तच्छ्रुतयोऽपि लेख्याः ।
कगं खघं कर्णमिती क्रमेण
खछं गचं लम्बमिती च तद्वत् ॥ ६८ ॥ ॥ ऋद्धिः उपजातिका ॥

kakhaṃ khagaṃ gāgham iti krameṇa
tryaṃśās tadā tacchrutayo'pi lekhyāḥ |
kagaṃ khaghaṃ karṇamitī krameṇa
khachaṃ gacaṃ lambamitī ca tadvat || 68 || || *ṛddhiḥ upajātikā* ||

[Now], there are three parts [to this arc]: *ka-kha*, *kha-ga* and *ga-gha* respectively. Then, may the hypotenuses related to them also be drawn. [Now] *ka-ga* and *kha-gha* are the measures of the two hypotenuses. And similarly *kha-cha* and *ga-ca* are the measure of the two perpendiculars.

भूमिः कघं वक्त्रमिहास्ति खागं
काखं गघं द्वौ क्रमशो भुजाख्यौ ।
एवं समानास्यभुजं महाक्ष्मं
चतुर्भुजं क्षेत्रमिहोहनीयम् ॥ ६९ ॥ ॥ बाला उपजातिका ॥

bhūmiḥ kaghaṃ vaktram ihāsti khāgaṃ
kākhaṃ gaghaṃ dvau kramaśo bhujākhyau |
evaṃ samānāsyabhujaṃ mahākṣmaṃ
caturbhujaṃ kṣetram ihohanīyam || 69 || || *bālā upajātikā* ||

Here, the base is *ka-gha*, the face is *kha-ga*, and *ka-kha* and *ga-gha* are the two sides respectively. Thus, may a four sided geometrical construction with a long base [and] whose face and sides are equal, be conceived.

Technical Analysis

Having given the recursive algorithm to compute the Sine of 1°, Nityānanda now seeks to validate the cubic equation by considering a suitable geometric construction that would lead to Equation (4). For this, first we need to draw a

[11] For details on the arc-hypotenuse, see Montelle, Ramasubramanian, and Dhammaloka (2016, 24–26).

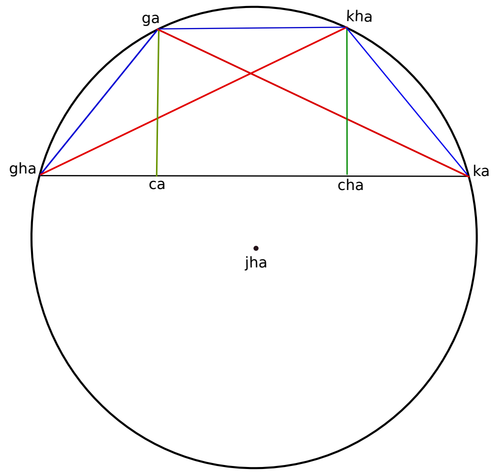


Figure 1: A reconstruction of the quadrilateral described in verses 67–69.

circle with a reasonably large radius (*trijyā*) so that the circle can comfortably accomodate various other geometrical constructions that need to be depicted inside it (see Figure 1).

The center of the circle is labelled with the letter *jha* (*jha-kendraka*). Now an arc of arbitrary length which is then divided into three equal parts is marked *ka-kha-ga-gha*. Then, *ka-gha* is joined up, effectively making a trapezium with three equal sides. The base of the trapezium, denoted by *ka-gha* is referred to by different names based on the context in which it is described. For instance, in verse 67, it is referred to as *cāpa-karṇa* when it is simply conceived as the chord of arc *ka-gha*. Whereas in verse 69, it is referred to as *mahā-kṣmā* (lit. the big base). The adjective *mahā* is used in order to distinguish it from the base (*kṣmā* or *kṣamā*) used to refer to the base of the triangle *ka-kha-ga* in a later verse (verse 70).

Now, the two diagonals of the trapezium *ka-ga* and *kha-gha*, as well as the two perpendiculars *ga-ca* and *kha-cha*—referred to as the *lambas* (perpendiculars)—are to be indicated in the diagram. It is further noted by Nityānanda that the face *kha-ga* and the two sides *ka-kha* and *ga-gha* of the trapezium are of the same dimension by using the phrase *samānāsya-bhuja* as an adjective for the word *caturbhuja* ("trapezium").

Although Nityānanda instructs the reader to draw an entire circle (*vṛtta*), only a half-circle appears in the various manuscripts that we were able to consult. The diagram found in one of the manuscripts is shown in Figure 2. A reconstructed version of it for the purpose of clarity is shown in Figure 3.[12]

[12] We guess restricting it to a semicircle could be in the interest of saving space.

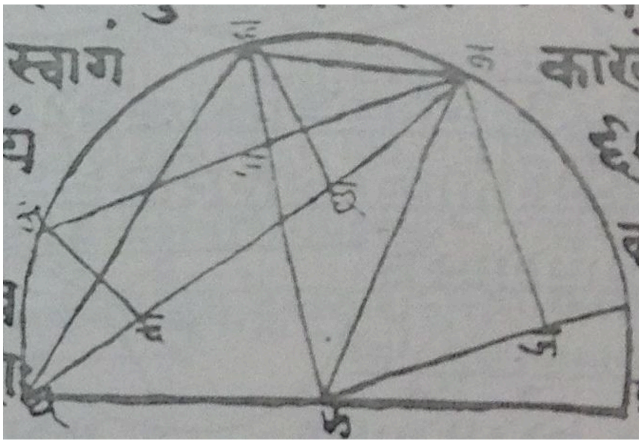


Figure 2: The diagram that accompanies this passage from the RORI manuscript (f. 13v).

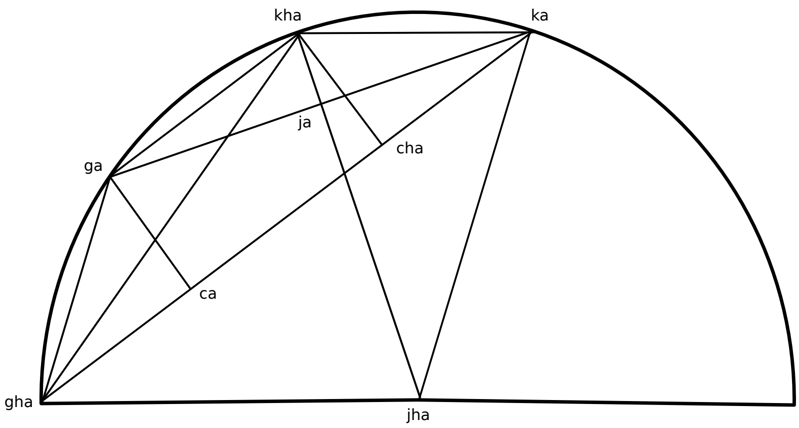


Figure 3: A reconstruction of the diagram in Figure 2.

III.2.2 Defining Triangles and Their Associated Elements Inside the Cyclic Quadrilateral

ततस्तदन्तर्गतमन्यदेतत्
कखागसंज्ञं त्रिभुजं प्रकल्प्यम् ।
कागं क्षमा तत्र भुजौ क्रमेण
कखं खगं लम्बमितिस्तु खाजम् ॥ ७० ॥ ॥ प्रेमा उपजातिका ॥

tatas tadantargatam anyad etat
kakhāgasaṃjñaṃ tribhujaṃ prakalpyam |
kāgaṃ kṣamā tatra bhujau krameṇa
kakhaṃ khagaṃ lambamitis tu khājam || *70* || || *premā upajātikā* ||

Then, may another triangle denoted by *ka-kha-ga* be conceived of inside that [quadrilateral]. There, *ka-ga* is the base (*kṣamā*[13]), the two sides are *ka-kha* and *kha-ga* respectively, [and] *kha-ja* is indeed the measure of the perpendicular.

अथात्र वृत्ते त्रिभमौर्विकास्ति
कझं खझं वा कजमेव दोर्ज्या ।
कोटिज्यका जाझमिता शरोऽपि
खजं सदा काखशरासनस्य ॥ ७१ ॥ ॥ प्रेमा उपजातिका ॥

athātra vṛtte tribhamaurvikāsti
kajhaṃ khajhaṃ vā kajam eva dorjyā |
koṭijyakā jājhamitā śaro'pi
khajaṃ sadā kākhaśarāsanasya || *71* || || *premā upajātikā* ||

Now in this circle, for the chord (*śarāsana*) *ka-kha*, *ka-jha* and *kha-jha* are certainly the Radii, *ka-ja* is indeed the Sine and *ja-jha* is the measure of the Cosine, [and] *kha-ja* is the arrow.

Technical Analysis

Having described the construction of a cyclic quadrilateral in great detail, Nityānanda considers the resulting construction in a slightly different way. This is just to highlight the quantities he will be dealing with in the next sections. He draws the attention of the reader to the non-right-angled triangle, *ka-kha-ga*, within the quadrilateral. He points out the base *ka-ga*, the two sides, *ka-kha* and *kha-ga*, and the perpendicular height of the triangle, here the line segment *kha-ja*. In two verses, he explicitly relates each of these line segments to their trigonometric equivalent. In particular, *ka-ja* is the Sine, *ja-jha* is the Cosine, and *kha-ja* is identified as the *śara* (arrow), or the Versine. These quantities are critical for the algebraic working that follows.

Before moving on to the next section, we may draw the attention of the readers to the word *śarāsana* appearing at the end of verse 71. The word literally means "the seat of the arrow." In this context, it refers to the Chord *ka-ga*.

[13] The word *kṣamā* literally means "forbearance." Since the earth is personified in Sanskrit literature as an embodiment of *kṣamā*, it seems the poet has taken the license to use *kṣamā* for *kṣmā* as the former fits the meter.

III.3 Algebraic Derivation

III.3.1 Expression for the Square of the Diagonal in a Cyclic Quadrilateral

Text and Translation

यावत्तावद्दोःप्रमाणं प्रकल्प्यं
तद्धीनक्ष्मा या दलं स्याच्च सन्धिः ।
सन्ध्यूना भूः पीठसंज्ञं च बाहोः
वर्गः साध्यः सन्धिवर्गेण हीनः ॥ ७२ ॥ ॥ शालिनी ॥
उत्पन्नोऽयं लम्बवर्गोऽथ तेन
युक्तः कार्यः पीठवर्गोऽपि तज्ज्ञैः ।
एवं जातो भूमिवक्त्रावघातः
दोर्वर्गाढ्यः कर्णवर्गस्स एव ॥ ७३ ॥ ॥ शालिनी ॥

yāvattāvaddoḥpramāṇaṃ prakalpyaṃ
taddhīnakṣmā yā dalaṃ syāc ca sandhiḥ |
sandhyūnā bhūḥ pīṭhasaṃjñaṃ ca bāhoḥ
vargaḥ sādhyaḥ sandhivargeṇa hīnaḥ || *72* || || *śālinī* ||
utpanno'yaṃ lambavargo'tha tena
yuktaḥ kāryaḥ pīṭhavargo'pi tajjñaiḥ |
evaṃ jāto bhūmivaktrāvaghātaḥ
dorvargāḍhyaḥ karṇavargās sa eva || *73* || || *śālinī* ||

Setting the measure of the chord to be the unknown (*yāvattāvat*,[14] i.e., the desired value), whatever is the [measure of the] base (*kṣmā*), [when that is] decreased by that [*yāvattāvat*], and halved, it is the *sandhi*.[15] The base (*bhū*) decreased by the *sandhi* is known as the *pīṭha*,[16] and the square of the side (*bāhu*) decreased by the square of the *sandhi* is to be determined.

The [result] thus obtained is the square of the perpendicular (*lamba*).[17] Now, the square of the *pīṭha* is to be increased by that (i.e., the square of the perpendicular) by those who are knowledgeable of that (i.e., geometry). Thus, what is obtained is the product of the base (*bhumi*) and the face (*vaktra*) increased by the square of the chord. This is indeed the square of the diagonal.

---

[14] Literally the term means "as much [desired] so much" (यावद्[इष्टं] तावत्). In the mathematical literature it is used as a technical term to refer to an "unknown" quantity denoted by *ka-kha* or *kha-ga* in Figure 1.

[15] This is also a technical term that refers to *ka-cha*.

[16] The significance of this term is explained in our notes.

[17] This refers to *cha-kha* or *ca-ga* in our diagram.

Technical Analysis

Nityānanda begins with the prescription that the length of the side of the cyclic quadrilateral (defined in the previous section), be referred to as *yāvattāvat*. Furthermore, he gives the mathematical relation for what he calls the *sandhi* which is the line segment *ka-cha*, and the *pīṭha*, which is the line segment *cha-gha* (see Figure 3).

$$\textit{sandhi} \; = \textit{ka-cha} = \frac{\textit{ka-gha} - \textit{kha-ga}}{2}$$

and

$$\textit{pīṭha} \; = \textit{cha-gha} = \textit{ka-gha} - \textit{ka-cha}$$

or alternatively,

$$\textit{pīṭha} \; = \textit{ka-ca} = \textit{ka-gha} - \textit{ca-gha}.$$

It may be noted from Figure 3 that the measure of the *yāvattāvat* (*kha-ga*) when mapped on the base (*ka-gha*) would be *ca-cha*, and this segment would simply be hanging in space without getting connected to the arc *ka-gha*, but for the two segments *ka-cha* and *ca-gha*. Since the latter two bring in the "union" with the arc, they are perhaps referred to as *sandhi*. The word *pīṭha* literally means a "seat" on which something rests. In the present context, the projection of *ka-ga* will rest on *ka-ca*, and maybe for this reason, the latter is referred to by the name *pīṭha*.

Then, by the Pythagorean theorem, the perpendicular (*lamba*) *cha-kha* is given by

$$(\textit{kha-cha})^2 = (\textit{ca-ga})^2 = (\textit{ka-kha})^2 - (\textit{ka-cha})^2. \tag{8}$$

Furthermore, the diagonal is computed via

$$(\textit{kha-gha})^2 = (\textit{kha-cha})^2 + (\textit{cha-gha})^2. \tag{9}$$

In the latter half of verse 73, Nityānanda states that the product of the base and the face increased by the square of the chord is the square of the diagonal. This gets easily verified as follows. Using Equation (8) in Equation (9), rewriting the first term in the RHS, we have

$$(\textit{kha-gha})^2 = (\textit{ka-kha})^2 - (\textit{ka-cha})^2 + (\textit{cha-gha})^2. \tag{10}$$

Now, rewriting the last two terms in the RHS of (10), we have

$$\begin{aligned}(\textit{kha-gha})^2 &= (\textit{ka-kha})^2 + (\textit{cha-gha} - \textit{ka-cha})(\textit{ka-cha} + \textit{cha-gha}),\\ &= (\textit{ka-kha})^2 + (\textit{cha-gha} - \textit{ca-gha})(\textit{ka-gha}),\\ &= (\textit{ka-kha})^2 + (\textit{ca-cha}) \cdot (\textit{ka-gha}),\\ &= (\textit{ka-kha})^2 + (\textit{kha-ga}) \cdot (\textit{ka-gha}),\end{aligned}$$

since $\textit{ka-cha} = \textit{ca-gha}$. This is the first expression for the square of the diagonal that Nityānanda presents in verse 73. Rewriting these in terms of the Sines and Chords, there results

$$(\textit{kha-gha})^2 = (\mathrm{Crd}\,\theta)^2 + \mathrm{Crd}\,\theta \cdot \mathrm{Crd}\,3\theta. \qquad (11)$$

III.3.2 Alternative Expression for the Diagonal of the Cyclic Quadrilateral

Text and Translation

शरेण हीना त्रिभसिञ्जिनी चेत्
तदा धनुःकोटिलवज्यका स्यात् ।
त्रिज्याकृतिस्तत्कृतिवर्जिता चेत्
तदा भुजज्याकृतिरेनयाढ्यः ॥ ७४ ॥ ॥ प्रेमा उपजातिका ॥
बाणस्य वर्गो भुजवर्गकः स्यात्
शराहतव्याससमो भवेत् सः ।
ततो भजेत्तद्भुजवर्गमानं
व्यासेन बाणो भवतीह लब्धिः ॥ ७५ ॥ ॥ माया उपजातिका ॥
तयोनितं व्यासदलं च कोटि-
ज्या जायते तत्कृतिवर्जिता च ।
त्रिज्याकृतिः स्याद्भुजमौर्विकाया
वर्गश्चतुर्घ्नः स तु कर्णवर्गः ॥ ७६ ॥ ॥ कीर्तिः उपजातिका ॥

śareṇa hīnā tribhasiñjinī cet
tadā dhanuḥkoṭilavajyakā syāt |
trijyākṛtis tatkṛtivarjitā cet
tadā bhujajyākṛtir enayāḍhyaḥ || 74 || *|| premā upajātikā ||*
bāṇasya vargo bhujavargakaḥ syāt
śarāhatavyāsasamo bhavet saḥ |
tato bhajet tadbhujavargamānaṃ
vyāsena bāṇo bhavatīha labdhiḥ || 75 || *|| māyā upajātikā ||*
tayonitaṃ vyāsadalaṃ ca koṭi-
jyā jāyate tatkṛtivarjitā ca |
trijyākṛtiḥ syād bhujamaurvikāyā
vargaś caturghnaḥ sa tu karṇavargaḥ || 76 || *|| kīrtiḥ upajātikā ||*

If the Radius is decreased by the Versine, then it is the Cosine (*koṭilavajyakā*).[18] If the square of the Radius is decreased by the square of that (the Cosine), then [there results] the square of the Sine; the square of the Versine increased by this is the square of the chord (*bhujavargaka*[19]). This will be equal to the Diameter multiplied by the Versine. Thence, one should divide the measure of the square of that Chord by the Diameter. The quotient (*labdhi*) here is the Versine.

Half the diameter diminished by that (i.e., the Versine) produces the Cosine and the square of the Radius decreased by the square of this (i.e., the Cosine) is the square of the Sine; that multiplied by four is indeed the square of the diagonal.

एवं प्रकारद्वयतोऽपि कुर्यात्
कर्णस्य वर्गौ गणकप्रवीणः ।
बीजक्रियातः पुनरत्र पक्षौ
तुल्यौ विदध्यात् कथितप्रकारैः ॥ ७७ ॥ ॥ इन्द्रवज्रा ॥

evaṃ prakāradvayato'pi kuryāt
karṇasya vargau gaṇakapravīṇaḥ |
bījakriyātaḥ punar atra pakṣau
tulyau vidadhyāt kathitaprakāraiḥ || *77* || || *indravajrā* ||

Thus, the expert mathematician may compute the squares of the diagonal by both the approaches (*prakāradvaya*). Through the use of algebra here again one should make the two sides equal by the methods that were stated earlier.

Technical Analysis

In order to arrive at the other expressions for the square of the diagonal, Nityānanda recounts a few well-known relations. He commences with

$$\begin{aligned} R - \operatorname{Vers}\theta &= (R - (R - \operatorname{Cos}\theta)),\\ &= \operatorname{Cos}\theta, \end{aligned} \tag{12}$$

and

$$R^2 - \operatorname{Cos}^2\theta = \operatorname{Sin}^2\theta.$$

Now, it is noted by Nityānanda (verse 75a) that the sum of the squares of the Sine and the Versine is equal to the product of the Diameter and the Versine.[20]

$$\operatorname{Sin}^2\theta + \operatorname{Vers}^2\theta = D\operatorname{Vers}\theta. \tag{13}$$

---

[18] The terms *koṭiḥ, koṭijyā, koṭijyakā* and *koṭilavajyakā* are used synonymously.

[19] Here, the term *bhuja* is used not to refer to the Sine but to denote the side of the quadrilateral, specifically, *ka-kha*.

[20] This identity has been given earlier in the work, in verse 32. See Montelle, Ramasubramanian, and Dhammaloka (2016, 24–26).

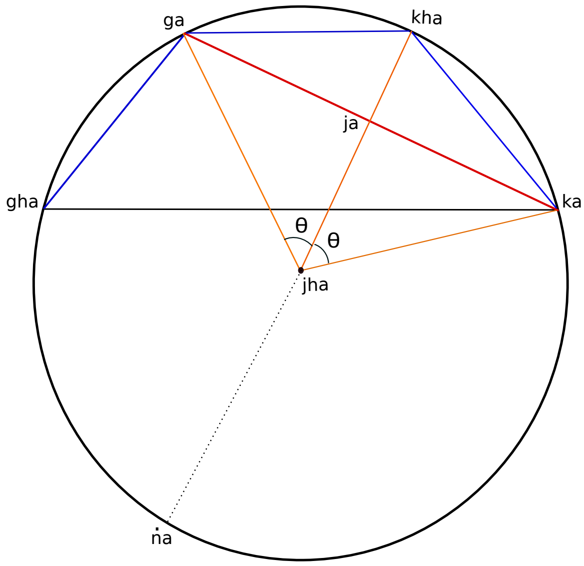


Figure 4: A reconstruction of the quadrilateral described in verses 67–69.

This is easily verified. Consider the RHS of the above equation

$$\begin{aligned} D\,\mathrm{Vers}\,\theta &= 2R\,\mathrm{Vers}\,\theta, \\ &= 2R(R - \mathrm{Cos}\,\theta), \\ &= R^2 + R^2 - 2R\,\mathrm{Cos}\,\theta, \\ &= \mathrm{Sin}^2\,\theta + \mathrm{Cos}^2\,\theta + R^2 - 2R\,\mathrm{Cos}\,\theta, \\ &= \mathrm{Sin}^2\,\theta + (R - \mathrm{Cos}\,\theta)^2, \\ &= \mathrm{Sin}^2\,\theta + \mathrm{Vers}^2\,\theta. \end{aligned} \tag{14}$$

Now we rewrite (13) as

$$\mathrm{Vers}\,\theta = \frac{\mathrm{Sin}^2\,\theta + \mathrm{Vers}^2\,\theta}{D}.$$

This is what has been stated in verse 75. Next in verse 76, Nityānanda instructs one to subtract both sides from R:

$$R - \mathrm{Vers}\,\theta = R - \frac{\mathrm{Sin}^2\,\theta + \mathrm{Vers}^2\,\theta}{D},$$

and then square both sides. Doing so, and simplifying the expression, we have

$$(R - \text{Vers}\,\theta)^2 = \left(R - \frac{\text{Sin}^2\theta + \text{Vers}^2\theta}{D}\right)^2,$$

$$\text{Cos}^2\theta = R^2 - 2R\cdot\frac{\text{Sin}^2\theta + \text{Vers}^2\theta}{D} + \left(\frac{\text{Sin}^2\theta + \text{Vers}^2\theta}{D}\right)^2,$$

$$R^2 - \text{Cos}^2\theta = 2R\cdot\frac{\text{Sin}^2\theta + \text{Vers}^2\theta}{D} - \left(\frac{\text{Sin}^2\theta + \text{Vers}^2\theta}{D}\right)^2,$$

$$= \text{Sin}^2\theta + \text{Vers}^2\theta - \frac{(\text{Sin}^2\theta + \text{Vers}^2\theta)^2}{4R^2}.$$

Multiplying through by 4, we get

$$4\,\text{Sin}^2\theta = 4(\text{Sin}^2\theta + \text{Vers}^2\theta) - \frac{(\text{Sin}^2\theta + \text{Vers}^2\theta)^2}{R^2}. \tag{15}$$

And as the diagonal *kha-gha* (which is the same as diagonal *ka-ga*) $= \text{Sin}\,\theta + \text{Sin}\,\theta = 2\,\text{Sin}\,\theta$, the LHS of the above equation is equal to *kha-gha*. Hence, Equation (15) becomes:

$$(\textit{kha-gha})^2 = 4(\text{Sin}^2\theta + \text{Vers}^2\theta) - \frac{(\text{Sin}^2\theta + \text{Vers}^2\theta)^2}{R^2},$$

$$= 4(\text{Crd}\,\theta)^2 - \frac{(\text{Crd}^2\theta)^2}{R^2}. \tag{16}$$

Now we have two equations, (11) and (16), that give the square of the diagonal of the cyclic quadrilateral. Nityānanda entreats us in verse 77 to equate the two sides and do some algebraic manipulation (*aparvartana*) in verse 78 (see the next section) in order to get the desired result.

III.3.3 Expression for the Measure of the Chord

Text and Translation

यावत्तावन्मानकेनापवर्त्यौ
यावत्तावन्मानजो यो घनोऽत्र ।
सोऽपि व्यक्ते पक्ष एवानुवेश्यो
व्यक्तं कार्यं गूढमानं ततोऽपि ॥ ७८ ॥ ॥ शालिनी ॥

yāvattāvanmānakenāpavartyau
yāvattāvanmānajo yo ghano'tra |
so'pi vyakte pakṣa evānuveśyo
vyaktaṃ kāryaṃ gūḍhamānaṃ tato'pi || 78 || || *śālinī* ||

[The two sides] are to be reduced by the measure of the desired quantity (*yāvattāvat*). Whatever cube is obtained here [by dividing throughout] by the desired quantity, even that one has to be placed on the side containing the known value (*vyaktapakṣa*). Further, the [algebraic] quantities whose magnitudes are unknown (*gūḍhamānaṃ*) have to be made known (*vyaktaṃ kāryaṃ*).

यावत्तावद्धनो योऽत्र त्रिज्यावर्गविभाजितः ।
तत्त्र्यंशयुक्तं भूत्र्यंशं बाहुमानं स्फुटं जगुः ॥ ७९ ॥ ॥ श्लोक ॥
yāvattāvadghano yo'tra trijyāvargavibhājitaḥ |
tattryaṃśayuktaṃ bhūtryaṃśaṃ bāhumānaṃ sphuṭaṃ jaguḥ || 79 || || *śloka* ||
Whatever here is the cube of the desired quantity (*yāvattāvat*) [is to be] divided by the square of the Radius. They state the third part of the base (*bhū*) increased by a third part of that to be the accurate measure of the chord.

Technical Analysis

In the previous section, it was demonstrated that Nityānanda derives two expressions for the diagonal (*kha-gha* which is equal to *ka-ga*) of the cyclic quadrilateral, namely Equations (11) and (16). Equating the two, we have

$$(\mathrm{Crd}\,\theta)^2 + \mathrm{Crd}\,\theta \cdot \mathrm{Crd}\,3\theta = 4(\mathrm{Crd}\,\theta)^2 - \frac{(\mathrm{Crd}^2\,\theta)^2}{R^2}. \tag{17}$$

Replacing all instances of $\mathrm{Crd}\,\theta$ with an unknown, say y, Equation (17) becomes

$$y^2 + y \cdot \mathrm{Crd}\,3\theta = 4y^2 - \frac{(y^2)^2}{R^2}.$$

Invoking the “unknown” (*yāvattāvat*, which is $\mathrm{Crd}\,\theta$), he states that the two sides are to be reduced by this amount. Reducing both sides by the unknown quantity (i.e., dividing each term by y) produces

$$y + \mathrm{Crd}\,3\theta = 4y - \frac{y^3}{R^2}.$$

Now, in the second and third quarter-verses of verse 78, Nityānanda alludes to the process of rearranging the terms in the above equation. He states that the cubic term here has to be moved to the other side of the formula where the numerical value of the quantity ($\mathrm{Crd}\,3\theta$) is known. This can be captured by modern symbolic rearrangement as follows:

$$\begin{aligned} 3y &= \mathrm{Crd}\,3\theta + \frac{y^3}{R^2}, \\ y &= \frac{\mathrm{Crd}\,3\theta}{3} + \frac{y^3}{3R^2}. \end{aligned} \tag{18}$$

Having arranged the equation in this form, Nityānanda says (in the last quarter of verse 78) that the unknown quantity (*gūḍhamāna*) in the RHS of Equation (18) which is a cubic term, has to be made known (*vyaktaṃ kāryam*). Verse 79 essentially presents Equation (18). In the following verses, how this is to be solved is described briefly once again, recounting the iterative procedure.

III.3.4 Iterative Procedure for Obtaining the Chord

Text and Translation

सामान्यतो भूमितृतीयभागो
व्यक्तस्य मानं परिकल्पनीयम् ।
कृत्वा घनं तस्य तु तत्त्रिभागं
त्रिभज्यकावर्गहृतं फलं यत् ॥ ८० ॥ ॥ इन्द्रवज्रा ॥
भूमित्रिभागेन युतं प्रकुर्यात्
मुहुर्मुहुः स्पष्टतरो भवेत् सः ।
प्रत्येकपङ्क्तिस्थितलब्धिजातं
घनं त्रिमौर्व्याः कृतिभाजितं तम् ॥ ८१ ॥ ॥ भद्रा उपजातिका ॥
तथा पुरोक्तेन घनेन हीनं
क्षिपेत् त्रिभक्तोर्वरिते मुहुश्च ।
एवं स्वबुद्ध्या गणको विदध्यात्
अव्यक्तमानस्य परिस्फुटत्वम्॥ ८२ ॥ ॥ माला उपजातिका ॥

sāmānyato bhūmitṛtīyabhāgo
vyaktasya mānaṃ parikalpanīyam |
kṛtvā ghanaṃ tasya tu tattribhāgaṃ
tribhajyakāvargahṛtaṃ phalaṃ yat || *80* || || *indravajrā* ||
bhūmitribhāgena yutaṃ prakuryāt
muhurmuhuḥ spaṣṭataro bhavet saḥ |
pratyekapaṅktisthitalabdhijātaṃ
ghanaṃ trimaurvyāḥ kṛtibhājitaṃ tam || *81* || || *bhadrā upajātikā* ||
tathā puroktena ghanena hīnaṃ
kṣipet tribhaktorvarite muhuśca |
evaṃ svabuddhyā gaṇako vidadhyāt
avyaktamānasya parisphuṭatvam || *82* || || *mālā upajātikā* ||

Grossly (i.e., as a first approximation), one third of the base is to be taken as the measure of the numerical value [of the desired measure, i.e., Crd θ]. And having computed the cube of that, that indeed has to be divided by three and by the square of the radius. Whatever is the result, it is to be added to the third part of the base iteratively (*muhurmuhuḥ*). That will produce increasingly more accurate values.

One should add that, [namely] the cube produced from the result obtained in the "result line" (*paṅkti*) at each stage, decreased by the previous cube [and] divided by the square of the Radius, to the remainder (*urvarita*) divided by the three. Thus, iterating (*muhuḥ*) as much as they desire, may the mathematician bring in accuracy to the unknown quantity.

Technical Analysis

This section closes with three verses recapitulating how each of the methods is to be implemented, now that the derivation has been made. The first step (verse 80, first half) is to use the initial approximation of one third of the base:

$$\frac{\mathrm{Crd}\,3\theta}{3} \tag{19}$$

This is the common step to be taken whatever iterative method one proceeds with. Methods two (described in subsection III.1.4) and three are summarised (verses 80, second half–81, first half). Then method one (described in subsection III.1.3) is summarised (verses 81, second half–82, first half). This group of verses ends with a note for the mathematician to continue iterating until the desired accuracy is achieved.

III.4 Worked Example (*udāharaṇa*)

Text and Translation

अत्रोदाहरणम् । भुजमानं यावत्तावत् या १ । तेनोनिता भूः [या १̇ भू १] । अस्या दलं [या १̇ भू १ / २] । जातोऽयं सन्धिः । सन्ध्यूना भूः पीठोऽयं [या १ भू १ / २] । बाहुवर्गः [याव १] । सन्धिवर्ग एषः [याव १ याभू २̇ भूव १ / ४] । अनयोः समच्छेदयोरन्तरं जातो लम्बवर्गोऽयं [याव ३ याभू २ भूव १̇ / ४] । अथ पीठवर्गः [याव १ याभू २ भुव १ / ४] । लम्बवर्गपीठवर्गयोर्योगोऽयमेव कर्णवर्गः [याव १ याभू १] । अयमेव भूमिवक्त्रावघातो भुजवर्गयुक्तः सम्पन्नः ।

अथ प्रकारान्तरेण कर्णवर्गः साध्यते । भुजवर्गमानं [याव १] व्यासहृतं जातो बाणः [याव १ / व्या १] । अनेन हीनं व्यासार्धम् [याव २̇ व्याव १ / व्या २] । इयं कोटिज्या । अस्याः कृतिरियं जाता । न्यासः [यावव ४ यावव्याव ४̇ व्यावव १ / व्याव ४] । अनया वर्जितो व्यासार्धवर्गो जातश्चतुर्गुणोऽयं कर्णवर्गः [यावव ४̇ यावव्याव ४ / व्याव १] ।

प्रथमानीतकर्णवर्गेण समः कार्यः । तेन समौ पक्षौ [यावव ० यावव्याव १ याभूव्याव १] [यावव ४̇ यावव्याव ४ या ००००] एतौ यावत्तावता अपवर्त्यौ । जातौ -- [याघ ० याव्याव १ भूव्याव १] [याघ ४̇ याव्याव ४ भू ०००] अत्र यावत्तावद्घनं रूपेष्वेव निवेश्य परस्परशोधनाल्लब्धं यावत्तावान्मानम् [याघ ४ भूव्याव १] [याव्याव ३] । भाज्यभाजकौ चतुर्भिरपवर्तितौ [याघ १ भूत्रिव १] [यात्रिव ३] ।

भूत्रिभागं सामान्यतो भुजमानमङ्गीकृत्य तस्य घनं कृत्वा त्रिभिर्विभज्य पुनस्त्रिज्यावर्गेण भक्ता भुजमाने निक्षिप्य तदेव भुजमानं पुनः स्वीकृत्य तस्य घनं कृत्वा त्रिभिर्विभज्य त्रिज्यावर्गेण च भक्ता भूत्रिभागे

क्षिप्त्वा मुहुः स्फुटं भुजमानयेत् प्रकारान्तरेण च ।

atrodāharaṇam | *bhujamānaṃ yāvattāvat yā 1* | *tenonitā bhūḥ* [*yā* $\dot{1}$ *bhū* 1]| *asyā dalaṃ* $\left[\frac{\textit{yā}\ \dot{1}\ \textit{bhū}\ 1}{2}\right]$ | *jāto'yaṃ sandhiḥ* | *sandhyūnā bhūḥ pīṭho'yaṃ* $\left[\frac{\textit{yā}\ 1\ \textit{bhū}\ 1}{2}\right]$ | *bāhuvargaḥ* [*yāva* 1] | *sandhivarga eṣaḥ* $\left[\frac{\textit{yāva}\ 1\ \ \textit{yābhū}\ \dot{2}\ \ \textit{bhūva}\ 1}{4}\right]$ | *anayoḥ samacchedayor antaraṃ jāto lambavargo'yaṃ* $\left[\frac{\textit{yāva}\ 3\ \ \textit{yābhū}\ 2\ \ \textit{bhūva}\ \dot{1}}{4}\right]$ | *atha pīṭhavargaḥ* $\left[\frac{\textit{yāva}\ 1\ \ \textit{yābhū}\ 2\ \ \textit{bhūva}\ 1}{4}\right]$| *lambavargapīṭhavargayor yogo'yam eva karṇavargaḥ* [*yāva* 1 *yābhū* 1] | *ayam eva bhūmivaktrāvaghāto bhujavargayuktaḥ sampannaḥ* |
atha prakārāntareṇa karṇavargaḥ sādhyate | *bhujavargamānaṃ* [*yāva* 1] *vyāsahṛtaṃ jāto bāṇaḥ* $\left[\frac{\textit{yāva}\ 1}{\textit{vyā}\ 1}\right]$ | *anena hīnaṃ vyāsārdham* $\left[\frac{\textit{yāva}\ \dot{2}\ \ \textit{vyāva}\ 1}{\textit{vyā}\ 2}\right]$ | *iyaṃ koṭijyā asyāḥ kṛtir iyaṃ jātā* | *nyāsaḥ* $\left[\frac{\textit{yāvava}\ 4\ \ \textit{yāvavyāva}\ \dot{4}\ \ \textit{vyāvava}\ 1}{\textit{vyāva}\ 4}\right]$ | *anayā varjito vyāsārdhavargo jātaś caturguṇo'yaṃ karṇavargaḥ* $\left[\frac{\textit{yāvava}\ \dot{4}\ \ \textit{yāvavyāva}\ 4}{\textit{vyāva}\ 1}\right]$ |
prathamānītakarṇavargeṇa samaḥ kāryaḥ | *tena samau pakṣau*
[*yāvava* 0 *yāvavyāva* 1 *yābhūvyāva* 1] [*yāvava* $\dot{4}$ *yāvavyāva* 4 *yā* 0000] *etau yāvattāvatā apavartyau* | *jātau* – [*yāgha* 0 *yāvyāva* 1 *bhūvyāva* 1][*yāgha* $\dot{4}$ *yāvyāva* 4 *bhū* 000] *atra yāvattāvadghanaṃ rūpeṣv eva niveśya parasparaśodhanāl labdhaṃ yāvāttāvānmānam*
[*yāgha* 4 *bhūvyāva* 1][*yāvyāva* 3] | *bhājyabhājakau caturbhir apavartitau*
[*yāgha* 1 *bhūtriva* 1][*yātriva* 3]|
bhūtribhāgaṃ sāmānyato bhujamānam aṅgīkṛtya tasya ghanaṃ kṛtvā tribhir vibhajya punas trijyāvargeṇa bhaktā bhujamāne nikṣipya tad eva bhujamānaṃ punaḥ svīkṛtya tasya ghanaṃ kṛtvā tribhir vibhajya trijyāvargeṇa ca bhaktā bhūtribhāge kṣiptvā muhuḥ sphuṭaṃ bhujam ānayet prakārāntareṇa ca |

Here, an example (*udāharaṇa*). As much as is the measure of the chord (*bhuja*), [that is denoted by] *yā*. The base (*bhū*) decreased by it is [*yā* $\dot{1}$ *bhū* 1]. Half of that is $\left[\frac{\textit{yā}\ \dot{1}\ \textit{bhū}\ 1}{2}\right]$. This produces the *sandhi*. The base [is to be] decreased by the *sandhi*. This is the *pīṭha* $\left[\frac{\textit{yā}\ 1\ \textit{bhū}\ 1}{2}\right]$. The square of the chord (*bāhu*) is [*yāva* 1]. The square of the *sandhi* is this $\left[\frac{\textit{yāva}\ 1\ \ \textit{yābhū}\ \dot{2}\ \ \textit{bhūva}\ 1}{4}\right]$. The difference of those two [converted] to the same denominator produces the square of the perpendicular, $\left[\frac{\textit{yāva}\ 3\ \ \textit{yābhū}\ 2\ \ \textit{bhūva}\ \dot{1}}{4}\right]$. Now, the square of the *pīṭha* is $\left[\frac{\textit{yāva}\ 1\ \ \textit{yābhū}\ 2\ \ \textit{bhūva}\ 1}{4}\right]$. The sum of the squares of the perpendicular and the *pīṭha* is indeed the square of the hypotenuse [*yāva* 1 *yābhū* 1]. This indeed produces the product of the base (*bhumi*) and the face (*vaktra*) increased by the square of the chord (*bhuja*).
Now by means of another method (*prakarāntara*) the square of the diagonal (*karṇa*) is established. The measure of the square of the chord (*bhuja*) is [*yāva* 1]. [This] divided by the Diameter produces the Versine (*bāṇa*) $\left[\frac{\textit{yāva}\ 1}{\textit{vyā}\ 1}\right]$. Half the Diameter diminished by this [yields] $\left[\frac{\textit{yāva}\ \dot{2}\ \ \textit{vyāva}\ 1}{\textit{vyā}\ 2}\right]$. This is the Cosine (*koṭijyā*). This is the square that is produced, [whose] lay-out (*nyāsa*) is: $\left[\frac{\textit{yāvava}\ 4\ \ \textit{yāvavyāva}\ \dot{4}\ \ \textit{vyāvava}\ 1}{\textit{vyāva}\ 4}\right]$. The square

of the Radius decreased by this, when multiplied by four, produces the square of the diagonal (*karṇa*). [*yāvava* $\dot{4}$ *yāvavyāva* 4].
vyāva 1
[This quantity] is to be equated with [the expression for] square of the diagonal obtained first. By this, these two sides are equal [*yāvava* 0 *yāvavyāva* 1 *yābhūvyāva* 1] [*yāvava* $\dot{4}$ *yāvavyāva* 4 *yā* 0000]. These [two sides] have to be reduced by a *yāvattāvat* [*yāgha* 0 *yāvyāva* 1 *bhūvyāva* $\dot{1}$] [*yāgha* $\dot{4}$ *yāvyāva* 4 *bhū* 0000]. The measure of the *yāvattāvat* is obtained by keeping the cube of *yāvattāvat* along with the side with the numerical constant, by reducing one by the other [*yāgha* 4 *bhūvyāva* 1] [*yāvyāva* 3]. The divisor and the dividend are to be reduced by four [*yāgha* 1 *bhūtriva* 1] [*yātriva* 3].
Having taken a third part of the base as the gross measure of the chord (*bhuja*), computing the cube of that, dividing by three, again dividing by the square of the Radius, adding it to the measure of the chord (*bhuja*), [and] again taking that as the measure of the chord (*bhuja*), computing the cube of that, dividing by three, and further dividing by the square of the Radius, [and] adding [that] to one third of the base (*bhū*), may one repeatedly obtain the refined value of the chord by the different method.

Technical Analysis

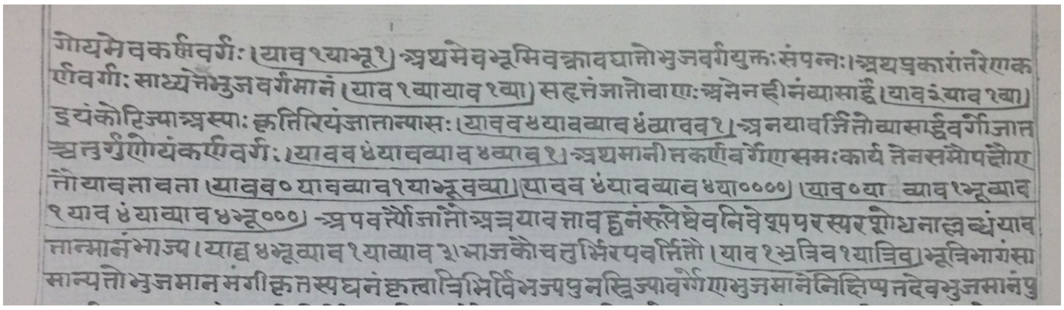


Figure 5: The text from the *R* (f. 14v) showing the way in which the equations have been laid down along with the prose text.

In this work, this is a rare instance of mathematics in prose. In addition, we find here a semi-symbolic style of reasoning laid out on the page (for examples, see Figures 5 and 6). This is obviously unfamiliar territory for the scribes as the manuscripts have frequent variant readings. By considering the mathematical consistency we could fix what the semi-symbolic expressions should be in a few instances where some crucial symbols such as a dot over the coefficient in the algebraic equations to denote that it is negative were missing. The symbols employed to denote the mathematical reasoning in the text are listed in Table 6.

Though all manuscripts contain this passage, MS *W* was found to provide the most reliable readings. In order to appreciate the reasoning, in what follows we give, in two columns, the mathematical steps in modern symbolic notation

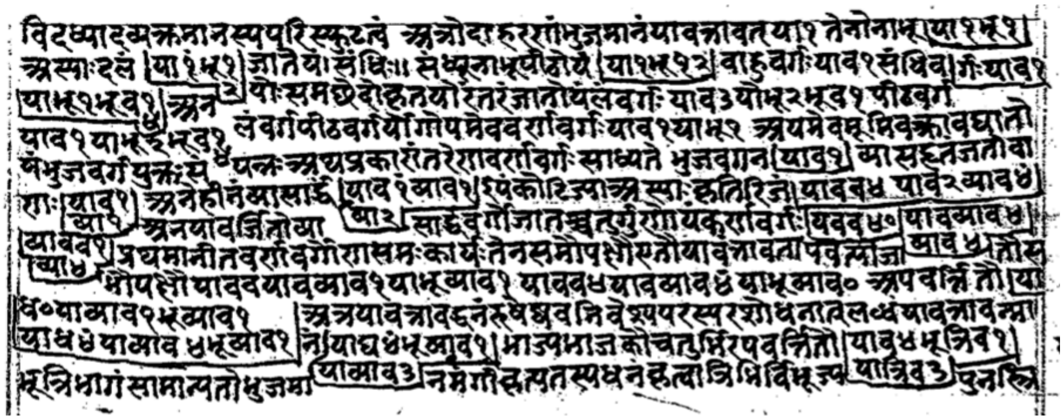


Figure 6: The text from the B_1 (f. 19v), essentially presenting the same passage as in figure 5 showing the way in which the equations have been laid down along with the prose text.

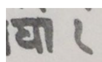
Nāgarī	Abbreviation	Concept	Translation	Notation
या	*yā*	*yāvattāvat*	Unknown	y
याव	*yāva*	*yāvattāvat-varga*	Square of unknown	y^2
याघ	*yāgha*	*yāvattāvat-ghana*	Cube of unknown	y^3
यावव	*yāvava*	*yāvattāvat-varga-varga*	Square-square of unknown	y^4
व्या	*vyā*	*vyāsa*	Diameter	D
व्याव	*vyāva*	*vyāsa-varga*	Square of Diameter	D^2
व्यावव	*vyāvava*	*vyāsa-varga-varga*	Square-square of Diameter	D^4
त्रिव	*triva*	*tribhajyā-varga*	Square of Radius	R^2
भू	*bhū*	*bhū*	Base	b

Table 6: Symbolic notations employed in the manuscripts to represent algebraic equations.

and a snippet from manuscript *W* to show how the scribe rendered the symbols. For convenience, we also provide a transliteration directly below.

The unknown, called *yāvattāvat* is represented by *yā*. We denote it by y. It is the measure of the chord (*bhuja*).

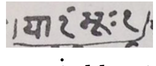
$$\text{Crd}\,\theta = \textit{ka-kha} = y$$



yā 1

Now, the base (*bhū*) is to be diminished by that:

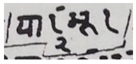
$$b - y$$



yā $\dot{1}$ *bhū* 1

Half of that is the part (*sandhi*):

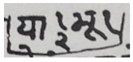
$$\frac{b-y}{2}$$



yā $\dot{1}$ *bhū* 1
2

The seat (*pīṭha*) is the base (*bhū*) diminished by the part (*sandhi*):

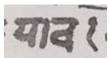
$$b-\left(\frac{b-y}{2}\right)=\frac{b+y}{2}$$



yā 1 *bhū* 1
2

Considering the square of the chord (*bāhuvarga*):

$$y^2$$



yāva 1

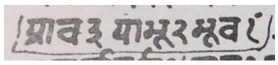
and the square of the part (*sandhi*):

$$\left(\frac{b-y}{2}\right)^2=\frac{y^2-2by+b^2}{4}$$

yāva 1 *yābhū* $\dot{2}$ *bhūva* 1
4

and the square of the perpendicular (*lamba*), when the square of the chord (*bāhu*) is diminished by the square of the part (*sandhi*), by making these two quantities have the same denominator (*samacheda*), we obtain:

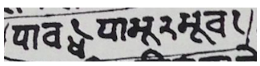
$$y^2-\left(\frac{b-y}{2}\right)^2=\frac{3y^2+2by-b^2}{4}$$



yāva 3 *yābhū* 2 *bhūva* $\dot{1}$
[4]

The square of the *pīṭha* (denoted by *ka-ca* in Figure 1) is:

$$\left(\frac{b+y}{2}\right)^2=\frac{b^2+2by+y^2}{4}$$

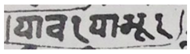


yāva 1 *yābhū* 2 *bhūva* 1
4

The sum of the square of the perpendicular (*lamba*; denoted by *ca-ga* in Figure 1) and the square of the *pīṭha* is thus given by:

$$\left(\frac{3y^2+2by-b^2}{4}\right)+\left(\frac{b^2+2by+y^2}{4}\right)$$

$$= y^2 + by$$



yāva 1 *yābhū* 1

This is the square of the hypotenuse (*karṇa*), denoted by *ka-ga*. That is,

$$(ka\text{-}ga)^2 = (ka\text{-}ca)^2 + (ca\text{-}ga)^2$$

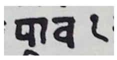
$$\text{or,} \qquad \text{hypotenuse}^2 = y^2 + by \tag{20}$$

It is to be noted that this is esentially the sum of the product of the base and the face and the square of the side (*bhujavarga*).

Now in order to obtain the desired cubic equation in y, Nityānanda proceeds to describe the procedure for obtaining a different expression for the square of the hypotenuse by another method. The two distinct expressions for this same hypotenuse can then be equated and after some simplification, the cubic expression can be derived.

This second expression finds its basis in the expression Nityānanda had given for getting the value of the Versine in terms of the chord and the diameter (see Equation 14). He begins by recalling the square of the chord (*bhuja*):

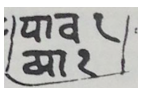
$$y^2$$



yāva 1

It can be easily seen that if y is the chord corresponding to an angle θ and D the diameter, then $\text{Vers}\,\theta$, which is referred to as *bāṇa* here, is given by[21]

$$b\bar{a}\d{n}a = \frac{y^2}{D}.$$

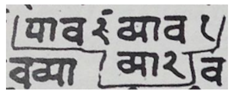


yāva 1
vyā 1

Now, subtracting this from the semi-diameter, we get $R - \text{Vers}\,\theta = \text{Cos}\,\theta$, referred to as *koṭijyā*. That is,

[21] This can be derived from a relation expressed earlier in the work, namely $\text{Crd}^2\,\theta = \left[2\,\text{Sin}\left(\frac{\theta}{2}\right)\right]^2 = \text{Sin}^2\,\theta + \text{Vers}^2\,\theta = D\,\text{Vers}\,\theta$ (Montelle, Ramasubramanian, and Dhammaloka 2016, verse 32).

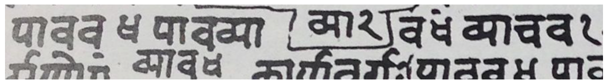
$$ko\d{t}ijy\bar{a} = \frac{D}{2} - \frac{y^2}{D} = \frac{-2y^2 + D^2}{2D}.$$



yāva $\dot{2}$ *vyāva* 1
vyā 2

Nityānanda states the square of this is

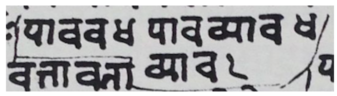
$$\left(\frac{-2y^2 + D^2}{2D}\right)^2 = \frac{4(y^2)^2 - 4y^2D^2 + (D^2)^2}{4D^2}.$$



yāvava 4 *yāvavyāva* $\dot{4}$ *vyāvava* 1
vyāva 4

Then asking us to subtract this from the square of the semi-diameter,[22]

$$\left(\frac{D}{2}\right)^2 - \frac{4(y^2)^2 - 4y^2D^2 + (D^2)^2}{4D^2} = \frac{4y^2D^2 - 4(y^2)^2}{4D^2}.$$



yāvava $\dot{4}$ *yāvavyāva* 4
vyāva 1

Nityānanda says that this expression multiplied by four (*caturguṇo'yaṃ*) produces the square of the hypotenuse (*karṇa*). This requires some additional explanation, which we provide below.

Consider the mid-point of the hypotenuse in Figure 4 denoted by *ja*. In this figure, *ka-ja* = *ga-ja* = $\mathrm{Sin}\,\theta$, and *ja-kha* = $R - \mathrm{Cos}\,\theta$. Now, by the rule of chords,

$$\begin{aligned} ka\text{-}ja \times ga\text{-}ja &= \dot{n}a\text{-}ja \times ja\text{-}kha\,, \\ &= (R + \mathrm{Cos}\,\theta)(R - \mathrm{Cos}\,\theta), \\ &= R^2 - \mathrm{Cos}^2\,\theta. \end{aligned}$$

Multiplying both sides of the above equation by four, we have

$$2\ ka\text{-}ja \times 2\ ga\text{-}ja = 4 \cdot (R^2 - \mathrm{Cos}^2\,\theta).$$

Since each of the two terms in the LHS corresponds to the hypotenuse (*ka-ga*), the above equation reduces to:

$$\text{hypotenuse}^2 = 4 \cdot \left[\left(\frac{D}{2}\right)^2 - \mathrm{Cos}^2\,\theta)\right]. \qquad (21)$$

[22] Here, the manuscript is in error. The number in the denominator should be 4 and not 1.

It may be noted that the operation prescribed by Nityānanda was to obtain an expression for the quantity in the square brackets in the above equation. This turned out to be

$$\frac{4y^2D^2 - 4(y^2)^2}{4D^2}.$$

It is obvious from (21) that this quantity multiplied by 4 gives the square of the hypotenuse, which is what Nityānanda states. Thus,

$$\text{hypotenuse}^2 = \frac{4y^2D^2 - 4(y^2)^2}{D^2}. \tag{22}$$

Now he instructs us to equate (20) and (22). Thus, we have

$$y^2 + by = \frac{-4y^4 + 4y^2D^2}{D^2}. \tag{23}$$

At this stage, the text makes it a point to explicitly mention that the two are equal in magnitude.

> *tena samau pakṣau*
>
> By this (i.e., for this reason), the two sides are equal.

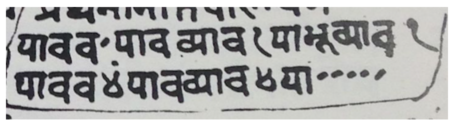
Perhaps the reason for making this statement is to emphasise the fact that though y is quadratic in the LHS and quartic in the RHS, the magnitudes of both sides are the same.

Next, we multiply both sides of Equation (23) by the square of the diameter, so that

$$D^2y^2 + D^2by = 4y^2D^2 - 4y^4,$$

or

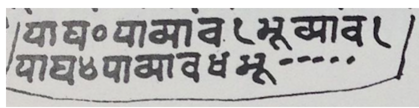
$$0 \cdot y^4 + y^2D^2 + D^2by = -4y^4 + 4y^2D^2 + 0 \cdot y.$$



yāvava 0 *yāvavyāva* 1 *yābhūvyāva* 1
yāvava $\dot{4}$ *yāvavyāva* 4 *yā* 0000

Dividing through by a factor of y we get

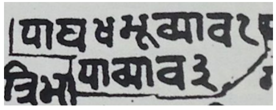
$$yD^2 + bD^2 = 4yD^2 - 4y^3.$$



yāgha 0 *yāvyāva* 1 *bhūvyāva* 1
yāgha $\dot{4}$ *yāvyāva* 4 *bhū* 000

Grouping like terms and rearranging, we have

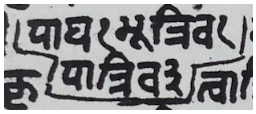
$$4y^3 + bD^2 = 3yD^2.$$



yāgha 4 *bhūvyāva* 1
yāvyāva 3

Dividing through by 4 and expressing D in terms of R, we finally have the desired cubic equation:

$$y^3 + bR^2 = 3yR^2.$$



yāgha 1 *bhūtriva* 1
yātriva 3

III.5 Determining the Sine of 0;01° from the Sine of 1°

Text and Translation

अथ प्रत्येकांशज्याज्ञाने सति प्रत्येककलाज्याज्ञानम् ।

atha pratyekāṃśajyājñāne sati pratyekakalājyājñānam |

Now, when the Sine of each degree is known, the knowledge of the Sine of each minute [is described].

चापार्धचापत्रिलवोत्थजीवा-
संसाधनार्थाभिहितप्रकारैः ।
संसाधयेत् पञ्चकाल्पोत्थजीवाम्
आद्यांशजीवात इह प्रवीणैः ॥ ८३ ॥ ॥ इन्द्रवज्रा ॥

cāpārdhacāpatrilavotthajīvā-
saṃsādhanārthābhihitaprakāraiḥ |
saṃsādhayet pañcakalpotthajīvām
ādyāṃśajīvāta iha pravīṇaiḥ || *83* || || *indravajrā* ||

Employing the methods that were described by the experts earlier (*praviṇaiḥ abhihitaprakāraiḥ*) for obtaining the Sine of half the arc and the Sine of three degrees [etcetera], here from the Sine of one degree (*ādyāṃśajīvātaḥ*), may the [desired] Sines that could be generated using the five fold approach be obtained.

अथ तिथिकलिकानां सिञ्जिनीतो विदध्यात्
विनृपलवकलायाः सिञ्जिनीमुक्तरित्या ।
पुनरपि लवमौर्व्या भागभूपांशजीवां
गगनरसविभक्तां साधयेत् तन्त्रविज्ञः ॥ ८४ ॥ ॥ मालिनी ॥

atha tithikalikānāṃ siñjinīto vidadhyāt
vinṛpalavakalāyāḥ siñjinīm uktarityā |

punar api lavamaurvyā bhāgabhūpāṃśajīvāṃ
gaganarasavibhaktāṃ sādhayet tantravijñaḥ || 84 || || *mālinī* ||

Then, from the Sine of 15 minutes (*tithikalikā*), may the Sine of one minute less a sixteenth part (*vinṛpalavakalā*) be determined by the method that was outlined earlier. And again, may the specialist in this discipline divide [the value of] the Sine of a sixteenth part of a degree–that which was obtained from the Sine of one degree–by 60 (*gaganarasa*).

एवं हि चापद्वयजातमौर्व्या
या जायते चापगुणैक्यजीवा ।
सैवाद्यलिप्तोद्भवसिञ्जिनी स्यात्
ततोऽन्यलिप्ताप्रभवा गुणाः स्युः ॥ ८५ ॥ ॥ बाला उपजातिका ॥

evaṃ hi cāpadvayajātamaurvyā
yā jāyate cāpaguṇaikyajīvā |
saivādyaliptodbhavasiñjinī syāt
tato'nyaliptāprabhavāguṇāḥ syuḥ || 85 || || *bālā upajātikā* ||

Indeed, from the Sine of the two arcs thus produced, whatever is the Sine value that is obtained by the sum of Sines of those two arcs, that in fact corresponds to the Sine arising from one minute [of arc]. From that, the Sines corresponding to other values of minutes can be obtained.

Technical Analysis

This last section, wherein Nityānanda concludes his treatment on Sines, contains an ingenious method for determining the Sine of 0;01° from the Sine of 1°.[23] This is a non-trivial problem because it is impossible to determine the Sine of 0;01° geometrically by using the half-angle approach or any other geometrical entities. The addition or subtraction formulae can not be used directly. Nor do there exist any analytic relation or established numerical technique that could be invoked to obtain the value of the Sine of such a small arc accurately. One has to resort to some sort of an approximation. One simpler way would be to argue that Sin 1° is very small, and hence using the linear approximation one could obtain

$$\mathrm{Sin}\, 0;01^\circ = \frac{1}{60}\, \mathrm{Sin}\, 1^\circ.$$

[23] Given the Sine of 0;01° is recorded in the Sine table of Ulug Beg, it is probable that he discussed how to compute it in his *Zīj*. We leave the exploration of this to future studies, simply noting that attempts to compute the Sine of 0;01° were also made by other authors in different cultures of inquiry, notably by medieval and early modern European mathematicians, including Levi ben Gerson, Regiomontanus, and Rheticus.

However, not being satisfied with this simplistic approximation, Nityānanda has set out a method based on the following relation:

$$\frac{1}{60} = \left(\frac{1}{4} + \frac{1}{10}\right)\frac{1}{16} = \frac{1}{64} + \frac{1}{60}\cdot\frac{1}{16}.$$

Since the Sine of 1° is known, $\frac{1}{16}$th of it can be obtained by a half-angle approach without using any approximation. Once this is obtained, linear approximation can be used to get $\frac{1}{60}$th, hence the error would be reduced by at least an order of magnitude.

Nityānanda's description of the method is rather condensed, contained in only a verse and a half. Crucial to reconstructing the method that Nityānanda is referring to here is the numerical data given in the Sine tables in his other work, the *Siddhāntasindhu*, which tabulate Sine values down to the precision of minutes. The precise value for the Sine of 0;01° given there helps us establish beyond doubt that our interpretation of these verses is correct.

The Sine table included in the *Siddhāntasindhu*[24] tabulates degrees along the horizontal axis and minutes along the vertical axis. In the reproduction here Sines for an argument of 0° to 14° are tabulated horizontally, and 0;00° to 0;19° vertically. Successive differences between minute values have been added in red, presumably for interpolation purposes for even more precise arc-measures.

The now familiar value for the Sine of 1° can be read off argument 1;00° as

$$1;02,49,43,11$$

. Likewise, the Sine of 0;01° can thus be read off argument 0;01° and is[25]

$$0;1,02,49,55$$

. We assume that the technique Nityānanda sets out here was the one used to produce the values that have been quoted by him in the tables. The method rests on two basic assumptions which we will cover in detail below. We start by describing the procedure outlined by him.

In the first half of verse 84, Nityānanda instructs us to establish the Sine of 0;01° of arc less a 16th part (*vinṛpalavakalā*). Since a 16th part of 0;01° is 0;00,03,45° we have:

$$\mathrm{Sin}(0;01,00,00° - 0;00,03,45°) = \mathrm{Sin}(0;00,56,15°).$$

[24] See Figure 7 in the Appendix for the first page of this multi-page table.

[25] These are both identical to those given in Ulug Beg's table (see Figure 8).

The magnitude of this can be easily determined by applying the half-arc formula repeatedly, a rule which was discussed in detail earlier on by Nityānanda (Montelle, Ramasubramanian, and Dhammaloka 2016, 22–26, verse 32):

$$\text{Sin}\left(\frac{\theta}{2}\right) = \frac{\sqrt{D \cdot \text{Vers}\,\theta}}{2}. \tag{24}$$

By applying this half-angle formula to the Sine of 1°, in just two steps one can get the Sine of 0;15° exactly. This is what is referred to by Nityānanda in the first quarter of verse 84 by the phrase *tithikalikānāṃ*. From this, we reconstruct the following half-arc amounts numerically, keeping a precision of six significant sexagesimal places:

n	$\text{Sin}\left(\frac{1}{2^{n\circ}}\right)$	Numerical value
1	Sin(0;30°)	= 0;31,24,55,53,00...
2	Sin(0;15°)	= 0;15,42,28,29,17...
3	Sin(0;7,30°)	= 0;07,51,14,18,41...
4	Sin(0;3,45°)	= 0;03,55,37,09,50...
5	Sin(0;1,52,30°)	= 0;01,57,48,34,59...
6	Sin(0;0,56,15°)	= 0;00,58,54,17,30...

Then in the second half of verse 84, Nityānanda instructs us to take the Sine of a sixteenth part of 1°, that is 0;03,45°, and divide it by 60. This prescription is based on the assumption that the arc is small enough so that it is reasonable to assume $\text{Sin}\,\theta = \theta$, which indeed is true, since we are dealing with measures that are an order of magnitude less than 1°. Therefore:

$$\frac{\text{Sin}(0;03,45°)}{60} = \text{Sin}(0;00,03,45°).$$

In other words:

$$\text{Sin}(0;00,03,45°) = \frac{\text{Sin}(0;03,45°)}{60} = \frac{0;03,55,37,09,50...}{60} = 0;00,03,55,37,09,50...$$

Now, with the "Sines of the two arcs," namely:

$$\text{Sin}(0;00,56,15°) \qquad \text{and} \qquad \text{Sin}(0;00,03,45°),$$

one can determine the Sine of 0;01° via the assumption:

$$\begin{aligned}
\mathrm{Sin}(0;01°) &= \mathrm{Sin}(0;00,56,15° + 0;00,03,45°)\\
&\approx \mathrm{Sin}(0;00,56,15°) + \mathrm{Sin}(0;00,03,45°)\\
&= 0;00,58,54,17,30... + 0;00,03,55,37,09,50...\\
&= 0;01,02,49,54,39...\\
&\approx 0;01,02,49,55 \qquad \text{(to four significant places)}
\end{aligned}$$

which is exactly the value given in the table for the Sine of 0;01°. Again this rests on an assumption that when the arcs are extremely small, the Cosine factors in the Sine of Sums formula are essentially equal to one, and thus the Sine of the sum of two arcs is simply the sum of the Sines of each of those arcs.

IV Concluding Remarks

Nityānanda's account of the derivation of the Sine of 1° is striking for a number of reasons. In addition to casting elaborate and technical procedures into the form of beautiful verses, Nityānanda gives expression, in the Sanskrit context, for new concepts concerning the circle geometry that underpin the cubic equation, as well as connecting in ever closer ways the geometry and algebraic analyses of the problem. Indeed, questions remain concerning where Nityānanda derived his inspiration for the contents of this passage, and the links they have to material relating back to al-Kashi and Ulug Beg are still to be investigated. However, one aspect that remains distinct about Nityānanda's exposition is the detailed and diverse ways that he derives and demonstrates the results. Within a handful of verses, he weaves together the specifics of how each algorithm is to be implemented with their underlying geometric derivation and their algebraic validation. Of particular note is the worked "example" he includes to derive the cubic equation. To the best of our knowledge, this prose passage with its semi-symbolic style of reasoning is unlike any other exposition of this sort in the history of Indian mathematics up to this point.

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Appendix: A Sample of Sine Tables Included in Various Manuscripts

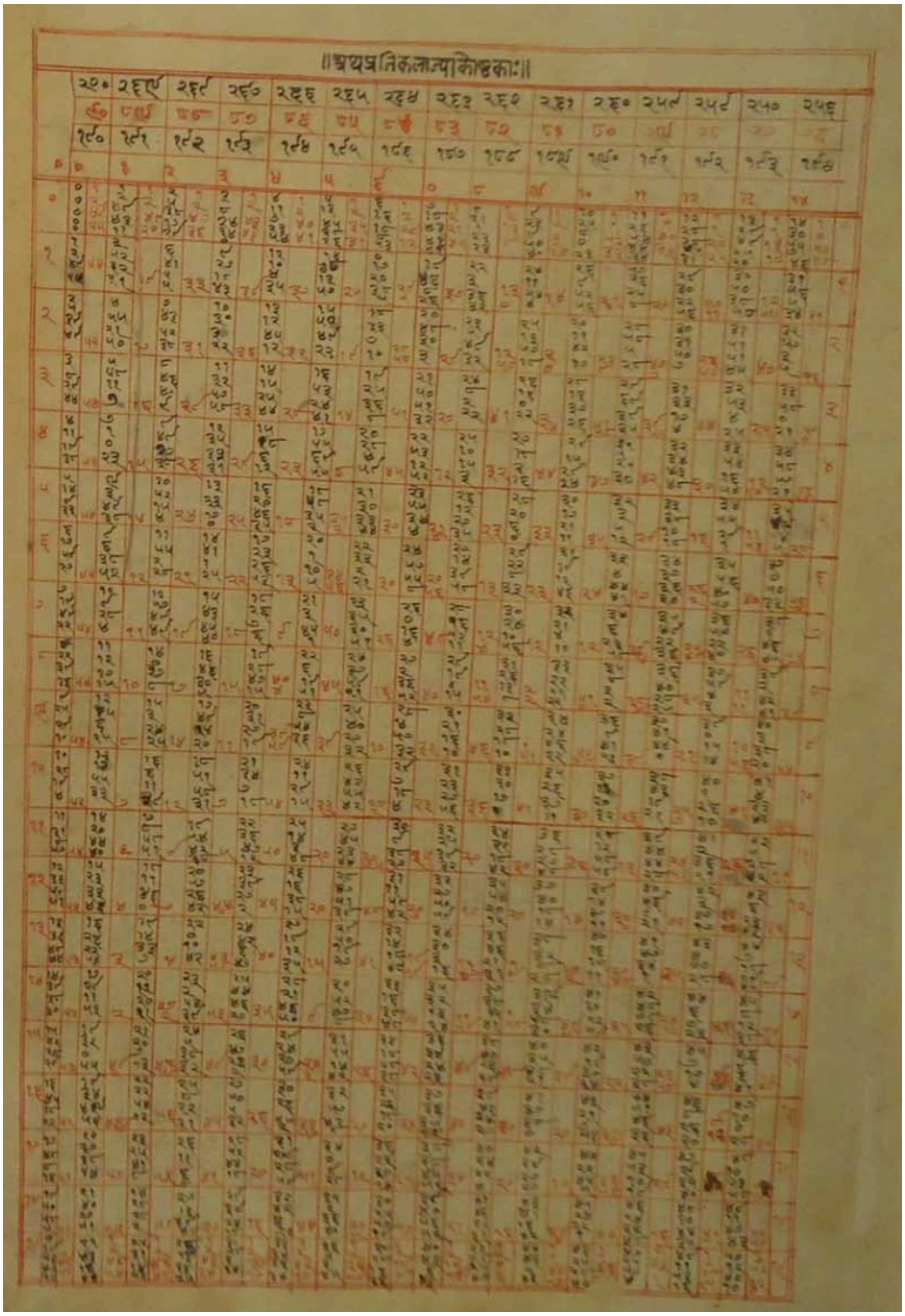

Figure 7: The first page of the Sine table from the *Siddhāntasindhu* of Nityānanda (City Palace Library, Jaipur 4962, f. 29r).

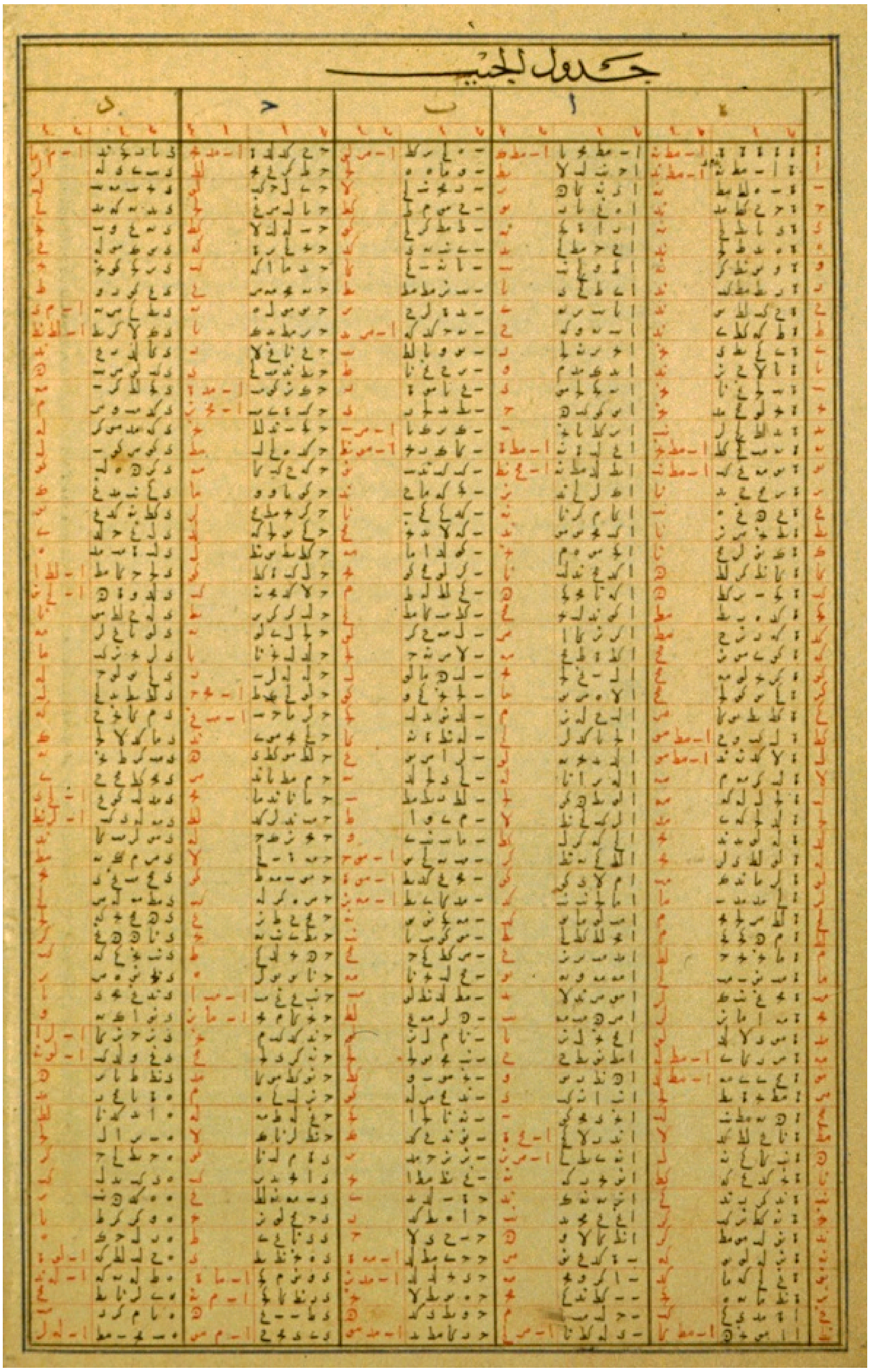

Figure 8: The first page of the Sine table from the *Zīj* of Ulug Beg.

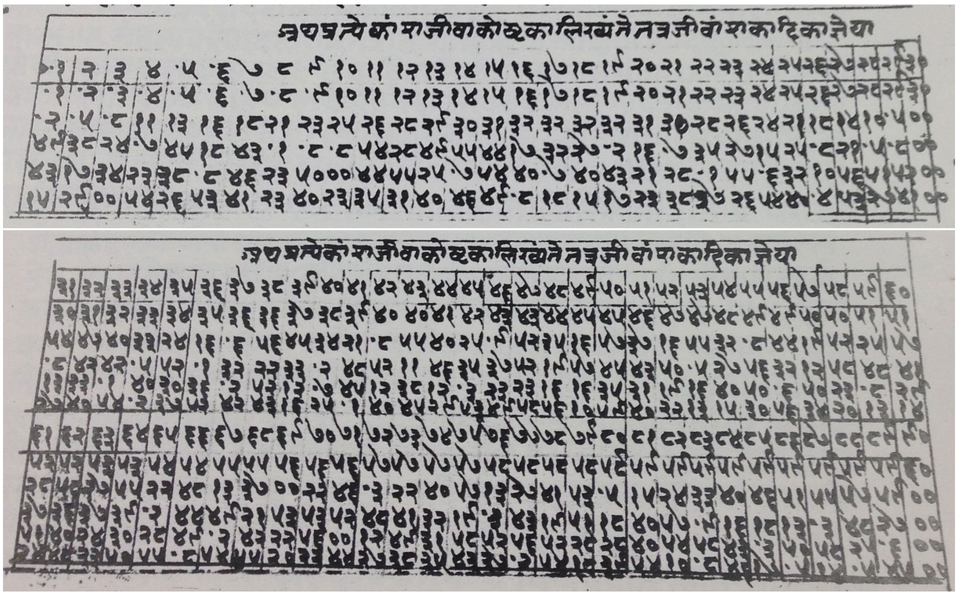

Figure 9: A complete Sine table from a manuscript of the *Sarvasiddhāntarāja* (Wellcome f. 18v–19r).

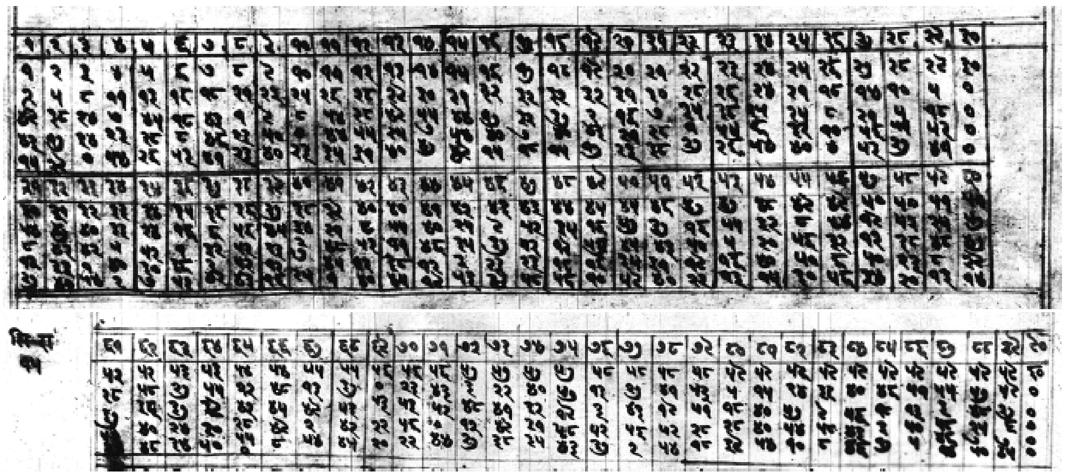

Figure 10: A complete Sine table from a manuscript of the *Sarvasiddhāntarāja* (Nepal B534 f. 15).

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