SCIAMVS 19 (2018), 103–166

Diagrams in the Arithmetical Books of Euclid's *Elements*

Ken Saito

Osaka Prefecture University (Professor Emeritus),
Seki Kowa Institute of Mathematics, Yokkaichi University

Abstract

In this paper we reproduce the diagrams in Books VII–IX of Euclid's *Elements* from four principal manuscripts. The diagrams in Heiberg's edition, reproduced in all the current translations, have little to do with the diagrams found in the manuscripts. The manuscript diagrams have the following features: (1) The lengths of lines are usually equal to each other, so that they do not show the relation of the relative largeness or smallness between the numbers they represent; (2) on the other hand, the arrangement of the lines can help readers' memory, exhibiting information about the relationship between the various numbers such as being continuously proportional (this feature of the diagrams is completely lost in Heiberg's edition); (3) some propositions in Books VII–IX that have logically corresponding propositions in Book X also have diagrams similar to those of their counterpart, suggesting that the diagrams found in the manuscripts already existed when those propositions were introduced. We also briefly treat the diagrams of Euclid's arithmetic in printed editions before Heiberg, and the specific numbers found in all the manuscript diagrams that are examples of numbers satisfying the proposition.

I Introduction

I.1 Diagrams in Euclid's *Elements*: Geometrical and Arithmetical Books

The text of Euclid's *Elements* is accompanied by diagrams. However, Heiberg's critical edition, on which all the current translations depend, does not reproduce the diagrams of the manuscripts. The diagrams of the geometrical books in Heiberg's *Elements* are copied from those in August (1826–1829), who considerably changed them, probably because of pedagogical concerns.[1]

---

[1] Reproduction of the manuscript diagrams of the geometrical books are available in (Saito 2006) for Book I, and for subsequent books (Books II–IV, VI, X-XIII) see http://www.greekmath.org/diagrams/diagrams_index.html.

August's diagrams are often different from those of the manuscripts and from those of previous editions. Heiberg's diagrams in the geometrical books are almost identical to August's, reproducing

The diagrams in Euclid's arithmetic (*Elements* VII–IX) have never been studied before. In this article, we examine and reproduce those in four of the manuscripts used by Heiberg, which are quite different from those found in his edition (Heiberg 1883–1888, vol. 2, which contains the three arithmetical books was published in 1884).

The first question is where Heiberg's diagrams come from. Unlike the diagrams in geometry, they are completely different from those in August's edition, nor do they resemble those in any manuscript used by Heiberg.[2] We have not been able to find any manuscript or edition prior to Heiberg with diagrams similar to his edition. So our previsionary conclusion is that Heiberg invented the diagrams for his edition of the arithmetical books.

I.2 Manuscripts Used by Heiberg and Our Reproduction of the Diagrams

The manuscripts used by Heiberg in his edition of the arithmetical books of the *Elements* are the following:

(P) Vatican Library (the only extant non-Theonine manuscript),
(B) Bodleian Library (Theonine),
(F) Laurentiana Library in Florence (Theonine),
(b) Archiginnasio Library in Bologna (Theonine),
(V) Austria National Library in Vienna (Theonine),
(p) National Library of France gr. 2466 (Book VII, Theonine), and
(q) National Library of France gr. 2344 (Books VIII–IX, Theonine).[3]

F is the best Theonine manuscript, but it is badly damaged, and of the arithmetical books only the beginning and the last parts are extant: the first 11 propositions of Book VII and the last 22 propositions of Book IX (IX.15–36). We have limited the reproduction of the diagrams to four manuscripts P, F, B and b, which we have judged to be the most important, for the sake of convenience of comparison. The reproduced diagrams are contained in Appendix 5. These images are not simply copied from the manuscripts, but are a sort of transcription of the principal features of the diagram.[4] The method and principle in the reproduction are described in Appendix 2.

---

even obvious errors in August's edition. So it is beyond any doubt that Heiberg copied August's geometrical diagrams (Saito and Sidoli 2012, 136–138). For further arguments about diagrams in geometry, see also (Saito 2012).

[2] For August's diagrams of the arithmetical books, see Appendix 1.

[3] See Appendix 3 for details about codices PBFb whose diagrams are reproduced in the present paper. For the other manuscripts (Vpq), see (Heiberg 1883–1888, I.viii–ix; II.v–vi).

[4] We use the program DRaFT (downloadable from http://www.greekmath.org/draft/draft_index.

II Two Types of Manuscript Diagrams in Euclid's Arithmetical Books

In Heiberg's edition, the three arithmetical books contain 102 propositions in total: 39 in Book VII, 27 in Book VIII, and 36 in Book IX. The so-called Theonine version (all the manuscripts except codex P) has two more propositions in Book VII, one before proposition 20 and the other before 21, and an alternative proof for proposition VII.31,[5] all appearing also in the margin of the non-Theonine manuscript P.[6] So we have reproduced 105 diagrams.

Unlike the geometrical diagrams, all the lines in the diagrams of the arithmetical books of the *Elements* are straight, and all of them are either vertical or horizontal—if we ignore slight inclinations, which certainly do not reflect the intention of the copyist. Most of the lines are vertical. In codex P, horizontal lines are found only in: (1) a part of the propositions of the so-called "pebble arithmetic" (IX.21–27), and (2) proposition VII.28. We will discuss these later in Sections II.2.4 and II.2.5. In other manuscripts, extra horizontal lines appear,[7] but we judge that these are due to later interventions, as we will argue in Section II.1.2.

We divide the 105 diagrams into two groups. In most propositions (80 of 105), the text assigns one Greek letter such as A, B, Γ to each number, which is represented by one line segment in the diagram. The assignment is in alphabetical order. Euclid assigns the first letter not yet used when a new number appears in the text. We call these propositions "type S" (separate).

In the remaining 25 propositions, however, the sum or difference of two numbers is treated, or a number is divided into two or more parts (divisors), and the operation of taking the sum/difference and dividing numbers into its parts, are shown by an analogous operation to the corresponding lines in the diagram. In this case, terminal points and dividing points of the lines are named, and the number is called by the two letters situated at the extreme points of the line segment that represents the number in the diagram. We call these diagrams "type R" (relational). Type R diagrams often contain lines of type S as well. We now give a more detailed discussion of each type of diagram.

---

html), which we have developed for reproducing geometric diagrams. Images of the manuscripts are available on the websites of the various libraries. See Appendix 4.

[5] We use the proposition numbers of Heiberg's edition, which is based on the non-Theonine codex P. For the propositions in the second half of Book VII (VII.20 and following), the proposition numbers in the Theonine codices are different. See the table in Appendix 4 for the correspondence of proposition numbers.

[6] The diagram accompanying the text in the margin is not reproduced in the appendix of the present paper.

[7] IX.15 in codex B and VII.25, 33, 34, 36, 38, 39 in codex b.

II.1 Diagrams of Type S

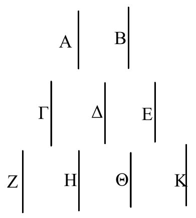
II.1.1 One Vertical Straight line, One Letter for One Number

We take proposition VIII.2 as an example of a diagram of type S. This is a problem in which a number and a ratio are given, and it is required to find the least numbers, as many as the given number, in continued proportion in the given ratio. Its *protasis* is:

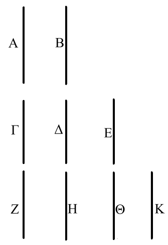
> To find numbers in continued proportion, as many as may be prescribed, and the least that are in a given ratio.[8]

For example, if the given number is 4 and the given ratio is 2 to 3, then it is required to find four numbers in continued proportion in the ratio of 2 to 3, and the four numbers must be the least among those having the same ratio. In this particular case, the solution is $2^3, 2^2{\cdot}3, 2{\cdot}3^2, 3^3$, and Euclid's solution consists of constructing such numbers without mentioning any concrete value.

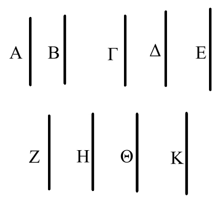
Proposition VIII.2



Codex P



Codex B



Codex b

Euclid takes A and B, which are the least numbers in the given ratio, then takes A squared, A times B, and B squared, which are Γ, Δ, E, respectively. Then, A multiplied by Γ, Δ, E, makes Z, H, Θ respectively (they would better be understood as A^3, A^2B and AB^2 in our algebraic thinking), and B multiplied by E, makes K (so K is B^3).

In the manuscript diagrams, the first two numbers A, B are arranged in the horizontal row at the top of the diagram, and the numbers Γ, Δ, E, which are generated by their multiplication, are arranged in the second horizontal row, in the order in which the numbers are in continued proportion. Then, the numbers generated from these numbers in the second row are arranged in the third row (Z, H, Θ, K) in such a way as to be in continued proportion.

This arrangement is shared by codices PB. Though codex b puts together in the first row the five lines in the first two rows of PB, the two groups (A, B and Γ, Δ,

[8] We quote (Heath 1925) for the English translation of the text of the *Elements*.

E) are separated by a larger interval, so this diagram can be interpreted as a result of trying to save space.[9] We may thus assume that the diagrams of this proposition come from a common archetype. Indeed, most of the other propositions admit the assumption of such an archetype.

II.1.2 Variants in Codex b

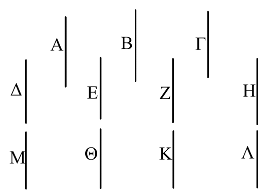
The most salient variants of type S diagrams are found in propositions VII.33–39 (VII.35–41 in the Theonine version), where codex b has diagrams with both vertical and horizontal lines. These can be explained as a result of intervention by someone in the tradition of this manuscript. Let us take proposition VII.33 (35 in the Theonine version) where two numbers Δ and M are represented by horizontal lines in codex b. In this proposition, Δ is the greatest common measure of A, B, Γ, so that (here we use the modern notation for multiplication)

$$E \times \Delta = A,\ Z \times \Delta = B,\ H \times \Delta = \Gamma.$$

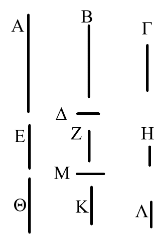
Similarly, M is supposed to be a common measure of A, B, Γ in the context of a demonstration by contradiction, so that

$$\Theta \times M = A,\ K \times M = B,\ \Lambda \times M = \Gamma.$$

Proposition VII.33



Codex P



Codex b

Codex b represents this particular property of numbers Δ and M by making the lines representing them horizontal. This difference between the manuscripts can better be attributed to some scribe in the tradition of b who altered the diagram to facilitate the understanding of the proposition, than to an independent origin of the diagram. Thus we can assume the existence of a common archetype for the diagrams in these propositions, as well.

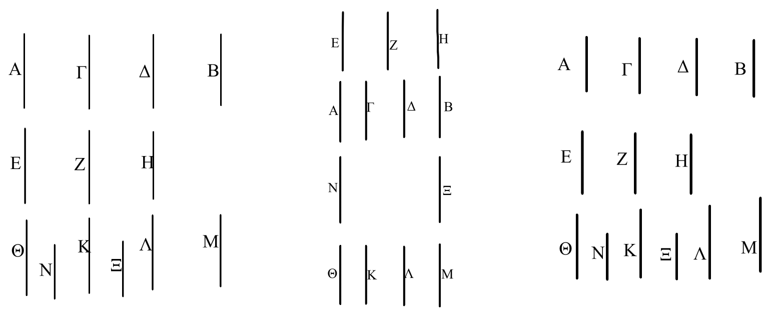
[9] In propositions VIII.9–12, which treat similar situations, codex b has diagrams with three rows, similar to those in P and B.

II.1.3 Uniform Length of Lines

There are two salient features of the diagrams of type S. First, all the lines representing numbers have approximately the same length, so that the length of lines gives no information about the largeness (or smallness) of numbers they represent. This feature is most consistent in codex P, somewhat less in B, while codex b sometimes exploits the length of lines to represent the quantitative relation between numbers. For the few exceptions of this principle in codex P, we can afford plausible explanations.

The lines in the lowest row are sometimes made shorter and placed in the previous row (proposition VIII.21, lines N and Ξ in codices P and b). This deformation can be explained by arguing that the scribe was trying to economize the space occupied by the diagram.

Proposition VIII.21



Codex P Codex B Codex b

When a proposition treats only one series of numbers in continued proportion, they are sometimes represented by increasing lines, that is, longer one after another (IX.3 and IX.8 in codex P). However, these increasing lines are not found in all such propositions, and different codices have increasing lines in different propositions.[10] Therefore, this feature should be attributed to later editors and copyists. We repro-

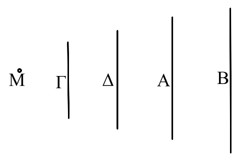
---

[10] All of the four propositions IX.8–11 have only the numbers in continued proportion beginning with the unit. In IX.8, all of the three codices reproduced here (P, B and b) show them by lines increasing one after another, while in IX.9 and 10, increasing lines are found only in codex B (and the last two lines of codex b in IX.9 are slightly longer than the others). In proposition IX.11, we see slightly increasing lines in codices P and B, while all the lines in codex b are of the same length.

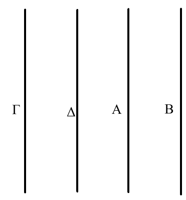
Probably it is worthwhile to refer to the diagrams in other manuscripts. Codex V uses increasing lines for IX.3 and 8, while all the lines are equal in IX.9–11. Codex p has always equal lines, but this may be due to the lack of vertical space, for the diagrams are drawn in the top or bottom margin in this manuscript. Codex q has a long lacuna from VIII.24 to IX.14, so it provides no information (Heiberg 1883–1888, II.xviii).

duce here the diagrams of IX.3, 8, and 10. For the diagrams of other propositions, see Appendix 5.

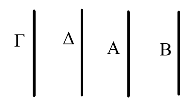
Proposition IX.3



Codex P

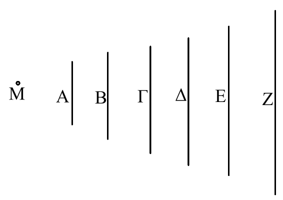


Codex B

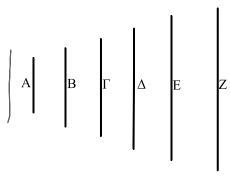


Codex b

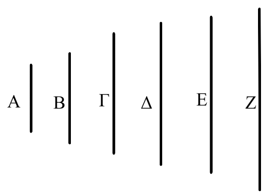
Proposition IX.8



Codex P

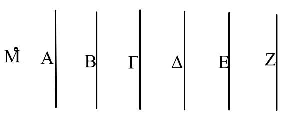


Codex B

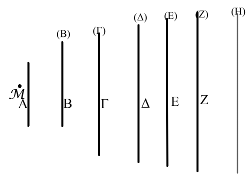


Codex b

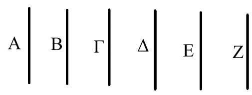
Proposition IX.10



Codex P



Codex B



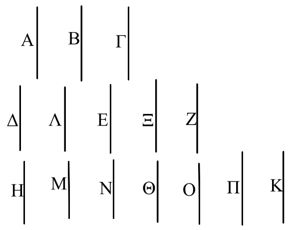
Codex b

Thus we conclude that the use of lines of equal length was the principle in diagrams of type S.

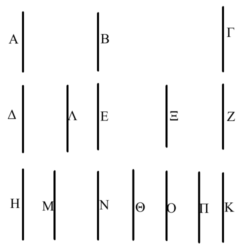
II.1.4 Arrangement of Lines

The second feature of diagrams of type S is that the arrangement of lines is significant and aids in reading the text. This is already obvious in proposition VIII.2, which we have seen above. Let us consider proposition VIII.13, which shows that if numbers are in continued proportion, their squares and cubes are also proportional and vice versa.

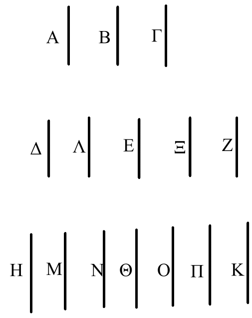
Proposition VIII.13



Codex P



Codex B



Codex b

In this proposition, there appear as many as fifteen numbers from A to Π. As explained above, Euclid assigns alphabetical names to the numbers simply in the order of their appearance in the text. Thus the names of numbers, such as A, B, Γ, contain in themselves no information about what they are, nor do they tell anything about their mutual relationship.

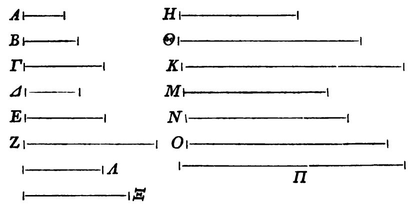
In this proposition, A, B, Γ are in continued proportion, and their squares are Δ, E, Z respectively, and Λ is the product of A and B, Ξ that of B and Γ, and so on. We would write A^2, B^2, Γ^2, AB, $B\Gamma$ instead of Δ, E, Z, Λ, Ξ. Our algebraic notation contains information about the numbers it represents. Euclid, however, did not employ such algebraic notations, and each time he created a new number by multiplication, he assigned to it the next letter of the alphabet. The assigned letter itself has nothing to do with the operation that gave rise to it. Thus one has to remember all of the relationships between the named numbers to be able to follow the argument—which is no easy task. In short, Euclidean arithmetic is quite demanding on the reader's memory.

But the manuscript diagrams provide a remedy. They suggest the relationships between the numbers appearing in the demonstration by the arrangement of lines representing those numbers—and thus they are of great help to the reader.

The manuscript diagrams arrange the three numbers A, B, Γ in the first row, then Δ, E, Z (squares of the first three numbers) in the second row in this alphabetical order, but Λ and Ξ are inserted between them. Indeed, these five numbers Δ, Λ, E, Ξ, Z are in continued proportion in this order, following the arrangement in the diagram. Also the numbers in the third row are arranged such that they are also in continued proportion. With this diagram, the reader sees those numbers in continued proportion in one glance—which is the key to the argument. The fact that Δ, E, Z in the second row are squares of A, B, Γ of the first row respectively, is best suggested in the diagram of codex B, but it is easy to remember this fact with those of other codices as well.

From a strictly logical point of view, the diagrams of type S is not indispensable for the demonstration, for all the properties of the numbers treated in the proposition are explicitly stated in the text, and the diagram has no additional information. This is the primary difference from the diagram in the geometrical books, where the diagram often provides information not explicitly contained in the text, such as the name of the intersection point of lines (not always specified in the text), the order of points on a line, and so on. However, it is obvious that the diagram of type S is nonetheless important and useful, given the limited competence of human memory. Indeed, it is difficult to follow the demonstration of this proposition VIII.13 with Heiberg's diagrams, which provides no information about the relationships between the numbers.[11]

Heiberg VIII.13

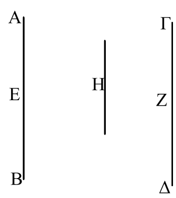


II.2 Diagrams of Type R

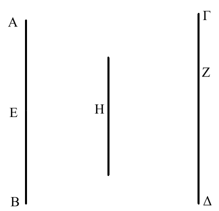
II.2.1 Example: Proposition VII.2

Typical examples of diagrams of type R are provided by the propositions at the very beginning of the arithmetical books, which make use of the so-called Euclidean algorithm to find the greatest common measure (GCM) of two given numbers.

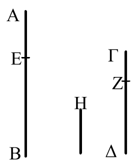
Proposition VII.2



Codex P



Codex B



Codex b

11 Arabic translations of the *Elements* preserve the features of the diagrams discussed here. (Bajri, Hannah and Montelle 2015) shows how al-Samaw'al made use of these features in his own work on arithmetic, and argues that such diagrams helped the development of inductive reasoning.

We give a paraphrase of part of Euclid's demonstration of proposition VII.2 with additional explanations for readers not familiar with the text of the *Elements*. For the moment it is convenient to consult the diagram of codex b. Later we examine the characteristics of codices PB (which seem to be nearer to the archetype).

The requirement of the proposition (problem) VII.2 is to find the greatest common measure of the two given numbers AB and $\Gamma\Delta$.[12] First, the lesser number $\Gamma\Delta$ is subtracted from the greater AB. If the remainder is still greater than $\Gamma\Delta$, then subtract $\Gamma\Delta$ again. Repeating such subtraction, the remainder AE will be lesser than $\Gamma\Delta$ at some point. In the diagram of codex b, the subtracted part BE looks equal to $\Gamma\Delta$, suggesting that $\Gamma\Delta$ is subtracted from AB only once to make the remainder AE smaller than $\Gamma\Delta$. In general, $\Gamma\Delta$ may be repeatedly subtracted from AB until the remainder AE becomes lesser than $\Gamma\Delta$, so BE is some multiple of $\Gamma\Delta$.[13]

Now $\Gamma\Delta$ is greater than AE, and the lesser AE is subtracted (repeatedly) from the greater $\Gamma\Delta$. At some point, the remainder ΓZ is smaller than AE. Then, the lesser remainder ΓZ is subtracted from the greater AE.

Here Euclid assumes that AE is some multiple of ΓZ and the mutual subtraction ends. Then Euclid shows that ΓZ measures both AB and $\Gamma\Delta$, so that it is a common measure of the two numbers. The line H is used in the second half of the proof, which demonstrates that there is no common measure greater than ΓZ.

In the diagrams, numbers are referred to by two labels (such as AB), which indicate the endpoints of the straight line representing the number. The procedure of taking the sum or difference of two numbers is transformed, in the diagram, into putting together two line segments or taking away a part of the larger line segment. In this proposition, VII.2, the affirmation that AE is the remainder of subtraction of BE from AB, corresponds to the position of the point E, between A and B on the straight line AB.

---

[12] Euclid first establishes (proposition VII.1) the condition that two numbers are prime to each other when they do not have a GCM greater than the unit (strictly, a GCM does not exist in this case, since in Euclid's arithmetic the unit is not a number). Then in VII.2, assuming that two given numbers are not prime to each other, he shows how to find their GCM. The proofs of these two propositions are quite similar to each other, and it is difficult for us to see the reason to treat them as two separate cases. Indeed, if we define the GCM as the unit (the number one) for two numbers prime to each other, we can unite the two propositions. Defining *number* as a multitude of units (VII.def.2), Euclid could not treat the unit (one) as a number. Here, we do not enter into details of the issue of the unit and numbers in Euclid's arithmetic.

[13] At the beginning of the proof, Euclid's treats the case in which AB is some multiple of $\Gamma\Delta$, so that there is no remainder AE. In this case, $\Gamma\Delta$ is obviously the GCM of AB and $\Gamma\Delta$. There is no diagram for this case.

Thus, the text of propositions with a diagram of type R depends on the diagram. Even though the diagrams of type R in Euclid's arithmetic are rather simple, so that it is not impossible to reconstruct them from the text, the truth of the statements in the text *logically* depends on the diagrams of type R. This is the difference from diagrams of type S, which do not play any logical role in the demonstration.

II.2.2 Lengths of Lines in Type R Diagrams

What is striking in the manuscript diagrams (especially in codices P and B), is that the length of a line has nothing to do with the size of the number it represents. In VII.2, which we have just examined, a line AB is divided at point E. What the diagram guarantees is that AE is the difference of AB and BE, and thus AE (and also BE) must be smaller than AB, but nothing more. Even if the point E is placed at the middle of the line AB in the diagram, as is the case in codices PB, this does not mean that AE is equal to EB. Indeed the argument in the text entails that BE (equal to or a multiple of $\mathit{\Gamma\Delta}$) is greater than AE, which is itself smaller than $\mathit{\Gamma\Delta}$. Nor is it possible to compare two segments of different straight lines in the diagram. The number $\mathit{\Gamma\Delta}$ is smaller than the number AB by hypothesis, but the line $\mathit{\Gamma\Delta}$ is equal to the line AB in the manuscript diagrams. Such indifference to metrical accuracy can often be found in diagrams of type R, which reminds us of similar indifference in the diagrams of the geometrical books of the *Elements*.[14]

It seems that the archetype of the diagrams in codices PB was drawn in the following way: two whole lines AB and $\mathit{\Gamma\Delta}$ are drawn with the same length, then as they have to be divided at a point, the dividing points E and Z are placed at the midpoint of each line. This hypothesis explains many of the type R diagrams, which are not metrically correct if we assume that the lengths of lines are proportional to the numbers they represent. Though we have no certain argument to explain why the line H is shorter, it represents a number introduced in the latter part of the proposition for the proof by contradiction, so it was natural to distinguish it from the other lines in some way.

Now, which of the two different diagrams of VII.2 (PB and b) is nearer to the archetype? Our suggestion is that a mathematically consistent diagram (in this case that of codex b) can always be drawn by anyone who reads and understands the arguments in the text. On the other hand, it is difficult to imagine a situation in which a change from the diagram of codex b to that of codices PB can take place.[15]

---

[14] For a discussion of indifference to visual accuracy in geometrical diagrams, see (Saito and Sidoli 2012, 143ff.).

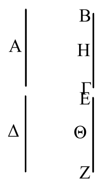
[15] (Carman 2018) shows, from a series of experiments of successive copying of geometrical diagrams (by today's students who participated in the experiments), that copyists tend to make lines and angles equal to each other, resulting in overspecified diagrams. Although this tendency may well explain a part of the overall presence of overspecified manuscript diagrams, in this specific case of

Besides, we think that the overall similarity of the diagrams in arithmetical books strongly suggests the existence of their archetype. So we conclude that the diagrams in the arithmetical books—at least those of P, B and b— derive from a common archetype, from which the tradition of b deviates more often and to a greater extent.

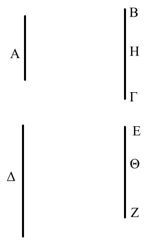
II.2.3 Lines of Equal Length: Propositions VII.5 and VII.7

Let us look at another example that corroborates our hypothesis of the "custom of drawing lines of the same length" in type R diagrams. In proposition VII.5,[16] the number A is equal to each of BH and $H\varGamma$. So if we are supposed to assume that the length of a line corresponds to the size of the number it represents, then a number would be equal to its half. Evidently, this is not what the diagram means. Here again, the manuscript diagrams suggest that the whole lines A, $B\varGamma$ (and $\varDelta$, EZ) were simply made equal when they were drawn. This kind of equality of the whole lines is maintained in codex P, while in other manuscripts the lines for larger numbers are sometimes slightly longer (but the lengths are not actually proportional to the numbers), as is the case of the diagram of codex b in this proposition.

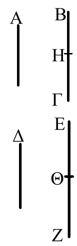
Proposition VII.5



Codex P



Codex B



Codex b

Proposition VII.7 looks like an exception to the principle of whole equal lines in codex P, where the line $H\varDelta$ is longer than AB. However, the numbers AB and $\varGamma\varDelta$ are set out first in the proof, then number $\varGamma H$ is introduced later. The line segments for numbers AB and $\varGamma\varDelta$, which appear at the beginning of the demonstration, are

---

VII.2, it is difficult to see how a diagram like that of codex b could be changed to that of codexes PB by successive copying.

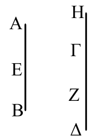
[16] Its *protasis* is as follows:

> If a number be a part (= divisor) of a number, and another be the same part of another, the sum will also be the same part of the sum that the one is of the one.

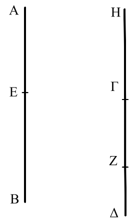
Using algebraic notation, it purports that if $na = b$ and $nc = d$, then $n(a + c) = b + d$. Propositions VII 5–8 prove such "trivial" property of numbers. The idea underlying these propositions deserves a careful study. In this article we concentrate on the diagrams.

nearly equal to each other in codices PB. Thus, this proposition is not an exception but another confirmation of the principle of drawing lines of equal length in codex P.

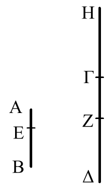
Proposition VII.7



Codex P



Codex B



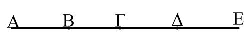
Codex b

II.2.4 Diagrams of Pebble Arithmetic

There is an interesting series of propositions (IX.21–27), all of which have a diagram of type R. They are rather rudimentary propositions concerning even and odd numbers, and are fairly isolated from the rest of Euclid's arithmetic.[17] These theorems are usually called pebble arithmetic, for their truth can be shown by arranging pebbles (*psephoi* in Greek), each of which represents a unit.[18]

Anyway, they certainly form a group that is somewhat isolated from the other propositions of Euclid's arithmetic, and they are also distinguished by their diagrams. The propositions IX.21–27 have diagrams of type R, with horizontal lines, while all the lines in other propositions are vertical, except for variants in some codices.[19] However, we hesitate to attribute this particularity of diagrams to different origins of the propositions. The horizontal line may well be due to economy of space, for each of the diagrams in IX.21–27 contains only one straight line, which cannot be very short because there are two or three intermediate points on it. Hence, it is natural to draw a horizontal line.

Proposition IX.21 (Codex P)



---

[17] The terms even and odd, and related terms such as even-times-even, etc., are defined in the definition at the beginning of Book VII, but they never appear before IX.21.

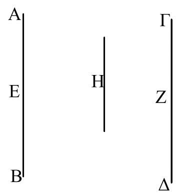
[18] However, the text and diagram of the *Elements* do not have any recourse to pebbles. Here we do not seek to evaluate the attribution of these propositions to the Pythagoreans by Becker (1936).

[19] Proposition VII.28 contains both vertical and horizontal lines (the only one in codex P). We will discuss this later.

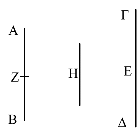
II.2.5 Type R Diagrams Common to Book VII and X: Propositions VII.1, 2 and 28

We have already had a look at the first two propositions of Book VII. What is striking in this group of propositions is that the diagrams of VII.1 and 2 are practically identical to those of propositions X.1, 2, and 3, where the same algorithm is applied to two magnitudes.

VII.2 and X.3 (Codex P)



VII.2



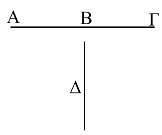
X.3

There is another proposition of the arithmetic theorems, VII.28, which has a counterpart, X.16, in book X. Here are their enunciations:

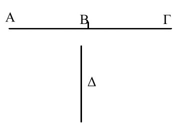
> VII.28: If two numbers be prime to one another, the sum will also be prime to each of them; and if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another.
> X.16: If two incommensurable magnitudes be added together, the whole will also be incommensurable with each of them; and, if the whole be incommensurable with one of them, the original magnitudes will also be incommensurable.

The parallelism of these two propositions is evident. X.16 proves practically the same property as VII.28, replacing numbers with magnitudes, and the relation "prime to each other" with "incommensurable with each other" are mathematically related. Their proofs in the *Elements* are also parallel. One could almost obtain the proof of VII.28 from that of X.16 by replacing "magnitude" and "incommensurable" by "number" and "prime to one another" (though the texts of their demonstration have other minor differences).

VII.28 and X.16 (Codex P)



VII.28



X.16

What is more striking is the similarity of their diagrams. They are practically the same. Moreover, VII.28 is the only one of the 105 propositions in the arithmetical books whose diagram has both horizontal and vertical lines.[20] We have no better explanation for this unique diagram of VII.28 than to assume that it has been "imported" from Book X along with its diagram. More research, which is beyond the scope of the present article, is desired to confirm this attractive hypothesis.[21]

III Spurious Elements in Diagrams

III.1 Representation of the Unit

The representations of the unit (our number "one") in the diagram is not consistent, and some of them may be later additions. When the unit appears in a proposition,[22] Euclid either assigns a name using a letter of the alphabet such as A or $\mathit{\Gamma}$ and a line in the diagram (case 1s), or a name of two letters in a diagram of type R (case 1r). However, sometimes he refers to the unit without assigning a specific name (case 2). The following is a list of each occurrence.

(1s)	VII.15, 16, 17, 37, 38; VIII.9, 10
(1r)	VII.1; IX.20, 22 alternative proof, 23, 25, 26, 27
(2)	IX.3, 8, 9, 10, 12, 13, 22, 32, 36

Case (1r) causes no problems. The unit corresponds to a certain segment of a line in the diagram, and is indicated by naming its two endpoints.

In case (1s), the diagram represents the unit by $\overset{\circ}{M}$ with a label such as A written beside it. Here we understand $\overset{\circ}{M}$ as the common abbreviation in the Greek manuscript tradition of mathematical texts for *monas*, the Greek word for the unit.

---

[20] Codices B and b contains some more propositions with both vertical and horizontal lines. See note 7.

[21] We are inclined to assume that the proposition VII.28 was "imported" from X.16. Here we limit ourselves to pointing out that it is difficult to claim that the diagrams were added later to a text originally without diagrams. Indeed, Cairncross and Henry (2015) have shown that a fragment from the Oxyrhynchus papyri contains several diagrams of propositions of Book I of the *Elements* that share the same characteristics as the diagrams of mediaeval manuscripts such as codices PB. Hence, one can no longer argue that the diagrams are later additions. Though a slight possibility may remain that only the propositions of geometry were accompanied by diagrams in antiquity, while diagrams in the arithmetical books are later additions. However, it is difficult to assume that the propositions with a diagram of type R were not accompanied by diagrams, for their text depends on the diagram, as we have argued.

[22] We exclude the locutions such as "let there be as many units in E as the times that $\mathit{\Gamma}$ measures A" (VII.21), for this does not necessitate a representation of the unit in the diagram.

Sometimes a line is drawn beside these symbols (for example, VII.15–17), but the lines are often clearly later additions (as in VII.17, codices PB).[23]

III.2 Example Numerical Values in Diagrams

In most diagrams of Euclid's arithmetic, some specific values (two, three, etc.) are written beside each of the line segments. These are examples of values that make the proposition true (the obvious exception are numbers that are introduced in proofs by contradiction). Though there is little doubt that they are later additions (for Euclid's text never mentions specific numbers), let us briefly describe them.

The numbers are written in Greek numerals in the manuscripts we examine in the present paper, but in other manuscripts Hindu-Arabic numerals can also be found. In older manuscripts PB (of ninth century), the numbers are written by later hands.

The same set of numbers are often found in different manuscripts. Not all of them can be the same by pure coincidence, so it seems that some of the numbers, especially those in later manuscripts or by later hands, were copied from one manuscript to another.

However, some of the numbers do differ, and this is another reason we think they are later additions. We show one example. In proposition VII.9, A, BH, $H\Gamma$ are equal to one another, and Δ, $E\Theta$, ΘZ are equal to one another. Codex P gives the value 4 for the former group, and 10 for the latter, while the values are 3 and 4 respectively in codex B.[24]

IV Conclusion

The most important conclusion from this quick examination is straightforward. The diagrams in the arithmetical books of the *Elements* have long been ignored, and the diagrams we find in Heiberg's edition (copied in all current modern language translations) have no value as a source. Unfortunately, they lack important, almost essential information provided by the manuscript diagrams: the arrangement of lines to help the memory of the reader, which suggest the relationships of the represented numbers.

Another important feature of the manuscript diagrams of Euclid's arithmetic is that the length of lines played little role. This is common to the geometrical diagrams, where visual accuracy was not important.

Strong similarity between the diagrams of parallel propositions in Book VII and

---

[23] However, the line beside the unit in VII.15 does not look different from other lines in the same diagram, and seems to have been drawn when the diagram was first made. It is not always easy to decide whether or not elements were added later to each diagram in the manuscripts.

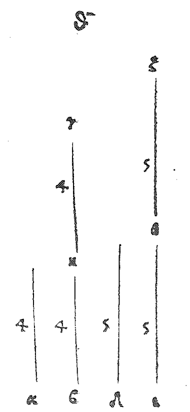
[24] The values 4 and 10 of codex P are common to many other manuscripts, including codices Fb. Codices Vp have these same values written in Arabic numeral.

Book X, suggests that the propositions along with the diagrams were "imported" from one book to another (probably from Book X to Book VII).

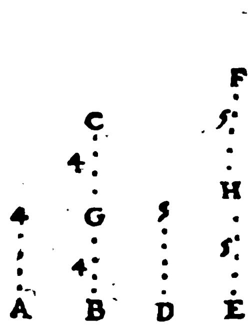
Appendix 1: Diagrams in Editions Before Heiberg

Here we show the diagrams of propositions VII.9, VIII.1 and 14 in some editions previous to Heiberg: Grynaeus (1533), Commandino (1572), Gregorius (1703) and August (1826–1829).

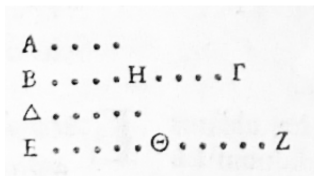
Proposition VII.9



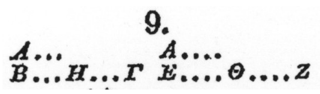
Grynaeus



Commandino

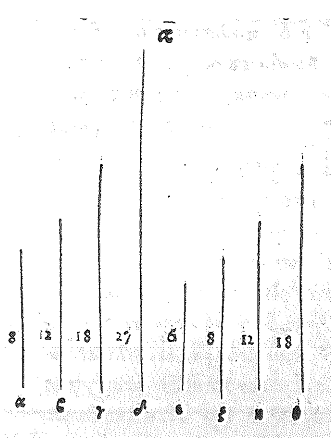


Gregorius

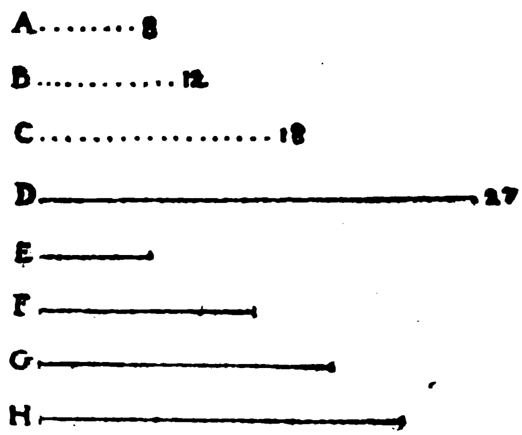


August

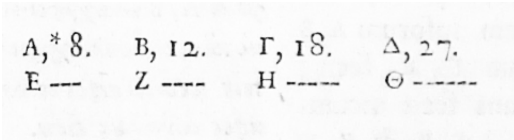
Proposition VIII.1



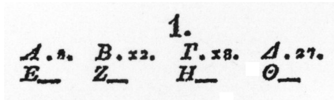
Grynaeus



Commandino

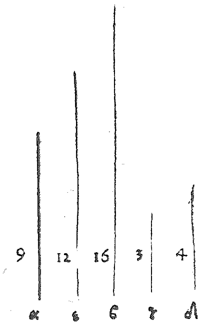


Gregorius

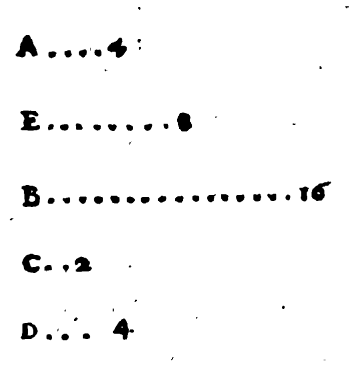


August

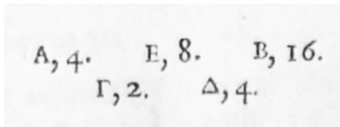
Proposition VIII.14



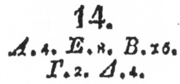
Grynaeus



Commandino



Gregorius



August

Example values in diagrams that exist in almost all the manuscripts (though they are additions at least in the oldest extant manuscripts; see section III.2) can be found in the *editio princeps* by Grynaeus, which, strangely, assigns values even in the impossible situations stipulated in proofs by contradiction, (see the diagram of VIII.1). Commandino's Latin translation, which has enjoyed a large distribution, uses dotted lines containing as many dots as the example values of the numbers, unless the value is too large (see Commandino's diagram of VIII.1). In these two editions, the arrangement of numbers in the manuscript diagrams is only partially preserved. In proposition VIII.14, the two rows A, E, B and $\Gamma\Delta$ in the manuscript diagram are reduced to one, but the order A, E, B is preserved.

Gregorius restores the arrangement of lines in the manuscript diagrams, but from Book VIII on, he replaced the straight lines in the manuscript (dotted lines in Commandino's edition) by example values. Indeed, we find this footnote at the beginning of Book VIII of his edition:

> In this and the following book, we put the corresponding numbers to the letters, so that a enormous number of points—whenever they must necessarily be used—would not annoy the readers.[25]

August largely follows Gregorius. Heiberg appropriately abolished the tradition of using example values in printed editions, but instead of restoring the manuscript

[25] In hoc et sequenti libro Literis adjecimus numeros illis respondentes; ne ingens punctorum multitudo, quandoque ex necessitate adhibenda, molestiam crearet lectoribus (Gregorius 1703, 171).

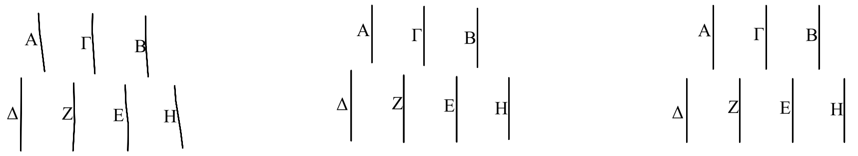
diagrams, he invented his own diagrams, in which he tried to make the length of lines somehow significant, but with little success. I wonder if the lack of interest in Euclid's arithmetic books on the part of today's historians is partly due to the difficulty of reading them with Heiberg's diagrams.

Appendix 2: Principles Used in Reproducing the Diagrams

1. The diagrams reproduced here are based on digital images scanned from microfilms. We have registered the positions of endpoints of each line, and joined them by straight lines of uniform width, putting the labels approximately at the same place where they are in the manuscript diagram. The labels in this reproduction have a constant size and are rather larger than those in the manuscripts.
2. We have reproduced one diagram for each proposition. Often other diagrams (mostly by a later hand) are added for various reasons. In a long proposition, which ends after a page turn, we often find, before the page turn, a rough copy of the "official" diagram at the end of the proposition. A copy of the diagram may appear also on the same page, which was perhaps drawn by a reader who tried to understand the proposition better. Marginal scholia may have an accompanying diagram. None of these diagrams are included in the reproductions in this paper.
3. In the reproduced image, all the lines are "standardized," that is, they are made either horizontal or vertical (except lines added afterwards), although in the scanned images they are often slightly inclined or curved, because of inaccuracy of the copyist and/or because the parchment was not pressed flat when it was photographed. Vertical lines in different rows that are nearly in a straight line, have been aligned. One could further make exactly equal the lengths of lines that are nearly equal in the diagram, and make the distances between parallel lines exactly equal.

   In the diagrams below, we show that for VIII.20 before the standardization in the left. The reproduced diagrams in Appendix 4 of this article is the central one, and further standardization would produce the diagram on the right.

Different extent of standardization of VIII.20



4. Example values (see §III.2) are not reproduced.

5. The lines that have clearly been added later, and their labels are reproduced in gray, and they are not standardized.
6. The width of lines is chosen to represent the average width of lines in the manuscript, and is constant for all the diagrams in one manuscript regardless of the real width of each line.
7. When a part or whole of a line is not visible in the image of the manuscript because of faded ink, the condition of the parchment, or erasure by a later hand, if both terminal points are visible, the whole line is shown in the reproduction. If one or both of the end parts of a line is not visible (or whole of the line is not visible, leaving only the trace of erasure), the non-visible part of the line is shown by a dotted line that ends at a presumed terminal point.
8. The manuscript diagrams often mark the terminal points of a line and the intermediate points to which labels are attached (in the case of diagrams of type R) by small filled circle, or a perpendicular short line segment, or a wedge-like triangle. The marks at the terminal points are *not* reproduced. The marks at an intermediate point are reproduced, either by a filled circle or a perpendicular short line segment. The type and size of the mark in the reproduction is determined according to the occurrence in the majority of the manuscripts and average size of the marks.

Appendix 3: Description of the Manuscripts

Codex P

Vat. Graec. 190. 9th century.[26]

The text is written in two columns. The diagram appears at the end of the proposition, occupying the whole width of one column. The terminal points of the lines are usually marked by a small wedge-like triangle. An example value for each number is written in different hand in Greek numerals.

Codex B

Bodleianus Dorvilianus 301 (in Heiberg, X,1 inf. 2,30), 888 CE.

The text is written in a single column. The diagram is drawn around the end of the text of the proposition, where the text occupies less width, leaving a space for the diagram to the right, which is one third to half of the width of the whole column. The terminal points of the lines are usually marked by a short, perpendicular line segment. The labels are written near the midpoint of each line. However, most of

[26] For the date of this codex, see (Mogenet and Tihon 1985, 23). Heiberg dated it to 10th century (Heiberg 1883–1888, VIII).

the original labels have been erased by a later hand that put example values in their place, copying the labels near the top of each line. In diagrams of type R, several original labels are preserved and the example values are written next to the line segments to which they refer (or above them, in the case of horizontal lines).

In the present reproduction, later hands are ignored and the original labels are restored. Where the original labels have been completely erased so that their positions are not certain, the reproduced labels show only an approximate indication of their position. When the original label is not visible (this is very often the case with this manuscript), we assume that the later hand correctly copies the label of the original hand, and we add a question mark to the label only when there is some reason to doubt the correctness of copying.

In some exceptional cases, the added label is reproduced between parentheses in IX.9–13. These propositions treat numbers in continued proportion beginning from the unit. In the diagram for proposition IX.11, it seems that the manuscript originally contained only the symbol $\overset{\circ}{M}$ for the unit and four vertical lines accompanied by labels B, $\varGamma$, $\varDelta$, E near the midpoint of each line, though all of these labels have been erased by a later hand, which puts in example numbers (1, 2, 4, 8 in Greek numerals, i.e., letters A, B, $\varDelta$ and H respectively), overwriting the original labels.[27] The labels by the later hand appear at the top of each line, and we have reproduced them in parentheses. The assignment of labels by the later hand is shifted, assigning the unit to the first line, then B, $\varGamma$, $\varDelta$ to the second, third, fourth line respectively.

Codex F

Laurentianus Plut.28.3, 11th century.

The text is written in a single column. The diagram is drawn in the outer margin, beside the final part of the text of the proposition. The diagram of IX.36, the last proposition of Book IX, is drawn after the colophon for the book, and the text of the beginning of Book X is written beside the diagram.

The codex is badly damaged, and the folios containing propositions VII.12–IX.15 are completely lost. They are supplemented by a later hand (Heiberg's φ), which has no value as a witness of the earlier tradition—hence, we have not reproduced the diagrams by this hand.

Codex b

Bibliothecae Communalis Bononiensis 18–19, 11th century.

The text is written in a single column. The diagram is drawn towards the end of the text of the proposition, where the text occupies less width, leaving a space for the diagram in the inner part of the column (usually one-third of the width of the

[27] We have not reproduced the example numbers. See item 4 in Appendix 2.

whole column). In most of the diagrams in Book VII, the terminal points of a line are marked by a wedge-like triangular shape, and the lines are drawn with a ruler. The diagrams of Books VIII and IX lack these marks at the terminal points, and the lines seem to have been drawn by freehand. The hand producing the diagrams has clearly changed between Book VII and Book VIII. In Book VII, the diagram is drawn by the same hand as that of the text, while in Book VIII and IX, the diagrams are drawn using different ink (visible in the images on the library website), and by a different hand from that of the text.

Appendix 4: Table of the Diagrams

The table indicates the folio number and column (for codex P only) in the manuscripts where each diagram reproduced in this paper appears ("Th" means Theonine tradition). The image of the manuscripts can be consulted on the websites of the libraries.

Codex P:	http://digi.vatlib.it/mss/detail/178321
	http://digi.vatlib.it/mss/detail/Vat.gr.190.pt.2
Codex B:	http://digital.bodleian.ox.ac.uk/inquire/p/06cfa3b7-2aad-465e-88ac-0ebe3f2b5d13
Codex F:	http://mss.bmlonline.it/Catalogo.aspx?Shelfmark=Plut.28.3
Codex b:	http://badigit.comune.bologna.it/books/A18/scorri.asp?Id=1
	http://badigit.comune.bologna.it/books/A19/scorri.asp?Id=1

Book VII

	P	B	F	b
VII.1	103va	124r	62r	135v
VII.2	104ra	125r	62r	136r
VII.3	104va	125v	63r	136v
VII.4	104vb	126r	63r	137r
VII.5	105ra	126v	63r	137v
VII.6	105va	127r	63v	137v
VII.7	105vb	127v	63v	138r
VII.8	106ra	128v	64r	138v
VII.9	106rb	129r	64r	139r
VII.10	106va	129v	64v	139v
VII.11	106vb	130r	64v	139v
VII.12	107ra	130r	lost	140r
VII.13	107rb	130v	lost	140r
VII.14	107va	131r	lost	140v
VII.15	107vb	131v	lost	141r
VII.16	108ra	132r	lost	141v
VII.17	108rb	132r	lost	141v
VII.18	108va	132v	lost	142r
VII.19	108vb	133r	lost	142v
VII.Th20	109r marg.	133v marg.	lost	142v
VII.20 (Th21)	109rb	133v	lost	143r
VII.Th22	109r marg.	134r	lost	143v
VII.21 (Th23)	109va	134v	lost	144r
VII.22 (Th24)	109vb	135r	lost	144r
VII.23 (Th25)	110ra	135v	lost	144v
VII.24 (Th26)	110rb	136r	lost	145r

	P	B	F	b
VII.25 (Th27)	110va	136v	lost	145r
VII.26 (Th28)	110vb	136v	lost	145v
VII.27 (Th29)	111ra	137v	lost	146r
VII.28 (Th30)	111va	138r	lost	146r
VII.29 (Th31)	111va	138r	lost	146v
VII.30 (Th32)	111vb	138v	lost	147r
VII.Th33 alternative	absent	139v	lost	147r
VII.31 (Th33)	112ra	139r	lost	147v
VII.32 (Th34)	112rb	139v	lost	147v
VII.33 (Th35)	112vb	140r	lost	148r
VII.34 (Th36)	113rb	141v	lost	149r
VII.35 (Th37)	113va	141v	lost	149r
VII.36 (Th38)	114ra	142v	lost	149v
VII.37 (Th39)	114rb	142v	lost	150r
VII.38 (Th40)	114rb	143r	lost	150r
VII.39 (Th41)	114va	143v	lost	150v

Books VIII, IX

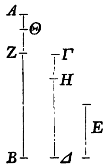
	P	B	F	b
VIII.1	115ra	144r	lost	151r
VIII.2	115va	145r	lost	151v
VIII.3	115vb	145v	lost	152r
VIII.4	116vb	147r	lost	153r
VIII.5	117rb	148r	lost	153v
VIII.6	117va	148v	lost	154r
VIII.7	117va	148v	lost	154r
VIII.8	118ra	149v	lost	154v
VIII.9	118vb	150v	lost	155r
VIII.10	119rb	151v	lost	156r
VIII.11	119vb	152r	lost	156v
VIII.12	120rb	153r	lost	157r
VIII.13	120va	153v	lost	157v
VIII.14	120vb	154r	lost	158r
VIII.15	121rb	154v	lost	158v
VIII.16	121rb	155r	lost	158v
VIII.17	121va	155r	lost	158v
VIII.18	122ra	156r	lost	159v
VIII.19	122vb	157r	lost	160v
VIII.20	123rb	158r	lost	161r
VIII.21	123vb	158v	lost	161v
VIII.22	124ra	159r	lost	161v
VIII.23	124ra	159r	lost	162r
VIII.24	124rb	159v	lost	162r
VIII.25	124va	160r	lost	162r
VIII.26	124va	160r	lost	162v
VIII.27	124vb	160v	lost	162v
IX.1	125ra	161r	lost	163r
IX.2	125rb	161v	lost	163r
IX.3	125vb	162r	lost	163v
IX.4	126ra	162r	lost	1r
IX.5	126rb	162v	lost	1r

	P	B	F	b
IX.6	126va	163r	lost	1v
IX.7	126vb	163r	lost	1v
IX.8	127ra	164r	lost	2r
IX.9	127va	164v	lost	2v
IX.10	128ra	165v	lost	3r
IX.11	128rb	165v	lost	3v
IX.12	129ra	166v	lost	4r
IX.13	129vb	168r	lost	5r
IX.14	130ra	168v	lost	5v
IX.15	130va	169r	lost	6v
IX.16	130vb	169v	88r	6v
IX.17	131ra	170r	88r	7r
IX.18	131va	170v	88r	7r
IX.19	132rb	171r	88v	7v
IX.20	132va	171v	88v	8r
IX.21	132va	171v	89r	8r
IX.22	132vb	172r	89r	8r
IX.23	132vb	172r	89r	8v
IX.24	133ra	172v	89r	8v
IX.25	133ra	172v	89r	8v
IX.26	133ra	172v	89r	9r
IX.27	133rb	173r	89r	9r
IX.28	133rb	173r	89v	9r
IX.29	133va	173r	89v	9r
IX.30	133va	173v	89v	9v
IX.31	133vb	173v	89v	9v
IX.32	134ra	174r	90r	10r
IX.33	134rb	174v	90r	10r
IX.34	134rb	174v	90r	10v
IX.35	134vb	175r	90v	11r
IX.36	136ra	176v	91r	11v

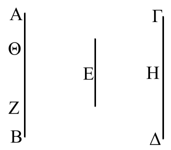
Appendix 5: Reproduced Diagrams of the Arithmetical Books of Euclid's *Elements*

Book VII

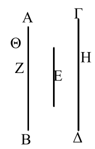
Proposition VII.1



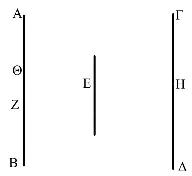
Heiberg



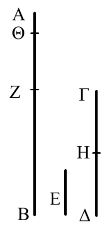
codex P



codex F

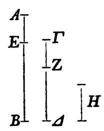


codex B

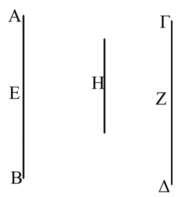


codex b

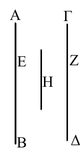
Proposition VII.2



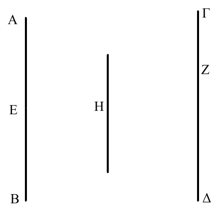
Heiberg



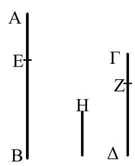
codex P



codex F

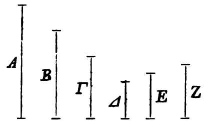


codex B

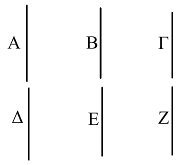


codex b

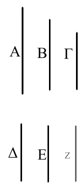
Proposition VII.3



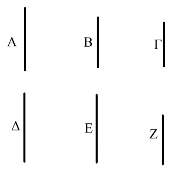
Heiberg



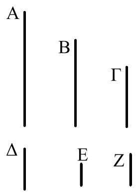
codex P



codex F

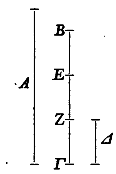


codex B

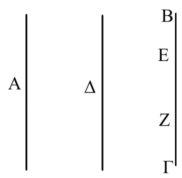


codex b

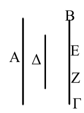
Proposition VII.4



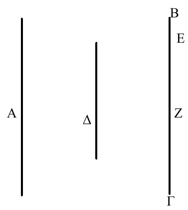
Heiberg



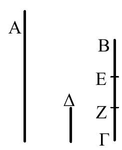
codex P



codex F

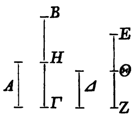


codex B

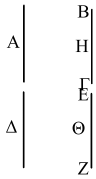


codex b

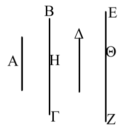
Proposition VII.5



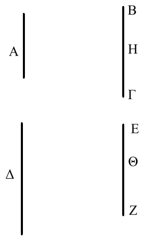
Heiberg



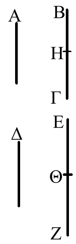
codex P



codex F

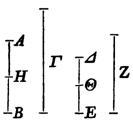


codex B

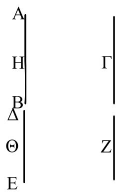


codex b

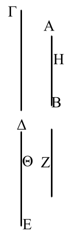
Proposition VII.6



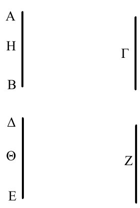
Heiberg



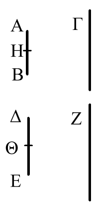
codex P



codex F

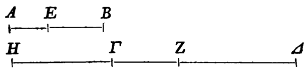


codex B

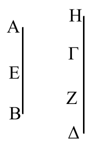


codex b

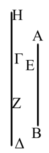
Proposition VII.7



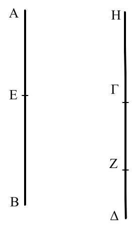
Heiberg



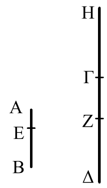
codex P



codex F

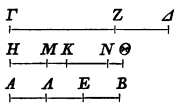


codex B

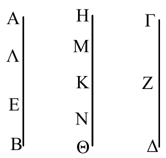


codex b

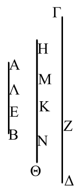
Proposition VII.8



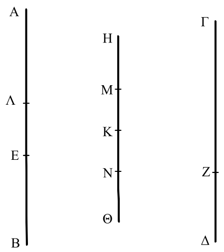
Heiberg



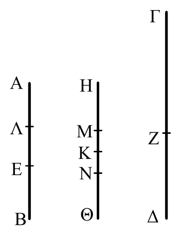
codex P



codex F

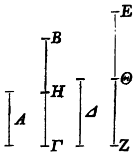


codex B

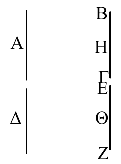


codex b

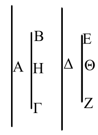
Proposition VII.9



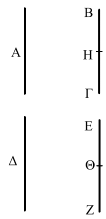
Heiberg



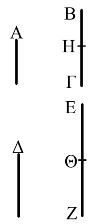
codex P



codex F

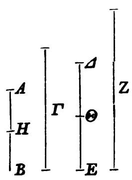


codex B

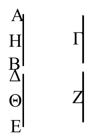


codex b

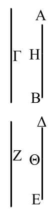
Proposition VII.10



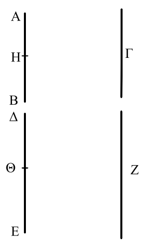
Heiberg



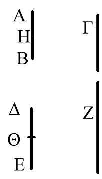
codex P



codex F

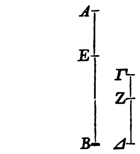


codex B

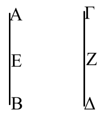


codex b

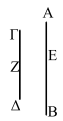
Proposition VII.11



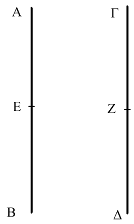
Heiberg



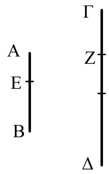
codex P



codex F

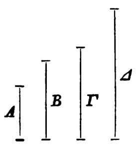


codex B

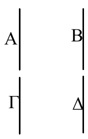


codex b

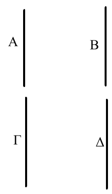
Proposition VII.12



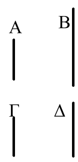
Heiberg



Codex P

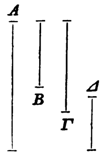


Codex B

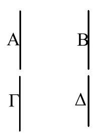


Codex b

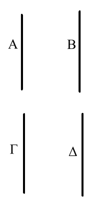
Proposition VII.13



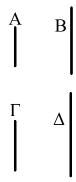
Heiberg



Codex P

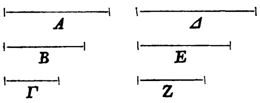


Codex B

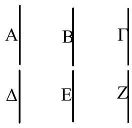


Codex b

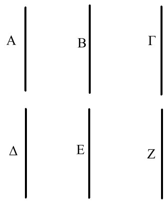
Proposition VII.14



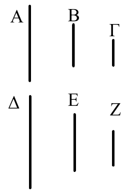
Heiberg



Codex P

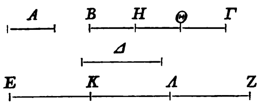


Codex B



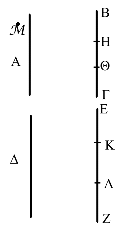
Codex b

Proposition VII.15

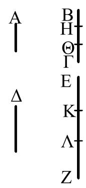


Heiberg

Codex P

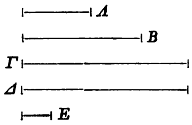


Codex B

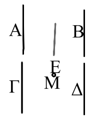


Codex b

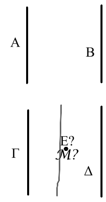
Proposition VII.16



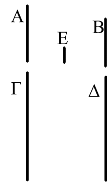
Heiberg



Codex P

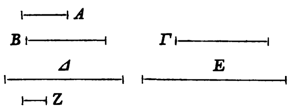


Codex B

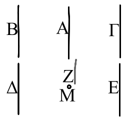


Codex b

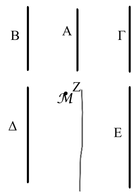
Proposition VII.17



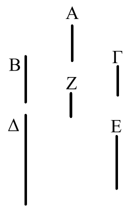
Heiberg



Codex P

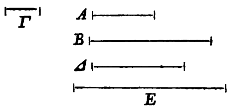


Codex B

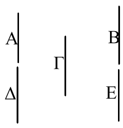


Codex b

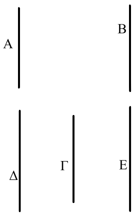
Proposition VII.18



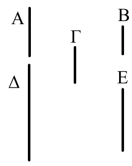
Heiberg



Codex P

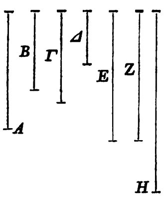


Codex B

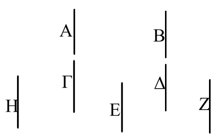


Codex b

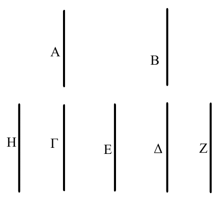
Proposition VII.19[28]



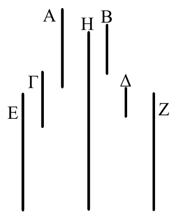
Heiberg



Codex P



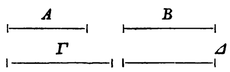
Codex B



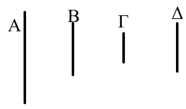
Codex b

[28] Codex b exchanges lines E and H.

Proposition VII.20 in the Theonine version[29]



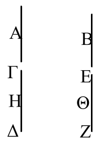
Heiberg



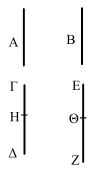
Codex b

Proposition VII.20 (21 in the Theonine version)

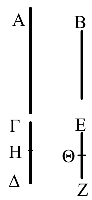
Heiberg



Codex P

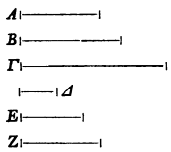


Codex B

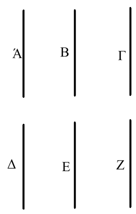


Codex b

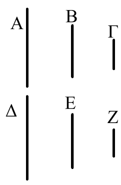
Proposition VII.22 in the Theonine version



Heiberg



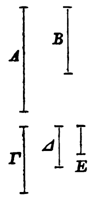
Codex B



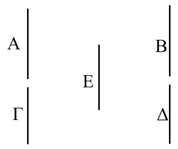
Codex b

[29] Codex B has this proposition in the margin, and its diagram is hardly visible.

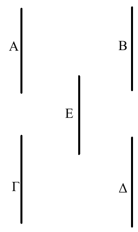
Proposition VII.21 (23 in the Theonine version)



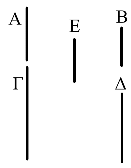
Heiberg



Codex P

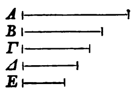


Codex B

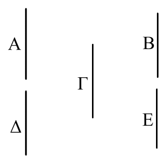


Codex b

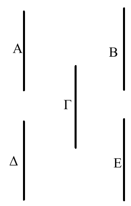
Proposition VII.22 (24 in the Theonine version)



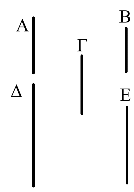
Heiberg



Codex P

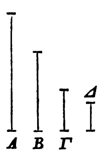


Codex B

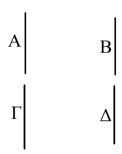


Codex b

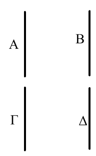
Proposition VII.23 (25 in the Theonine version)



Heiberg



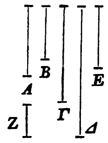
Codex P



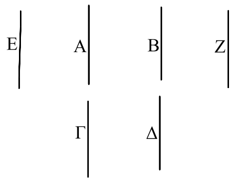
Codex B

Codex b

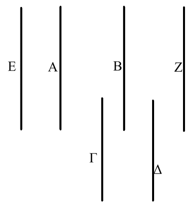
Proposition VII.24 (26 in the Theonine version)



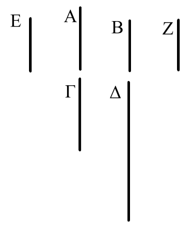
Heiberg



Codex P

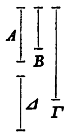


Codex B

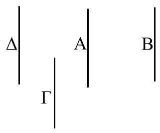


Codex b

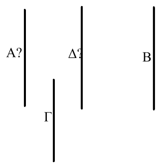
Proposition VII.25 (27 in the Theonine version)



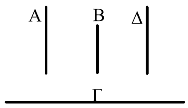
Heiberg



Codex P

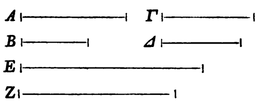


Codex B

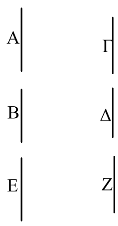


Codex b

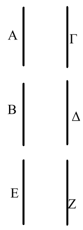
Proposition VII.26 (28 in the Theonine version)



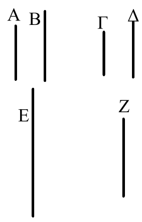
Heiberg



Codex P

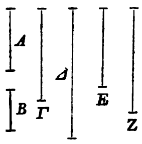


Codex B

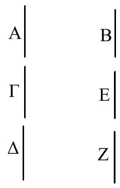


Codex b

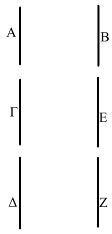
Proposition VII.27 (29 in the Theonine version)



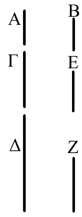
Heiberg



Codex P

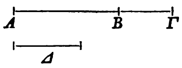


Codex B

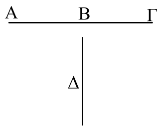


Codex b

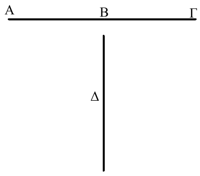
Proposition VII.28 (30 in the Theonine version)



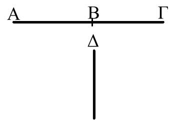
Heiberg



Codex P

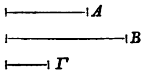


Codex B

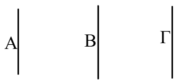


Codex b

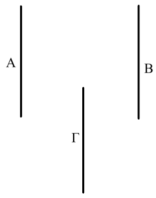
Proposition VII.29 (31 in the Theonine version)



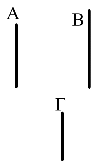
Heiberg



Codex P

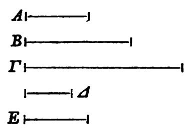


Codex B

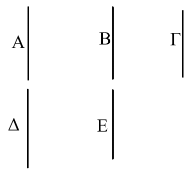


Codex b

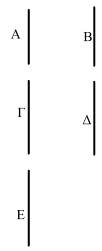
Proposition VII.30 (32 in the Theonine version)



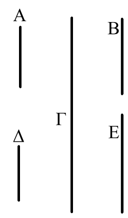
Heiberg



Codex P



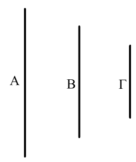
Codex B



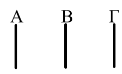
Codex b

Proposition VII.31 (33 in the Theonine version), alternative proof

Heiberg

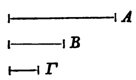


Codex B

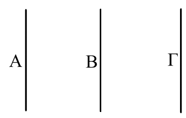


Codex b

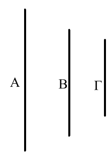
Proposition VII.31 (33 in the Theonine version)



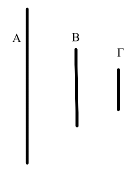
Heiberg



Codex P



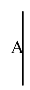
Codex B



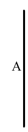
Codex b

Proposition VII.32 (34 in the Theonine version)

Heiberg



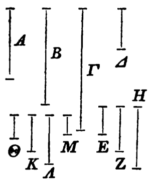
Codex P



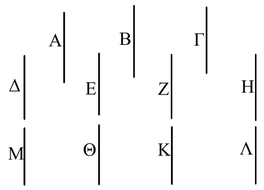
Codex B

Codex b

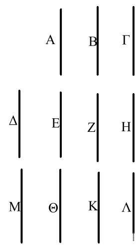
Proposition VII.33 (35 in the Theonine version)



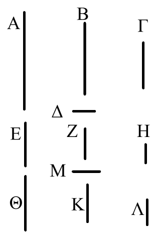
Heiberg



Codex P

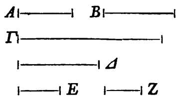


Codex B

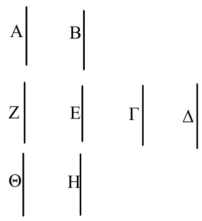


Codex b

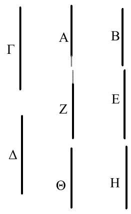
Proposition VII.34 (36 in the Theonine version)



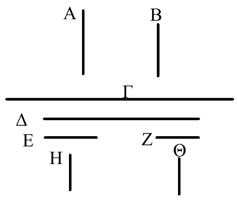
Heiberg



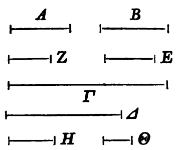
Codex P



Codex B

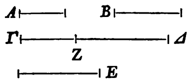


Codex b

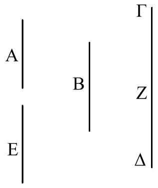


Heiberg (second case)

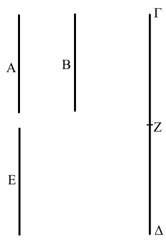
Proposition VII.35 (37 in the Theonine version)



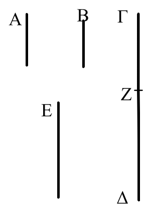
Heiberg



Codex P

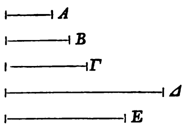


Codex B

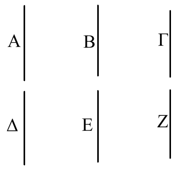


Codex b

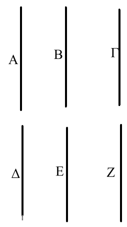
Proposition VII.36 (38 in the Theonine version)[30]



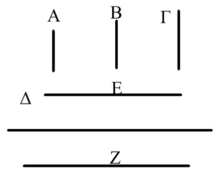
Heiberg



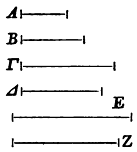
Codex P



Codex B

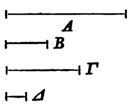


Codex b

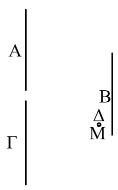


Heiberg (second case)

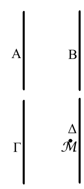
Proposition VII.37 (39 in the Theonine version)



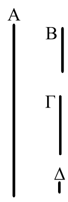
Heiberg



Codex P

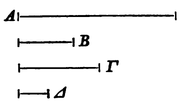


Codex B

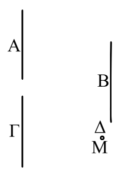


Codex b

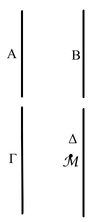
a Proposition VII.38 (40 in the Theonine version)



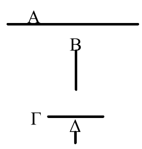
Heiberg



Codex P



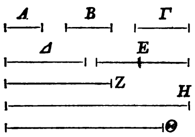
Codex B



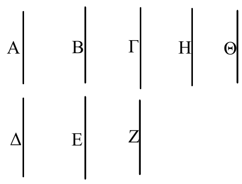
Codex b

[30] Codex b probably misplaces the label *E*.

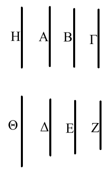
Proposition VII.39 (41 in the Theonine version)



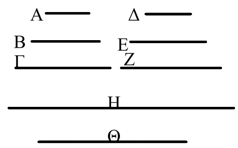
Heiberg



Codex P



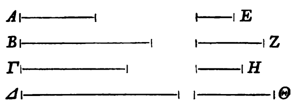
Codex B



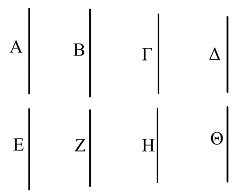
Codex b

Book VIII

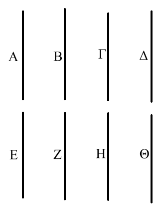
Proposition VIII.1



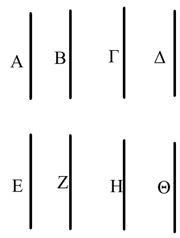
Heiberg



codex P

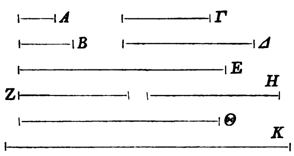


codex B

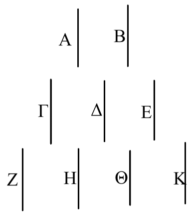


codex b

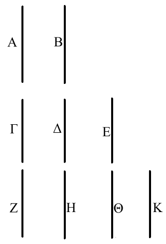
Proposition VIII.2



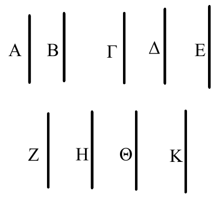
Heiberg



codex P

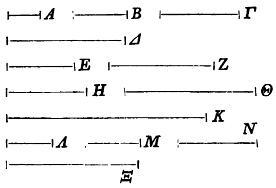


codex B

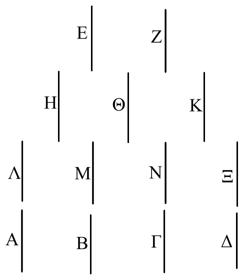


codex b

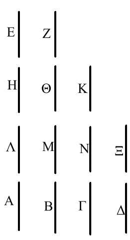
Proposition VIII.3



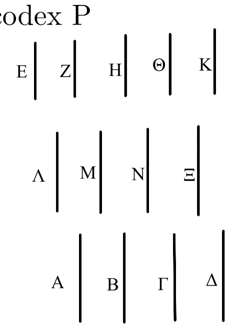
Heiberg



codex P

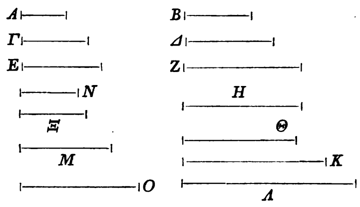


codex B

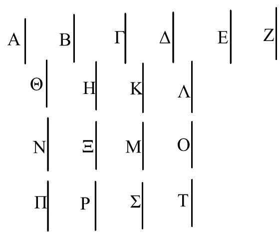


codex b

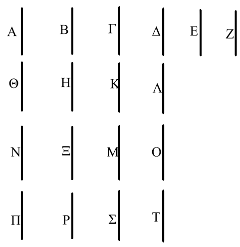
Proposition VIII.4



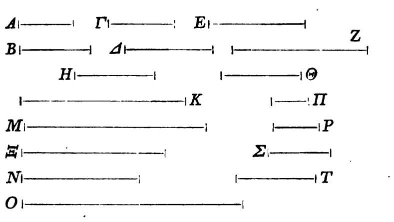
Heiberg



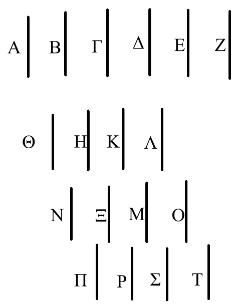
codex P



codex B

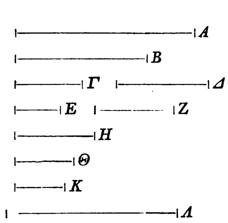


Heiberg (second case)

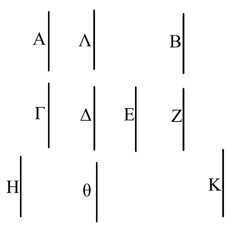


Codex b

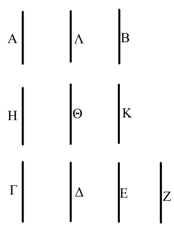
Proposition VIII.5



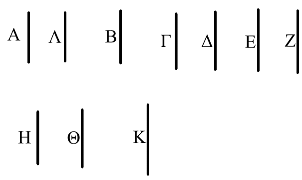
Heiberg



Codex P

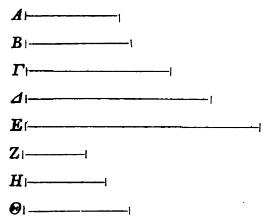


Codex B

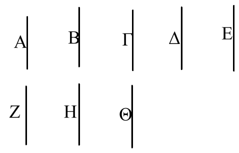


Codex b

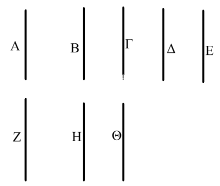
Proposition VIII.6



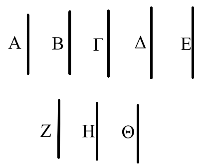
Heiberg



Codex P

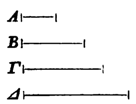


Codex B

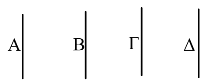


Codex b

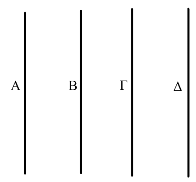
Proposition VIII.7



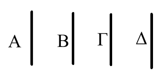
Heiberg



Codex P

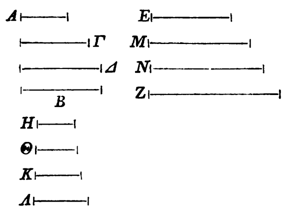


Codex B

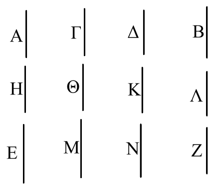


Codex b

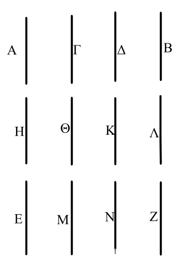
Proposition VIII.8



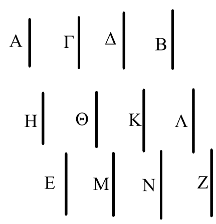
Heiberg



Codex P

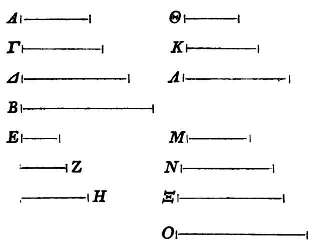


Codex B

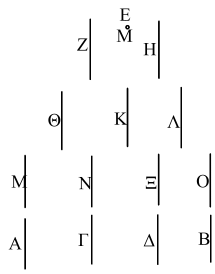


Codex b

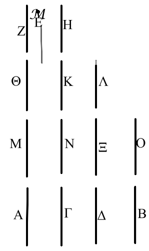
Proposition VIII.9



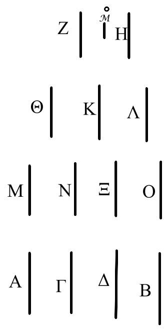
Heiberg



Codex P

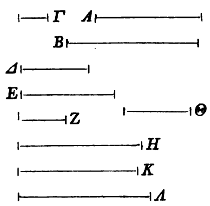


Codex B

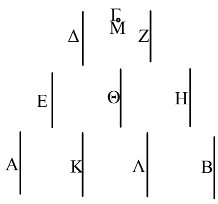


Codex b

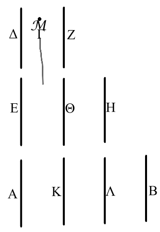
Proposition VIII.10



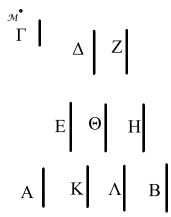
Heiberg



Codex P

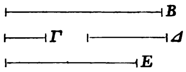


Codex B

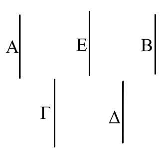


Codex b

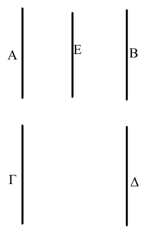
Proposition VIII.11



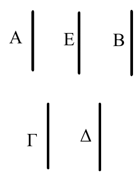
Heiberg



Codex P

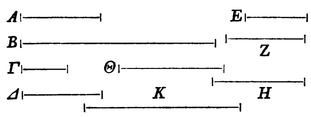


Codex B

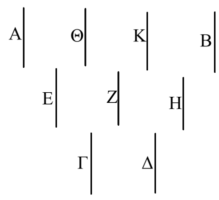


Codex b

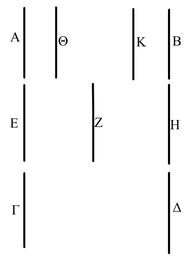
Proposition VIII.12



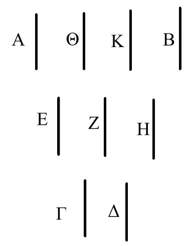
Heiberg



Codex P

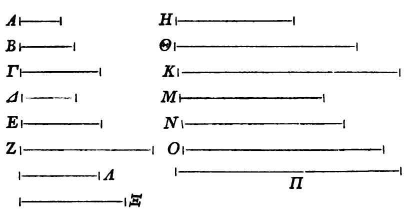


Codex B

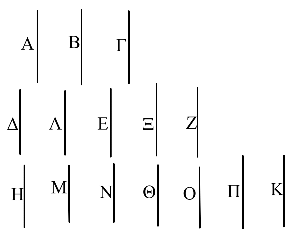


Codex b

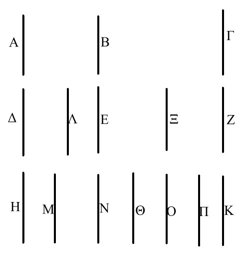
Proposition VIII.13



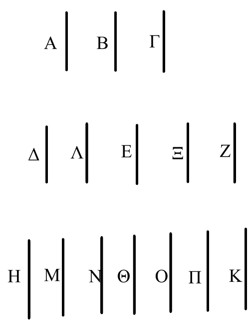
Heiberg



codex P

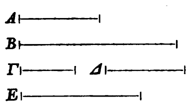


codex B

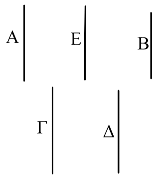


codex b

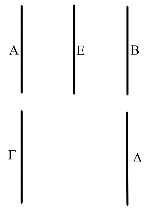
Proposition VIII.14



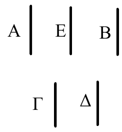
Heiberg



Codex P

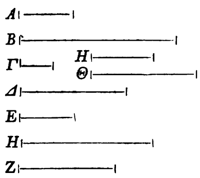


Codex B

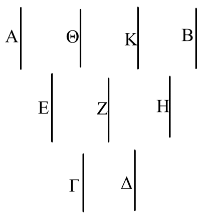


Codex b

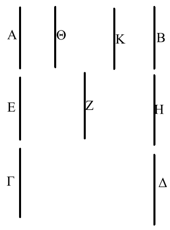
Proposition VIII.15



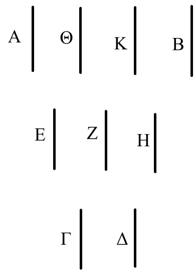
Heiberg



Codex P

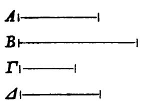


Codex B

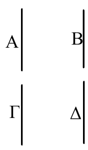


Codex b

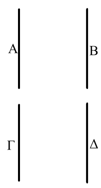
Proposition VIII.16



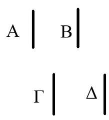
Heiberg



Codex P

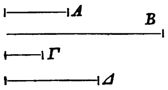


Codex B

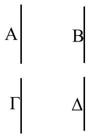


Codex b

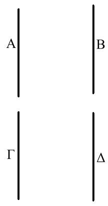
Proposition VIII.17



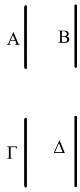
Heiberg



Codex P

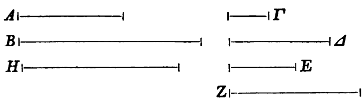


Codex B

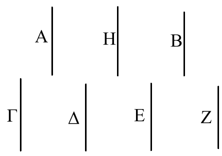


Codex b

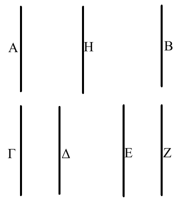
Proposition VIII.18



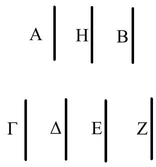
Heiberg



Codex P

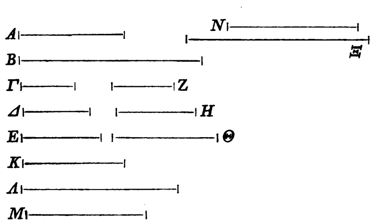


Codex B

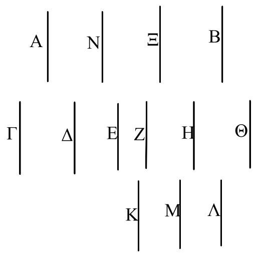


Codex b

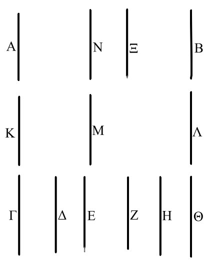
Proposition VIII.19



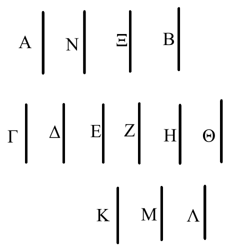
Heiberg



codex P

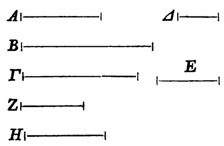


codex B

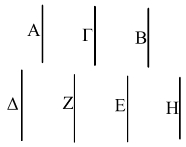


codex b

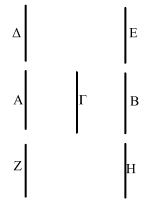
Proposition VIII.20



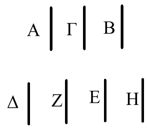
Heiberg



Codex P

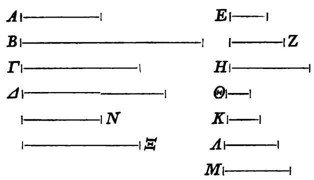


Codex B

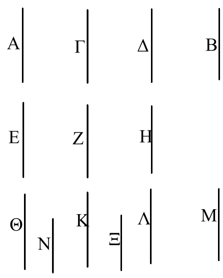


Codex b

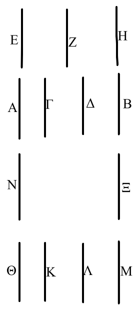
Proposition VIII.21



Heiberg



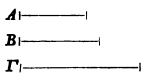
Codex P



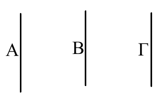
Codex B

Codex b

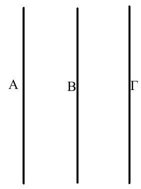
Proposition VIII.22



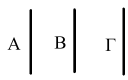
Heiberg



Codex P

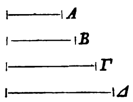


Codex B

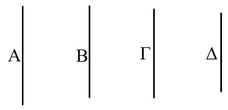


Codex b

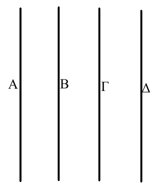
Proposition VIII.23



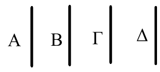
Heiberg



Codex P

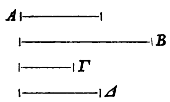


Codex B

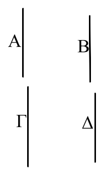


Codex b

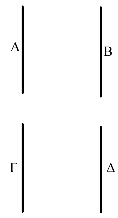
Proposition VIII.24



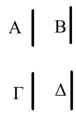
Heiberg



Codex P

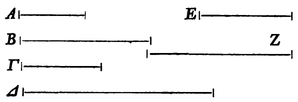


Codex B

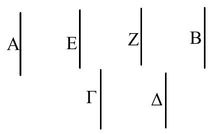


Codex b

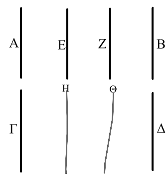
Proposition VIII.25



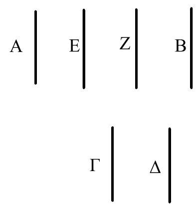
Heiberg



Codex P

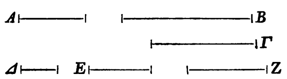


Codex B

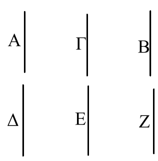


Codex b

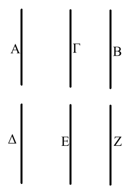
Proposition VIII.26



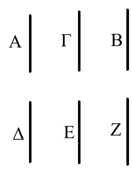
Heiberg



Codex P

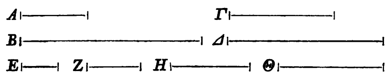


Codex B

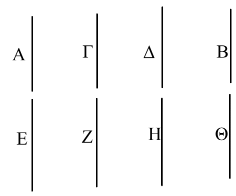


Codex b

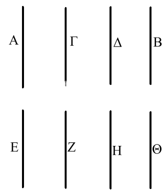
Proposition VIII.27



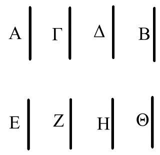
Heiberg



codex P



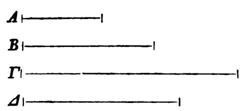
codex B



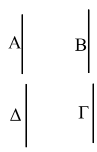
codex b

Book IX

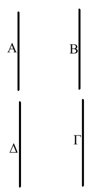
Proposition IX.1



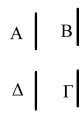
Heiberg



Codex P

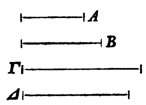


Codex B

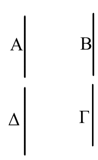


Codex b

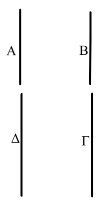
Proposition IX.2



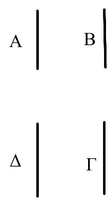
Heiberg



Codex P

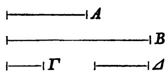


Codex B

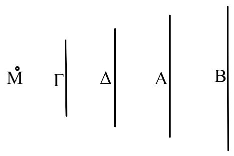


Codex b

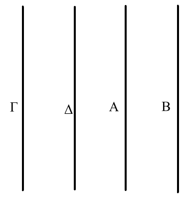
Proposition IX.3



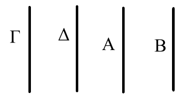
Heiberg



Codex P

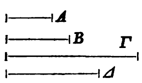


Codex B

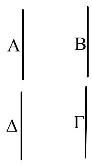


Codex b

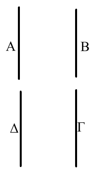
Proposition IX.4



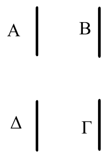
Heiberg



Codex P

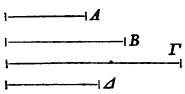


Codex B

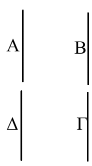


Codex b

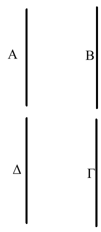
Proposition IX.5



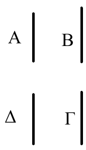
Heiberg



Codex P

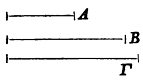


Codex B

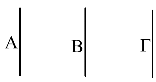


Codex b

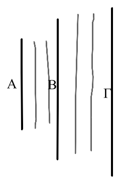
Proposition IX.6



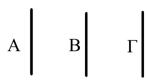
Heiberg



Codex P

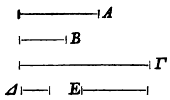


Codex B

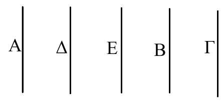


Codex b

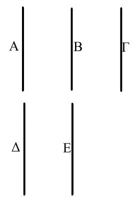
Proposition IX.7



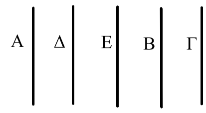
Heiberg



Codex P

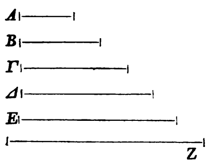


Codex B

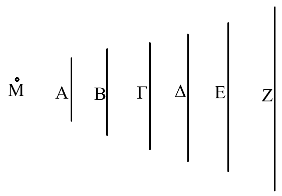


Codex b

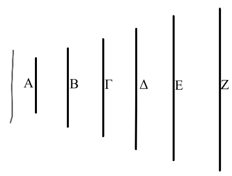
Proposition IX.8



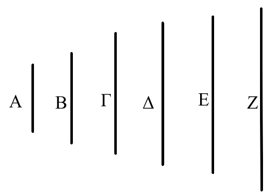
Heiberg



Codex P

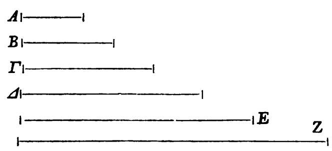


Codex B

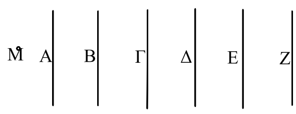


Codex b

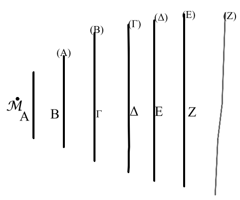
Proposition IX.9



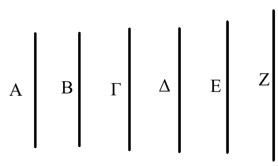
Heiberg



codex P

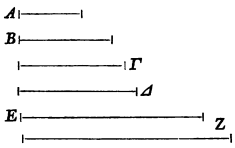


codex B

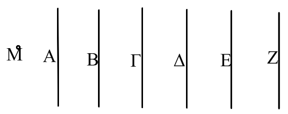


codex b

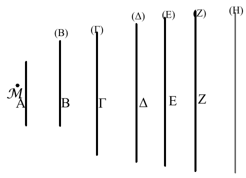
Proposition IX.10



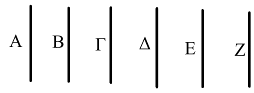
Heiberg



Codex P

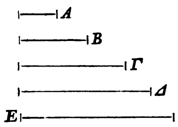


Codex B

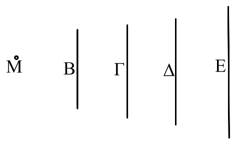


Codex b

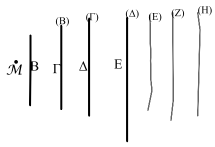
Proposition IX.11



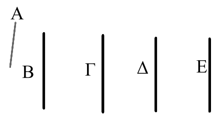
Heiberg



Codex P

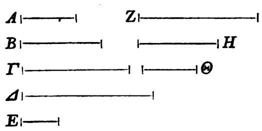


Codex B

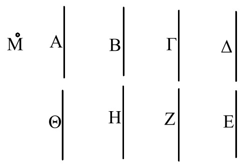


Codex b

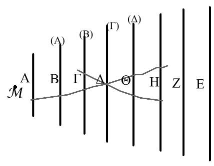
Proposition IX.12



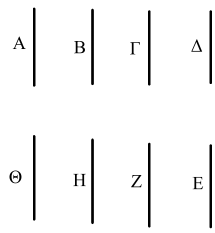
Heiberg



Codex P

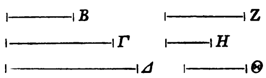


Codex B

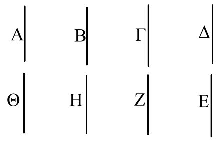


Codex b

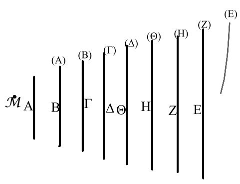
Proposition IX.13



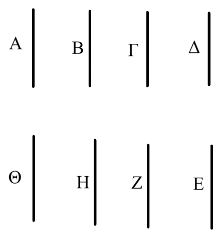
Heiberg



Codex P

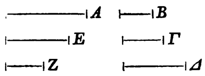


Codex B

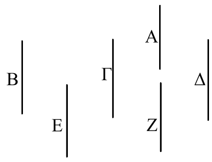


Codex b

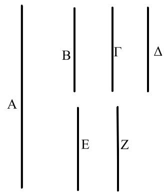
Proposition IX.14



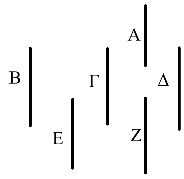
Heiberg



Codex P

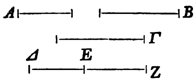


Codex B

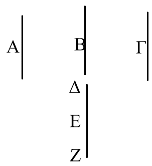


Codex b

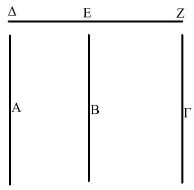
Proposition IX.15



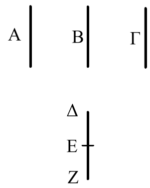
Heiberg



Codex P

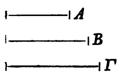


Codex B

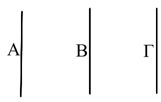


Codex b

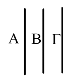
Proposition IX.16



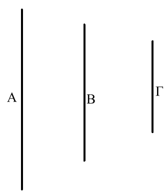
Heiberg



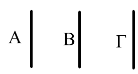
codex P



codex F

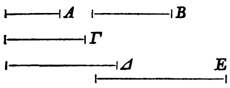


codex B

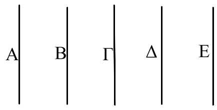


codex b

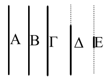
Proposition IX.17



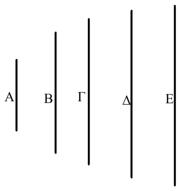
Heiberg



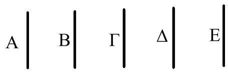
codex P



codex F

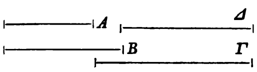


codex B

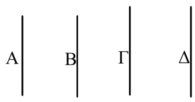


codex b

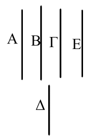
Proposition IX.18[31]



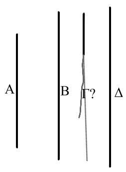
Heiberg



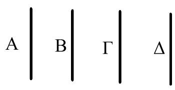
codex P



codex F



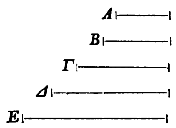
codex B



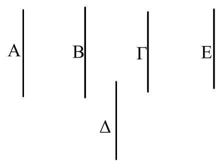
codex b

[31] F seems to show here the diagram for IX.19.

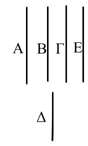
Proposition IX.19



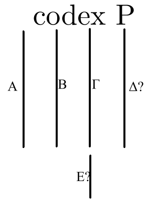
Heiberg



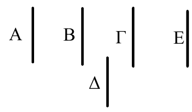
codex P



codex F

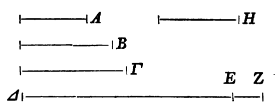


codex B

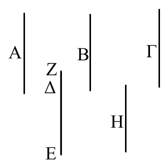


codex b

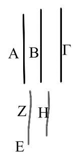
Proposition IX.20



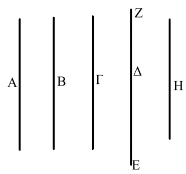
Heiberg



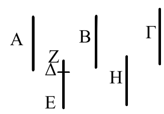
codex P



codex F

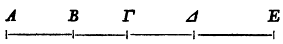


codex B

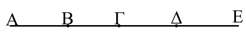


codex b

Proposition IX.21

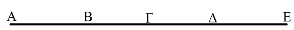


Heiberg

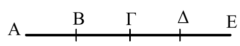


codex P

codex F

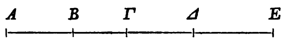


codex B

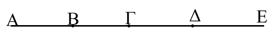


codex b

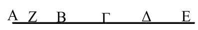
Proposition IX.22[32]



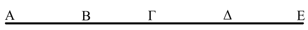
Heiberg



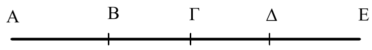
codex P



codex F

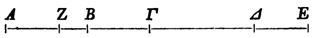


codex B



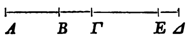
codex b

Proposition IX.22 alternative proof

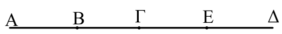


Heiberg

Proposition IX.23[33]



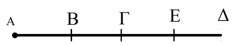
Heiberg



codex P

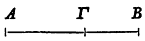
codex F

codex B

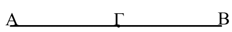


codex b

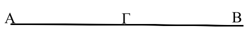
Proposition IX.24



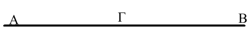
Heiberg



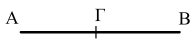
codex P



codex F



codex B



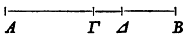
codex b

---

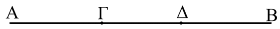
[32] The diagram in Codex F includes the point Z, which is introduced in the alternative proof found only in this codex. (The alternative proof refers to the same diagram as the main proof.) Heiberg provides an independent diagram for this alternative.

[33] F seems to try alternative arguments similar to alternative of IX.22.

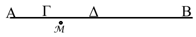
Proposition IX.25



Heiberg

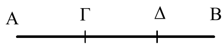


codex P



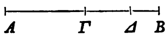
codex F

codex B

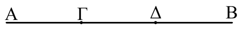


codex b

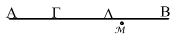
Proposition IX.26



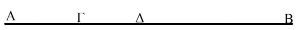
Heiberg



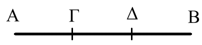
codex P



codex F

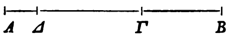


codex B

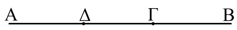


codex b

Proposition IX.27



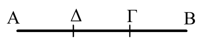
Heiberg



codex P

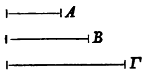
codex F

codex B

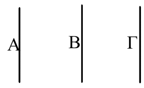


codex b

Proposition IX.28

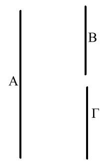


Heiberg

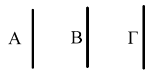


codex P

codex F

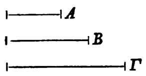


codex B

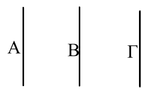


codex b

Proposition IX.29

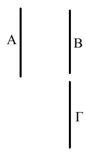


Heiberg

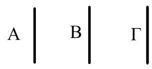


codex P

codex F

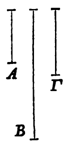


codex B

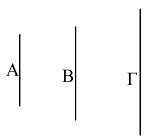


codex b

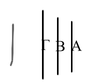
Proposition IX.30



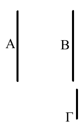
Heiberg



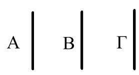
codex P



codex F

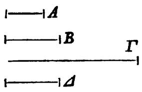


codex B

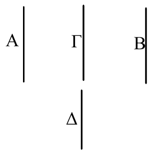


codex b

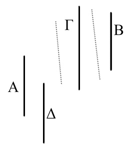
Proposition IX.31[34]



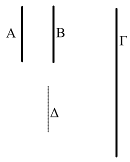
Heiberg



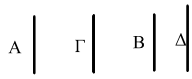
codex P



codex F

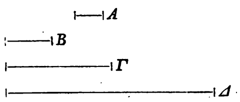


codex B

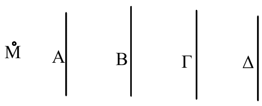


codex b

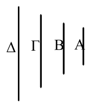
Proposition IX.32



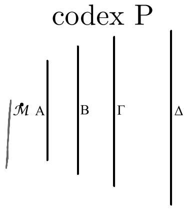
Heiberg



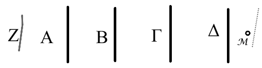
codex P



codex F



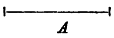
codex B



codex b

34 B has a trace of an erased line which seems to have been Δ.

Proposition IX.33



Heiberg

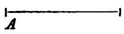
codex P

codex F

codex B

codex b

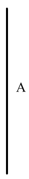
Proposition IX.34



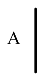
Heiberg

codex P

codex F

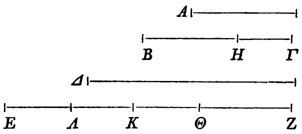


codex B

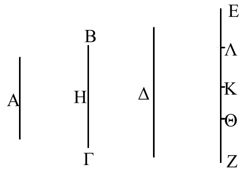


codex b

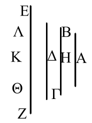
Proposition IX.35



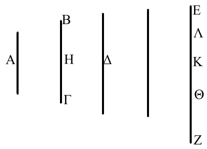
Heiberg



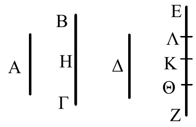
codex P



codex F

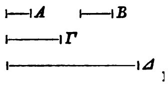


codex B

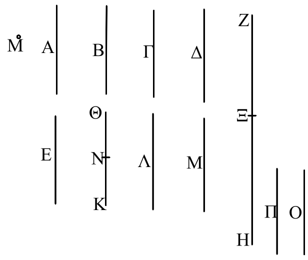


codex b

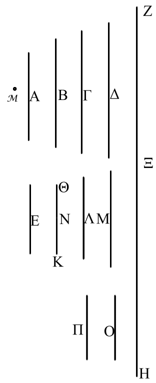
Proposition IX.36



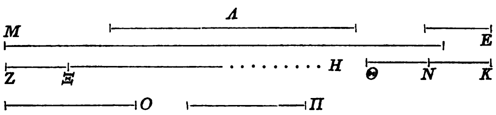
Heiberg



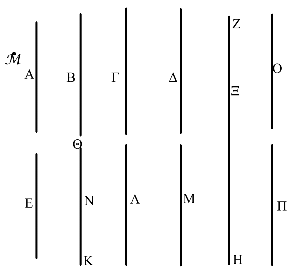
codex P



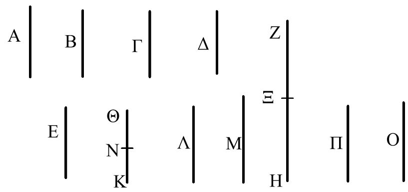
codex F



Heiberg



codex B



codex b

Acknowledgements

This research on diagrams was supported by Japan Society for the Promotion of Science (Grant nos. JP17300287, JP21300325). The author would like to express his gratitude to the referees for their valuable comments and suggestions.

References

Original Sources

B: Oxford, Bodleian Libraries, *Dorv.* 301, 888 CE.

b: Bologna, Biblioteca comunale dell'Archiginnasio, A 1819, 11th century.

F: Florence, Biblioteca Medicea Laurenziana, *Plut.* 28.3, 11th century.

P: Vatican, Biblioteca Apostolica Vaticana, *Vat. gr.* 190, 9th century.

Modern Scholarship

August, E.F., 1829. *Euclidis Elementa ex optimis libris in usum tironum*, 2 vols., Berlin.

Bajri, S., Hannah, J., Montelle, C., 2015. "Revisiting Al-Samaw'al's table of binomial coefficients: Greek inspiration, diagrammatic reasoning and mathematical induction," *Archive for History of Exact Sciences* 69, 537–576.

Becker, O., 1936. "Die Lehre vom Geraden und Ungeraden im Neunten Buch der Euklidischen Elemente. (Versuch einer Wiederherstellung in der ursprünglichen Gestalt)," *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*, Band 3, 533–553.

Cairncross, A., Henry, W.B., 2015. "Euclid, *Elements* I.4 (diagram), 8–11, 14–25 (without proofs)," *The Oxyrhynchus Papyri*, LXXXII (Graeco-Roman memoires, No. 103), 23–38.

Carman, C.C., 2018. "Accounting for overspecification and indifference to visual accuracy in manuscript diagrams: A tentative explanation based on transmission," *Historia Mathematica* 45, 217–236.

Gregory, D., 1703. *Euclidis quae supersunt omnia ex recensione Davidis Gregorii*, Oxford.

Grynaeus, S., 1533. ΕΥΚΛΕΙΔΟΥ ΣΤΟΙΧΕΙΩΝ ΒΙΒΛ. ΙΕ. ΕΚ ΤΩ ΘΕΩΝΟΣ ΣΥΝΟΥΣΩΝ, Basileae apud Ioan, Hervagium.

Heath, T.L., 1925. *The Thirteen Books of the Elements, Translated with Introduction and Commentary by Heath, Thomas L. (Sir)*, 2nd ed., 3 vols., Cambridge. (Reprinted: New York, 1956.)

Heiberg, J.L., 1883–1888. *Euclidis Elementa: edidit et latine interpretatus est I.L. Heiberg*, 5 vols., Leibzig.

Mogenet, J., Tihon, A., 1985. *Le Grand Commentaire de Théon d'Alexandrie aux tables faciles de Ptolémée, Livre I., Histoire du texte, édition critique, traduction,* *Revues et complétées par Anne Tihon, commentaire par Anne Tihon*, Studi e Testi 315, Biblioteca Apostolica Vaticana.

Saito, K., 2006. "A preliminary study in the critical assessment of diagrams in Greek mathematical works," *SCIAMVS* 7, 81–144.

——— 2012. "One diagram for multiple cases," *Hvmanistica: An International Journal of Early Renaissance Studies* 7, 17–25.

Saito, K., Sidoli, N., 2012. "Diagrams and arguments in Ancient Greek mathematics: Lessons drawn from comparisons of the manuscript diagrams with those in modern critical editions," in Chemla, K., ed., *The History of Mathematical Proof in Ancient Traditions*, Cambridge, 135–162.

(Received: January 18, 2017)

(Revised: October 16, 2018)