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Al-Shīrāzī's "Proofs" of Euclid's Postulates in Arabic and Persian

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Abstract

This article discusses a largely unnoticed medieval treatise setting out "proofs" of Euclid's postulates, composed by Quṭb al-Dīn al-Shīrāzī (d. 710 AH / 1311 CE). The mathematical content of most of the proofs can be traced back to antiquity. In this treatise these proofs are brought together into a single unit. The proofs are extant in both Arabic and Persian transmissions of Euclid. In this paper, we present an edition of the Arabic and Persian texts and situate these "demonstrations" within the history of attempts to prove Euclid's postulates.

I Introduction

We examine a short untitled Arabic treatise offering "proofs" for the six postulates (*al-uṣūl al-mawḍūʿa*) that traditionally follow the definitions of Book I of the *Elements* in the Arabic transmission. These "demonstrations" have been described by De Young (2007) in his study of the geometrical section of the Persian treatise *Durrat al-tāj li-ghurrat al-Dabbāj* (The pearl of the crown for the illustrious <one> of al-Dubbāj <family>), an encyclopedic survey of Aristotelian philosophy composed by Quṭb al-Dīn Maḥmūd b. Maṣʿūd al-Shīrāzī (634–710 AH / 1236–1311 CE).[1] We have edited both the Arabic and the three Persian versions of these proofs of the Euclidean postulates and have translated them into English. We situate these demonstrations within the broader historical landscape in relation to other demonstrations of the postulates whose origins, for the most part, can be traced back to Hellenistic Greek discussions of the *Elements*. We conclude with a brief consideration of the context in which these "demonstrations" appear to have been read.

In a ground-breaking study of the Persian transmission of Euclid's geometry, Brentjes (1998) identified the version of Euclid translated by al-Shīrāzī as the first Persian edition of Euclid's classic mathematical work. She pointed out (1998, 75) that not all manuscript copies of *Durrat al-Tāj* include the section on mathematics (geometry, astronomy, arithemetic, music).[2] Following up on Brentjes pioneering

[1] For a succinct summary of al-Shīrāzī's scientific oevre, see Nasr (1975).

[2] Similarly, the omission of the mathematical section is also a common occurrence in numerous manuscripts of the earlier Arabic philosophical compendium, *Kitāb al-Shifāʾ*, composed by Ibn Sīnā,

study, De Young (2007) noted that the geometrical summary contained in the Persian compendium, although based on the already mentioned *Taḥrīr* of al-Ṭūsī, was not simply a translation of al-Ṭūsī's Arabic treatise into Persian. Al-Shīrāzī felt free to add to and delete from al-Ṭūsī's text in order to construct his own version of the *Elements*. Among the additions that al-Shīrāzī included were these "demonstrations" of the Euclidean postulates, although they are not found in al-Ṭūsī's Arabic edition.

II Proving Postulates

Why, we might ask, did earlier scholars feel it necessary or even desirable to include demonstrations of the postulates? For many of us who studied Euclidean geometry in secondary school, the very meaning of the term "postulate" seems antithetical to the concept of demonstration. We were taught that postulates and axioms are statements meant to be assumed as self-evident, true. A typical example of this view is expressed succinctly by Spector (2020) when he states at the beginning of his discussion of the primitives of Euclidean geometry:

> It is not possible to prove every statement. … Nevertheless, we should prove as many statements as possible. Which is to say, the statements we do not prove should be as few as possible. They are called the First Principles. They fall into three categories: Definitions, Postulates, and Axioms or Common Notions.

If postulates belong among the first principles, then why is there so much interest in proving or at least justifying their assumption? We suggest that the answer may lie, at least in part, in the terminology used by Aristotle when discussing first principles of any science and the terminology used in Euclid's *Elements*.

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on which al-Shīrāzī seems to have modeled his own philosophical compendium. Some indication of the frequency of copies of Ibn Sīnā's work that include the mathematics section can be gleaned from the census of manuscripts by Bertolacci (2008), which is updated when necessary on his website: http://www.avicennaproject.eu/index.php?id=33. The information in this census concerning the presence of the mathematical section of Avicenna's compendium is sometimes incorrect, however. It appears that in some cases Bertolacci was making a very rapid survey of the contents of manuscript copies containing the *Illāhiyyāt* (metaphysics) section and may have been mislead by the presence of several diagrams in the section on logic.

II.1 First Principles in Euclid's *Elements*

The *Elements* opens with statements of fundamental principles that correspond in some ways to Aristotle's (ἀρχαί) in the discipline of physics.[3] These fundamental principles are the beginning points of mathematical science. They "constitute the points of departure for chains of deductive arguments" within Euclid's text (Vitrac 1990, 117). They are sometimes denoted in the Arabic secondary transmission using the term *muṣādarāt*.[4]

These first principles are divided into three classes. We find in the first place, definitions (ὅροι), in Arabic *ḥudūd*. Euclid placed twenty-three definitions at the beginning of book I.[5]

The definitions of book I are followed by postulates (αἰτήματα), which are called in the Arabic primary transmission *al-ashyāʾ allatī taḥtāj ilā al-ittifāq ʿalayhā* (things that one must agree to accept).[6] This terminology is not widely used in the Arabic secondary literature, though. The early commentary on the *Elements* by al-Nayrīzī uses the term *al-muṣādarāt*, as does the commentator Ibn al-Haytham in his *Ḥall shukūk Kitāb Uqlīdis*, but these are the only two examples that we know in which the postulates are denoted using this term. Later authors, such as al-Samarqandī and Naṣīr al-Dīn al-Ṭūsī and the Pseudo-Ṭūsī use the term *al-uṣūl al-mawḍūʿa*.[7]

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[3] Although these fundamental principles are not given a specific label in the Arabic transmission of the *Elements*, they appear to function much like what Ibn Sīnā called *al-mabādīʾ* in his analysis of Aristotelian physics. For example, the first chapter in the section on the physics in *Kitāb al-Najāt* is titled: "On the first principles (*al-mabādīʾ*) which the <science of> physics assumes" (Ibn Sīnā 1331 H., 159; see also Lammer 2018, 81).

[4] For example, the Arabic commentary on the premises of Euclid's *Elements* by Ibn al-Haytham is titled *Sharḥ muṣādarāt Kitāb Uqlīdis fī al-Uṣūl* (Sude 1974, 6). His commentary considers the definitions, postulates, and axioms. In this context, the Arabic term, as a third-stem verbal noun, conveys the idea of a request or a demand and thus is comparable to the Greek αἰτήματα in its general sense (Lammer 2018, 82).

[5] Not all the definitions are located in book I, however. Euclid apparently decided to place at the beginning of each book the definitions of entities that first appear in that section of the treatise. Thus we find definitions at the beginning of nearly all thirteen books of the *Elements*. There are two exceptions to this general procedure. All the definitions of entities used in the arithmetical books (VII–IX) have been placed at the beginning of book VII. Similarly, in the case of the stereometrical books (XI–XIII), all the definitions have been collected at the beginning of book XI.

[6] In the Duneden University Library MS De Beer 8, the copyist adds an alternative title: *al-muṣādarāt*, perhaps influenced by the secondary literature.

[7] The Arabic verbal root *waḍʿ* has the meaning to put or to place (something), and by extension to posit (something as something). Thus it is frequently used to translate the Greek verb τιθέναι.

After these, Euclid states several Common Notions (κοιναὶ ἔννοιαι, denoted in the Arabic primary transmission using the term *ʿilm ʿāmm muttafaq ʿalayhā* (general principles that must be assumed).[8] In the secondary Arabic literature on Euclid, these Common Notions are much more frequently termed *al-ʿulūm al-mutaʿārifa* (common principles). These principles are Common Notions in the sense that they are shared by more than one discipline. There are five Common Notions mentioned in the Greek edition of Heiberg, but nine are traditionally given in the medieval Arabic transmission of the *Elements*.

These three classes of mathematical primitives together constitute the fundamental principles in Euclid's treatise. It appears that from the time the *Elements* was transmitted into Arabic, the commentators and editors felt a pull toward using Aristotelian terminology, perhaps attempting to draw clearer parallels between Aristotle's *Physics* and Euclid's *Elements*. To explain this verbal parallelism, we digress briefly to consider some points of Aristotle's philosophy and how its concepts were translated into Arabic.

II.2 Aristotle on First Principles of Science

Aristotle, in a well-known passage at the beginning of his *Posterior Analytics* (I, 10), seems to place the first principles of any science beyond proof (Heath 1926, I, 117–118):

> By first principles in each genus I mean those the truth of which it is not possible to prove. What is denoted by the first (terms) and those derived from them is assumed; but, as regards their existence, this must be assumed for the principles but proved for the rest. Thus what a unit is, what the straight (line) is, or what a triangle is (must be assumed); and the existence of the unit and of magnitude must also be assumed, but the rest must be proved. …
>
> For every demonstrative science has to do with three things, (1) the things that are assumed to exist, namely the genus (subject-matter) in each case the essential properties of which the science investigates, (2) the common axioms so-called, which are the primary source of demonstration, and (3) the properties with regard to which all that is assumed is the meaning of the respective terms. …

Aristotle goes on to explain the difference between a hypothesis and a postulate (Heath 1926, I, 118–119):

To convey the meaning of the Greek ὑπόθεσις Arabic translators used *al-uṣūl al-mawḍūʿa* (Lammer 2018, 84).

[8] The copyist of Duneden University Library, De Beer 8, has added an alternative heading: *ʿulūm mutaʿārifa*.

> Now that which is *per se* necessarily true, and must necessarily be thought so, is not a hypothesis nor yet a postulate. … Now, anything that the teacher assumes, though it is matter of proof, without proving it himself, is a hypothesis (ὑπόθεσις) if the thing assumed is believed by the learner, and it is moreover a hypothesis, not absolutely, but relatively to the particular pupil; but if the same thing is assumed when the learner either has no opinion on the subject or is of a contrary opinion, it is a postulate (αἴτημα).

Thus the same statement may be either a hypothesis or a postulate, depending on whether or not the student believes it to be valid. And whether considered as hypotheses or postulates, these statements are susceptible to proof, even though the teacher asks the student to accept them without proof for the moment.

Aristotle's *Posterior Analytics* was translated into Arabic by Abū Bishr Mattā ibn Yūnus (d. 328 AH / 940 CE), whose work was based on a Syriac version by the famous translator Isḥāq ibn Ḥunayn. As Lammer (2018, 84) has pointed out, Abū Bishr translated the Greek ὑπόθεσις into Arabic using the term *al-aṣl al-mawḍūʿ*. And he translates the Greek αἴτημα using the Arabic term *muṣādara*. As Aristotle uses these terms in the *Posterior Analytics*, they refer to statements that are susceptible to proof, even if no proof is immediately offered.

When the same Greek technical terms are used both by Aristotle and by Euclid, there is the potential that the meaning of the terms can coalesce and be read the same (Aristotelian) way in both treatises. Although this potential for confusing technical terms does not appear in the Arabic primary transmission, we see already early in the secondary transmission that Aristotelian terminology begins to invade geometry. And even though the Arabic terms used to denote Euclidean postulates undergoes a distinct change between the 10th and 12th centuries, both terms commonly used are derived from the Aristotelian tradition. And both the terms are used by Aristotle to refer to principles that are capable of being proved but that the student is asked to accept. The difference in terminology reflects whether the student accepts the statement because he agrees with it or whether he accepts the statement provisionally even though he has some doubt about or even disagrees wth the premise. The key point to notice is that both the common terms in the Arabic secondary transmission parallel the Greek terms that, in the *Physics*, suggest the premise is capable of proof. And it may well be this parallelism that sparked the attempts by the Hellenistic and Arabic commentators to prove Euclid's postulates.

III Al-Shīrāzī's Collection of Proofs

Al-Shīrāzī's collection of proofs are known in both Arabic and Persian. The Arabic appears only as an independent treatise within collections of works devoted to Euclid's *Elements*. The Persian collection appears in three different forms: (1) as an inclusion in al-Shīrāzī's Persian translation of the *Taḥrīr Kitāb Uqlīdis* of Naṣīr al-Dīn al-Ṭūsī; (2) as an inclusion in the geometrical section of al-Shīrāzī's encyclopedic *Durrat al-Tāj li-ghurrat al-Dubāj*, which was also based on al-Ṭūsī's text; and (3) as an independent treatise.

III.1 Arabic Version

Following the traditonal *basmalah*, the Arabic treatise begins with a short preamble:

> I want to make known the postulates (*al-uṣūl al-mawḍūʿa*) of the subject <of geometry> that the author, may God have mercy on him, quoted (*naqala ʿan*) at the beginning, that is, his statement "We may connect a straight line between <any> two points" through his statement "the two of them meet on that side if extended."

In this brief statement, al-Shīrāzī informs us that the focus of his treatise is going to be the postulates (*al-uṣūl al-mawḍūʿa*) of book I. These postulates are more explicitly identified by two brief quotations ("from his statement … through his statement …") from the treatise of the *muṣannif* (author). Use of these quotations to delimit the quoted section suggests that these fundamental principles have been extracted from a larger treatise. The phrasing of these brief quotations corresponds precisely to the first postulate and the concluding phrase of the last postulate as formulated in Naṣīr al-Dīn al-Ṭūsī's *Taḥrīr* of the *Elements*.[9]

Furthermore, we observe that al-Shīrāzī mentions a *muṣannif* (author) upon whom he invokes God's mercy. This pious invocation is always applied to the dead, so we know that the author responsible for the original treatise on which a-Shīrāzī proposes to base his discussion was already dead when he began his Arabic text. Even though al-Shīrāzī does not identify this dead author by name, we can guess from the formulation of the quoted postulates that it is most probably al-Ṭūsī (d. 672 AH / 1274 CE). If this hypothesis is correct, the logical conclusion would be that the Arabic treatise was composed sometime after al-Ṭūsī's death and before al-Shīrāzī completed his translation of al-Ṭūsī's *Taḥrīr*.

[9] We have examined numerous manuscripts of both the primary and secondary Arabic transmission and have found the wording used in these quotations only in al-Ṭūsī's *Taḥrīr* of the *Elements*. Although the same formulation of the introductory postulate is also found in the widely-read *Ashkāl al-Taʾsīs* by Shams al-Dīn al-Samarqandī (active in the second half of the 7th century AH / 13th century CE), the concluding postulate is formulated differently, so the text from which al-Shīrāzī is quoting cannot be this popular treatise.

III.2 Inclusion in the Persian Translation of al-Ṭūsī's *Taḥrīr*

Al-Shīrāzī's proofs also appear as an inclusion in his Persian translation of al-Ṭūsī's *Taḥrīr*.[10] This translation begins with a somewhat lengthy preamble in which we find its dedication to Amīr Shāh ibn Tāj al-Dīn Muʿayn ibn Ṭāhir (d. 701 AH / 1302 CE).[11] The preamble also includes the name of the author / translator, Maḥmūd ibn Masʿūd al-Shīrāzī. The colophon of Tehran, Majlis Shūrā, Sinā 226 gives the date of copying as 698 AH / 1298–1299 CE.

We have consulted two manuscripts of this Persian translation in our edition:[12]

- New York, Columbia University Library, Plimpton Or 282
- Tehran, Majlis Shūrā Library, Sinā 226

III.3 Inclusion in the Geometrical Section of *Durrat al-Tāj*

Al-Shīrāzī's collection of proofs also exists as an inclusion in the geometrical section of his encyclopedic Persian treatise, *Durrat al-Tāj li-ghurrat al-Dubāj* (Pearl of the Crown for the outstanding Dubāj). This treatise, completed in 705 AH / 1305 CE near the end of his life, was dedicated to Dubāj ibn Ḥusām al-Dīn Fīl-Shāh ibn Sayf al-Dīn Rustam ibn Dubāj Isḥāqāwand, ruler of Bayah Pas in Gīlān province of Iran (Savage-Smith 2005, 67).[13]

Al-Shīrāzī's philosophical encyclopedia has many parallels to the earlier scientific and philosophical encyclopedia *Kitāb al-shifāʾ* of Ibn Sīnā (d. 428 AH / AD 1037), although al-Shīrāzī's treatise is less voluminous. Its organization mirrors that of *Kitāb al-shifāʾ*, using the same terminology to name the divisions and subdivisions of the text. The parallels are not surprising. Quṭb al-Dīn had studied the writings of Ibn Sīnā for many years and had been heavily influenced by the Aristotelian approach of Ibn Sīnā. Rather than follow the lead of Ibn Sīnā and create a condensation of

[10] Doostgharin (2008–2009) has published in modern Persian an overview of this translation and its distinctive characteristics.

[11] This is the same ruler to whom Shīrāzī dedicated his *al-Tuḥfa al-Shāhiyya* in 684 AH / 1285 CE. We thank an anonymous referee for pointing out to us the dedication of this earlier treatise to Amīr Shāh.

[12] Storey (1958, 1) reports that Istanbul, Yeni Cami 796 is also a copy of al-Shīrāzī's translation. We have not been able to inspect this manuscript. Several additional manuscripts are reported in Iranian libraries (Ghassemlou 1387 AH, 161).

[13] Brentjes (1998, 78) gives the date of composition as 1282 CE (or 680 AH). The statement is made without citation of any sources. The manuscript evidence seems to us to favor a later date. The colophon at the end of Istanbul, Ragip Pasa 9744 indicates that the text was completed on 12 Rajab 705 AH / 28 January 1306 CE.

the *Elements* based on the Arabic primary transmission, al-Shīrāzī used a lightly edited version of his translation of al-Ṭūsī's *Taḥrīr*.[14] As an inclusion within this larger encyclopedic work, the preamble found in the independent translation has been omitted, along with its dedication and author statement.

There are numerous manuscript copies of al-Shīrāzī's encyclopedic *Durrat al-Tāj li-ghurrat al-Dubāj*. As one finds also in the case of ibn Sīnā's *Kitāb al-Shifāʾ*, the mathematical section is sometimes omitted by the copyists. We have examined a dozen copies that include the mathematical section in preparing our edition.

III.4 Independent Treatise Including the "Proofs" of al-Shīrāzī

Al-Shīrāzī's "proofs" in Persian also exist as an independent treatise. The preamble found in the three copies we have been able to examine does not include any dedicatory statement but this introduction explicitly names the author of these demonstrations as al-Shīrāzī. These independent treatises include both the demonstrations of the six postulates ascribed to Euclid in the medieval transmission of the *Elements* as well as the summary diagram and its explanation that al-Shīrāzī added at the end of book I.[15] The independent version differs from the other Persian versions primarily in that the demonstration of the last postulate (Euclid's parallel lines postulate) has been truncated.

III.5 Relationships Among the Versions

The relationship between the Arabic version and the Persian versions is unclear. There are no obvious patterns of variants that could link the Arabic to one or more of the Persian versions. Nor is it clear whether the Arabic was prior to the Persian chronologically. Since the primary language of mathematics instruction at that time was Arabic, and because the main sources on which al-Shīrāzī depended (al-Nayrīzī and al-Ṭūsī) were both written in Arabic, it is tempting to speculate that these "demonstrations" were produced first in Arabic. It may stem from al-Shīrāzī's time at Marāgha with al-Ṭūsī, but without some additional evidence to corroborate our suspicions, we can do little to resolve this question. The earliest datable copies of the Arabic version come from the 9th / 15th century, which leaves the prehistory of the Arabic version clouded in obscurity.

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[14] Pourjavady and Schmidtke (2004, 313), citing Sayyid Muḥammad Mishkāt (1317–1320 AH/ 1938–1941 CE), 69–71), who edited *Durrat al-Tāj* (with the exception of the mathematical section), assert incorrectly that al-Shīrāzī's geometrical section was based on his Persian translation of the *Taḥrīr* of Muḥya al-Milla wa-l-Dın Yaʿqūb b. Muḥammad al-Maghribī al-Andalusī al-Qurṭubī, who died between 680 AH / AD 1281 and 690 AH / 1291 CE.

[15] The appendix containing the summary diagram has been translated into English by Doostgharin (2012) and by De Young (2013). This appendix was also included in the lithograph edition of the commentary of Muḥammad Barakāt on book I of al-Ṭūsī's *Taḥrīr* (De Young 2012b).

Because the Persian translation of the *Taḥrīr* is dedicated to a specific individual, whose reign can be dated fairly accurately, we can be quite sure that this treatise dates from the time of al-Shīrāzī's residence in Anatolia serving as a judge in Sivas. This translation of al-Ṭūsī's classic text was probably an attempt by al-Shīrāzī to gain patronage from the government. Al-Shīrāzī may have created this discussion of the postulates in Persian prior to completing the translation of al-Ṭūsī's treatise into Persian. But we think it more probable that he took an Arabic version, translated it into Persian and added it to the translation of the *Taḥrīr*. Either scenario would have involved not just translating but also editing the text since there are passages in Arabic that are not present in Persian, as well as passages in the Persian that are not present in the Arabic.

This Persian translation of the *Taḥrīr* was later incorporated into al-Shīrāzī's encyclopedic *Durrat al-Tāj*. Again in this case we know from the dedication quite precisely when the treatise was composed. When reusing his earlier work, al-Shīrāzī introduced some modest editing, such as changing some specific vocabulary to more distinctly Persian terms, changing some verb tenses from present to past, and similar editorial interventions. He also added a few explanatory statements to his demonstrations of the postulates. These revisions can be located through the apparatus notes. We have also noted the more mathematical interventions in notes to the translation.

The independent Persian version exists only in a few copies, all of which are quite late. Textually, one can see in the apparatus, as well as from the notes to the translation, this independent version shares several stylistic features of the Persian translation of the *Taḥrīr*. This suggests to us that the independent version may have been created after the lifetime of al-Shīrāzī based on his translation of the *Taḥrīr*. Its most distinctive features are (a) the reformulation of the statement of each postulate and (b) the omission of the alternative demonstration for the last postulate (Euclid's parallel lines postulate).

IV Authorship

The author responsible for these "demonstrations," Quṭb al-Dīn Abū al-Thanāʾ Maḥmūd ibn Masʿūd ibn Muṣliḥ al-Shīrāzī, was born into an illlustrious family of Shīrāz in Ṣafar 634 AH / October–November 1236 CE.[16] His father, Ḍiyāʾ al-Dīn Maḥmūd ibn Muṣliḥ al-Kāzarūnī, was a physician who headed the ophthalmology section of the Muẓaffarī Hospital.[17] He became his father's apprentice at the hospital and, when his father died, he succeeded him as ophthalmologist, although he

[16] Al-Shīrāzī typically states his name as Maḥmūd ibn Masʿūd.

[17] Quṭb al-Dīn included an autobiographical sketch at the beginning of his commentary on the *General Principles* of the *Canon of Medicine* (*Sharḥ Kulliyyāt al-Qānūn*) by Ibn Sīnā, sometimes

was only fourteen years old. He spent the next decade in an intensive study of Ibn Sīnā's *General Principles*.

His studies left him unsatisfied, and in 658 AH / 1260 CE he gave up his position in the hospital and left Shīrāz in search of further education. He traveled first to Marāgha, where Naṣīr al-Dīn al-Ṭūsī, under the patronage of the Mongol Īl Khān, had begun the construction of an observatory and teaching institution. Although initially disappointed that al-Ṭūsī's lack of practical medical experience limited his ability to teach medicine, al-Shīrāzī was quickly drawn into the study of mathematical astronomy (*hay'a*) and philosophy.

Between 665 and 667 AH (1267 and 1269 CE) he accompanied al-Ṭūsī on a book-buying expedition to Khurāsān and Quhistān. Some time thereafter, he left Marāgha and went to Baghdād, being eager to learn more of the religious sciences.[18] By 673 AH / 1274 CE he had journeyed to Anatolya. He visited Konya, where he studied ḥadīth and related religious sciences. He was appointed judge in Malatya and Sivas by the Ṣāḥib Parvāna, Muʿayn al-Dīn, who had been appointed by the Mongol court to administer their Anatolian territories, some time before 676 AH (when the Parvāna was administratively killed, ostensibly for plotting with the Mamlūk ruler, Baybars).[19] Most of the administrative work of his position was done by his assistants, allowng time for writing and teaching. It was during this period that al-Shīrāzī completed his *Nihāyat al-idrāk fī dirāyat al-aflāk* and his *Al-Tuḥfat al-shāhiyya fī al-hay'a*, in which he expounded his views on mathematical astronomy and cosmography.

Al-Shīrāzī remained in Sivas for several years, although he seems to have made several visits to the court in Tabriz. In 681 AH / 1281 CE al-Shīrāzī was drafted by the Īl Khān to head a delegation to the Mamlūk court in Cairo to attempt to make peace between the two rival powers. The political mission was a failure, but al-Shīrāzī was able to visit libraries in Egypt where he found additional commentaries on Ibn Sīnā's *General Principles*. He now felt that he had finally understood Ibn Sīnā's work and set out to write his own commentary on the text, which he completed after his return to Anatolya.

Although details are scarce, it appears that al-Shīrāzī left Sivas and took up residence in Tabriz, the capitol of the Īl Khān rulers. Although he seems to have devoted himself primarily to research and writing, he continued to have contact

known as *al-Tuḥfat al-Saʿdiyya*. We have generally followed the summary included in Walbridge (1992).

[18] Some sources suggest he may have had a falling out with al-Ṭūsī. Whether or not this may have contributed to his decision to leave Marāgha, he always referred to al-Ṭūsī in terms of highest respect in his own writings.

[19] Niazi (2013, 30) asserts that al-Shīrāzī's Persian translation of al-Ṭūsī's *Taḥrīr* of the *Elements* is dedicated to this Ṣāḥib Parvāna, although the internal evidence does not seem to support the claim.

with several Īl Khān potentates. In 697 AH / 1298 CE, Rashīd al-Dīn was named vizier. He had never liked al-Shīrāzī and began to make his life uncomfortable (for example, by having his state pension reduced by more than fifty percent). It seems that about 705 AH / 1306 CE al-Shīrāzī had decided to go to the still-independent Isḥāqid principality ruled by Amīr Shāh of the Dubāj family in search of a new patron. It was to this Amīr that he dedicated his encyclopedic *Durrat al-Tāj li-ghurrat al-Dubāj.*

Al-Shīrāzī's Persian encyclopedia of Peripatetic philosophy was modeled on the *Kiāb al-Shifāʾ* of Ibn Sīnā. Like Ibn Sīnā, al-Shīrāzī divides his treatise into four sections, each dealing with one of the main Aristotelian division of the philosophical sciences: logic, physics, mathematics, metaphysics. Although al-Shīrāzī's treatise is in Persian, it is not a Persian translation of Ibn Sīnā's work. Whereas Ibn Sīnā had summarized original sources, al-Shīrāzī incorporated already existing Persian treatises. So in his section on Euclidean geometry, he adapted his own Persian translation of al-Ṭūsī's *Taḥrīr* of the *Elements* (including the demonstrations of the postulates), with only slight editorial changes.

His attempt to win a new patron was largely unsuccessful since the Īl Khāns annexed Dubāj's principality less than a year later, leaving al-Shīrāzī little option but to return to Tabriz. He died in Ramaḍān 710 AH / February 1311 CE.

V Al-Shīrāzī's "Demonstrations" in Context

In this section we situate the "demonstrations" of al-Shīrāzī within the broader landscape of attempts at demonstrating Euclid's postulates. This history takes its starting point in demonstrations introduced by several Greek commentators, most notably Proclus. Al-Shīrāzī viewed his own work as a further link in this chain of demonstrations, unifying and perfecting earlier efforts. These Greek and early Arabic attempts at demonstrating Euclid's postulates were also influential in some of the early Latin translations of Euclid.

V.1 Postulates 1–3

The three first postulates are:

- To connect between any two points with a straight line.
- To extend any limited (finite) straight line rectilinearly.
- About any point and with any radius to draw a circle.

These first three postulates are not given a formal geometrical demonstration by al-Shīrāzī. Rather, they are explained or justified in a verbal quasi-philosophical argument that relies on moving a point in the imagination. This justification through use of imagined motion of geometrical entities was already introduced as early as the Greek commentary of Proclus on book I of the *Elements*, who seems to have suggested that one could imagine moving points to generate a straight line and rotating lines about a fixed endpoint in order to generate a circle (Morrow 1970, 145–147).[20]

This technique of imagining motion of points and lines appears early in the Arabic transmission in the commentary of al-Nayrīzī, who ascribes it to Simplicius (died after 533 CE).[21] Ibn al-Haytham, another early commentator, also used motion of a point in the imagination to explicate these postulates (Sude 1974, 84–91).[22] The theme of motion in the imagination continued into the later period of the Arabic transmission in the *Iṣlāḥ* of the *Elements* by Athīr al-Dīn al-Abharī (d. 663 AH / 1265 CE).[23] The notion of the motion of a point (although it was not specified that it be in imagination) to generate lines was also used in the *Taḥrīr* of the *Elements* by an anonymous author usually denominated as Pseudo-Ṭūsī (Pseudo-Ṭūsī 1594, 6–7).[24] Although these demonstrations were not included in the *Ashkāl al-Ta'sīs* of al-Samarqandī, they were paraphrased from the formulation of the Pseudo-Ṭūsī by Mūsā al-Bursawī, who is more usually known by his professional title, Qāḍīzāde

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[20] Heath (1926, I, 195) suggests that this appeal to imagination may be a response to the criticism made by Aristotle (*Anal. post.* I. 10, 76 b 41) that geometers cannot draw a perfectly straight line using the imperfect material instruments of the draughtsman. Hence a true straight line can be constructed in imagination only and not in actuality. A similar sentiment is expressed by Simplicius, as quoted by the early Arabic commentator, al-Nayrīzī (Besthorn and Heiberg 1897, 18; Arnzen 2002, 44; Curtze 1899, 31; Tummers 1994, 28; Lo Bello 2003b, 92).

[21] The Arabic text has been edited by Arnzen (2002) based on the two existing Arabic manuscripts and the Latin translation attributed to Annaritius (Tummers 1994). Doostgharin (1391 SH) has investigated al-Shīrāzī's demonstrations in relation to these early Arabic demonstrations.

[22] Not all early Arabic commentators relied on motion in imagination. The brief commentary ascribed to Thābit ibn Qurra, for example, seems to describe the production of a straight line or a circle as an actual construction, rather than an imagined motion (see Tehran, Malik MS 3586, 6–7).

[23] A number of extant manuscript copies are reported. We have used the only copy available: Dublin, Chester Beatty Library 3424. See Sezgin (1975, 111) and Rosenfeld and Ihsanoğlu (2003, 209–210) for additional biobibliographical information. Tehran, Sipahsālār 540, despite the note on its title page, is not a copy of al-Abharī's treatise but rather a handwritten copy from the Pseudo-Ṭūsī *Taḥrīr* printed in Rome in 1594 (De Young 2012a, 281–283).

[24] The remarkable features of this first printed Arabic redaction of Euclidean geometry have been outlined by Cassinet (1993) and have been further explored by De Young (2012a). The author of this *Taḥrīr* also includes a number of other postulates not traditionally found in the Arabic transmission, leading up to a porism that it is not possible to continue a straight line rectilinearly by two straight lines (Pseudo-Ṭūsī 1594, 7).

al-Rūmī, an important founder of Ottoman scientific studies, in his commentary on the *Ashkāl al-Ta'sīs* (Qāḍīzāde 1856, 9; Souissi 1984, 49).[25]

Proclus seems to have regarded the rectilinear extension of a line (Euclid's postulate 2) as simply an onward motion of its extremity along the shortest path (Morrow 1970, 145).[26] Simplicius (as reported by Arabic commentator al-Nayrīzī), on the other hand imagines the extension as an attaching of two lines together so that an endpoint of one is superimposed on an endpoint of the other. In this case, the lines can either be attached to one another rectilinearly or not rectilinearly. But there can only be one line that is attached to another rectilinearly (such that they form a single line). To demonstrate this, we must make use of the third postulate. Given line AB, let us assume that two different lines (BG and BD) can be attached to it in order to extend it rectilinearly as lines ABG and ABD. With point B as a center and distance AB as a radius, we construct circle AGD. Then if ABG and ABD are each straight lines, they would each be a diameter of the circle. Then arc AGD, the greater, would be equal to arc AG, the smaller because each diameter bisects the circle.[27] Since our assumption leads to a contradiction, there is only one line that can extend a given line rectilinearly (Besthorn and Heiberg 1897, 17–20; Arnzen 2002, 44–45; Lo Bello 2003b, 92–94; Curtze 1899, 31–32).[28] This is also the approach of Pseudo-Ṭūsī (1594, 6).[29]

Ibn al-Haytham, perhaps not wishing to assume validity of the third postulate without first demonstrating it, takes a somewhat different approach. He asserts, in

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[25] The commentary continued to be copied for centuries and appears to have had a place in the curriculum of the Ottoman madrasa system (Ihsanoğlu 2004, 14–15). The commentary was printed in Istanbul in 1858 (De Young 2012c, 13–16).

[26] This is also the approach taken in the *Taḥrīr* printed in Rome in 1594 (Pseudo-Ṭūaī 1594, 6). This demonstration is followed by a porism that a straight line cannot be continued rectilinearly by more than one straight line. Al-Shīrāzī places this porism, with an identical demonstration, following postulate 5. The demonstration, in both cases, is that used by Simplicius to prove the second postulate.

[27] This argument rests on a visual inspection of the diagram in order to know which arc is bigger and which is smaller. The diagram in Leiden University Library, Or. 399.1, folio 3a is the same as that used by al-Shīrāzī in his porism to postulate 5 except that it interchanges points G and D and point E is missing – see the diagram in section 5–3 in the translation, below.

[28] The first two postulates are combined into one in the Latin commentary attributed to Albertus Magnus (Tummers 1994, II, 19–20; Lo Bello 2003a, 24), and the proof offered is identical to that of Simplicius, as rendered into Latin by Annaritius (Curtze 1899, 31–32) except that Albertus relies on the physical construction of a circle—described in the Latin as done using a compass, rather than relying entirely on imagination as did his predecessors.

[29] Qāḍīzāde presents first the formulation of Pseudo-Ṭūsī, but then quotes the formulation of al-Abharī as an alternative (Qāḍīzāde 1858, 10; Souissi 1984, 49).

agreement with Simplicius, that a straight line extending an existing straight line must form a single straight line with it. He then explains that if they do not form a single straight line, but rather produce an angle, we need only to rotate, in our imagination, the attached line about the point of attachment until the angle between them disappears (which occurs at 180°).[30] At that point the required rectilinearity will be achieved (Sude 1974, 88–90). Both al-Abharī and al-Shīrāzī use Ibn al-Haytham's approach in their demonstrations. Moreover, both al-Abharī and al-Shīrāzī conclude with a brief porism: "In this way, it is possible that a line may be extended indefinitely," although al-Abharī adds the condition "in imagination (*bi-t-tawahhum*)," which is not mentioned in either the Arabic or the Persian versions of al-Shīrāzī. The primary argument in these "demonstrations" again depends on the concept of motion—in this case, motion of a line.

The "demonstration" of the third postulate, according to Proclus (Morrow 1970, 145) also depends on motion of geometric entities—in this case, motion of the endpoint of a line segment that is rotated around a fixed endpoint in order to produce the circumference of a circle.[31] The argument based on rotation of the line about a fixed endpoint is attributed to Simplicius by al-Nayrīzī (Besthorn and Heiberg 1897, 20; Arnzen 2002, 46; Curtze 1899, 32; Tummers 2004, 29–30; Lo Bello 2003b, 94). The same argument is developed by Ibn al-Haytham in a somewhat more detailed discussion (Sude 1974, 90–91). Al-Abharī (Chester Beatty Lib., MS arab. 3424, f. 2b) and al-Shīrāzī also use the same argument in their demonstrations, specifying that the line segment is moved in imagination.[32]

But the use of motion as a technique for demonstration was also criticized by several later Arabic commentators, most notably al-Ṭūsī (Sabra 1972, 202) and

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[30] If we rotate the line in the opposite direction (to make the angle 0°) the two lines will be superimposed, not extended.

[31] The Pseudo-Ṭūsī *Taḥrīr* does not include this third postulate among its postulates. Rather, the author attaches to the definition of the circle a porism stating: "We may draw about any point and with any radius a circle." His demonstration is essentially the same as that for al-Shīrāzī's lemma to his demonstration of postulate 5—the rotation of a half-diameter of the circle about the diameter (Pseudo-Ṭūsī 1594, 4)—see section 5–1 in the translation, below. Qāḍīzāde, who had been following the formulation of Pseudo-Ṭūsī, does so also in his discussion of the third postulate. But even though he follows the verbal formulation of Pseudo-Ṭūsī, he places his demonstration in the section dealing with the postulates. Moreover, he does not quote the demonstration of Pseudo-Ṭūsī but rather the demonstration of al-Abharī (Qāḍīzāde 1858, 10; Souissi 1984, 50).

[32] In the Latin transmission, Albertus Magnus uses a similar argument, but makes reference specifically to use of a compass one of whose legs is fixed at a point (the center) and with the distance equal to any desired line (Tummers 1984, II, 20; Lo Bello 2003a, 25). His use of constructivist language seems to move away from the idea of imagined motions that was implicit in the Greek commentators and explicit in the Arabic (and Persian) transmission.

al-Khayyām (Vitrac 2005). This critique of the use of motion may be one reason why there are no demonstrations attached to the postulates in the *Taḥrīr* of the *Elements* composed by al-Ṭūsī. And so, from the very beginning of his treatise, al-Shīrāzī has adopted a position that his teacher did not accept as valid or appropriate for demonstrating geometrical ideas.

Although al-Shīrāzī's "demonstrations" often appear simliar to those of al-Abharī, there are some important differences in structure and diction and technical vocabulary that clearly distinguish the two. Structurally, al-Abharī places each demonstration immediately following the postulate that it demonstrates, while al-Shīrāzī has placed all the demonstrations in a single block following the postulates. Like al-Abharī, al-Shīrāzī, in both the Arabic and Persian versions, demonstrates the first postulate through imagination (*takhayyul*) of a point superimposed (*munṭabiq*) upon another point, then moved in imagination until it is superimposed on the other point, creating a straight line.[33] But in the "demonstrations" both the Arabic and Persian versions use the expression "we assume" (*nafriḍu*) instead of Abharī's "we imagine" (*natawahhamu*) when describing this point as moved in order to draw a straight line or a circle.

V.2 Postulate 4

To demonstrate the fourth postulate ("All right angles are equal to one another") al-Shīrāzī uses a proof by contradiction. It relies on moving the lines forming sides of the given angle until points and lines defining the angle are superimposed upon the known right angle and showing that if we assume the two angles are not completely superimposed, a contradiction results. This contradition argument is the same approach that had been used since the time of Proclus in his commentary on book I of the *Elements* (Morrow 1970, 147–148).[34] The demonstration appears to have been known in the Arabic transmission quite early since it is quoted by al-Nayrīzī (Besthorn and Heiberg 1897, 20–23; Arnzen 2002, 46–48; Curtze 1899, 32–34; Tummers 2004, 30–31; Lo Bello 2003b, 94–95) with an ascription to Simplicius.[35] This argument was also used by Ibn al-Haytham, although his "demonstration," unlike that of al-Nayrīzī, was purely verbal and did not include a geometrical diagram (Sude

[33] The role of imagination in the process of intellection also plays an important part in the philosophical discussion of epistemology (including knowledge of mathematical entities) found in the metaphysics section of the *Kitāb al-shifā'* of Ibn Sīnā (Ardeshir 2008, 53–58).

[34] As Heath (1926, I, 200) has pointed out, the demonstration proposed by Proclus is not convincing because it assumes without justification that lines CB and GB can only be extended in one direction and that line BK always falls outside angle ABH. (See Figure 1.)

[35] The same demonstration appears also in the Latin commentary ascribed to Albertus (Tummers 1984, II, 20–21; Lo Bello 2003a, 25–26).

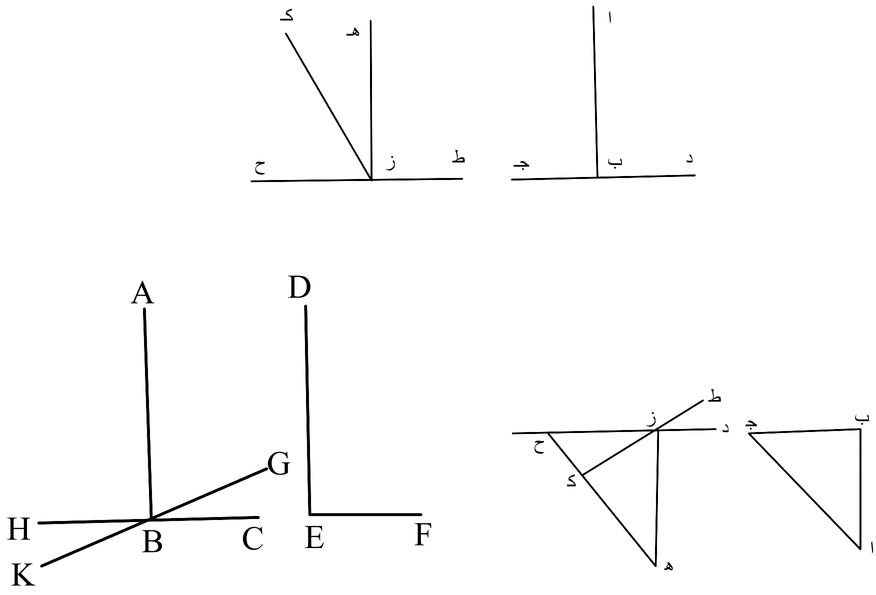

Figure 1: Diagrams for Postulate 4. Top, al-Shīrāzī, edited from Munich, Bayerische StaatsBibliothek Cod. arab. 2697, f. 184a. Below left, Commentary of Proclus, edited from Morrow (1970, 148); below right, Commentary of al-Nayrīzī, edited from Leiden 399.1, f. 3b.

1974, 91–93). Al-Abharī, in his *Iṣlāḥ* has used an identical contradition argument based on motion of the lines bounding the right angle. A mathematical demonstration comparable to that of Proclus is also included in the Pseudo-Ṭūsī *Taḥrīr* (1594, 7–8).[36]

The diagrams accompanying this demonstration exhibit differences in architecture that are unexpected since the verbal content of the demonstration is always the same. The diagram used by Proclus (Morrow 1970, 148) is geometrically identical to that used in the Latin translation of the commentary ascribed to Annaritius (Curtze 1899, 33). This is somewhat surprising because the diagram of this demonstration in the Arabic commentary ascribed to al-Nayrīzī is drawn in the form of right triangles rather than intersecting lines (Besthorn and Heiberg 1897, 23; Lo Bello 2003b, 95) (See Figure 1).[37] The diagram for the demonstration in the *Taḥrīr* of the Pseudo-Ṭūsī (1594, 7) is a variant of the diagram of Proclus and its letter labels are assigned differently, suggesting that it has probably been modified from

[36] Qāḍīzade has used a close paraphrase of the demonsrtation given by al-Abharī (Qāḍīzāde 1858, 10–11; Souissi 1984, 50–51).

[37] Since editors of modern printed editions of early mathematical works have been known to silently redraw diagrams as they thought these diagrams should appear (see Saito 2012; Saito and Sidoli 2012), we may wonder whether this difference in diagram architecture is the result of modern editing. But in this case we find the diagram drawn in the same form in Leiden 399.1, f. 3b. (See Figure 1.) Unfortunately, most of the diagrams, including this one, are missing from Qum, Kitābḫāna-i ʿUmūmī 6256, the only other known manuscript of al-Nayrīzī's commentary, according to the report of Arnzen (2002, XVII), making comparisons impossible.

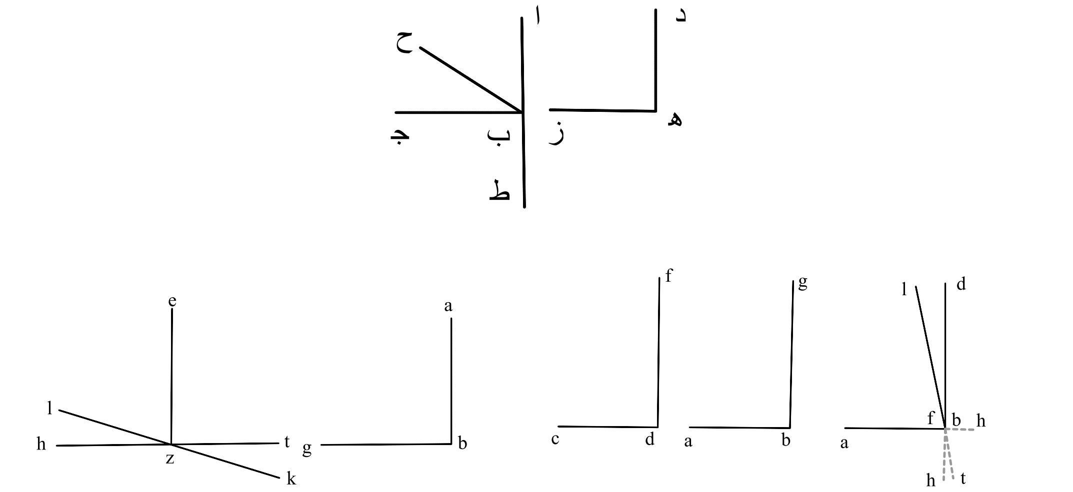

Figure 2: Diagrams for Postulate 4. Above, Pseudo-Ṭūsī *Taḥrīr*, edited from Pseudo-Ṭūsī (1594, 7). Below left, Latin commentary attributed to Annaritius, edited from Curtze (1899, 33); below right, Albertus Latin commentary, edited from Tummers (1974, II, 20).

the diagram of Proclus. The diagram in the Latin commentary of Albertus (Tummers 1984, II, 20; Lo Bello 2003a, 26) is also a variant of the diagram of Proclus but differs from the diagram found in the Latin translation ascribed to Annaritius in both form and labeling (see Figure 2).

Proclus added a discussion of the converse of this postulate—that an angle equal to a right angle will also be a right angle, which is only possible when the angles are both rectilinear (Morrow 1970, 148–150). Here Proclus is reporting an argument that he attributes to Pappus, who had showed that if one right angle is rectilinear and the other is lunular, for example, the two right angles will not be equal to one another in the sense that they will not be capable of being superimposed one upon the other. This converse was also known early in the Arabic transmission, for it is present in the commentary on the *Elements* by al-Nayrīzī (Besthorn and Heiberg 1897, I, 22–25; Arnzen 2002, 48–49; Curtze 1899, 71; Tummers 2004, 31; Lo Bello 2003b, 48–49).[38] This converse is not discussed by Ibn al-Haytham or al-Abharī or al-Shīrāzī and it is also omitted from the commentary of Qāḍīzāde on al-Samarqandī's *Ashkāl al-Taʾsīs*.[39]

Al-Shīrāzī's Arabic and Persian versions add that the same method (superimposition) can be used to prove two further porisms: (1) "When a straight line falls on

[38] A similar discussion is found in the Latin commentary of Albertus (Tummers 1984, II, 21; Lo Bello 2003a, 26–27). The diagram of Albertus, although displaying the same architectural structure, is a mirror image of the diagram in the Arabic of al-Nayrīzī and its Latin translation.

[39] Perhaps they omitted this discussion because lunular right angles do not play a significant role within the *Elements*, although they may be encountered from time to time in higher mathematics.

a straight line, the two angles that are produced on the two sides of the incident line are either two right angles or are equal to two right angles" and (2) "a rhomboid surface has sometimes two right angles and sometimes acute and obtuse angles."[40] The second porism is apparently the work of al-Shīrāzī since it is not found elsewhere in the Greek or Arabic transmission. Both porisms are absent from al-Abharī's *Iṣlāḥ*.

V.3 Postulate 5

Al-Shīrāzī's fifth postulate ("Two straight lines do not <together> bound a surface (area)") is also al-Ṭūsī's fifth postulate—but it is the sixth in al-Abharī's list of postulates. (Al-Abharī's fifth postulate states that two straight lines cannot continue a single straight line rectilinearly, which al-Shīrāzī had made a porism to his own fifth postulate.) It is also the sixth postulate in the commentary of al-Nayrīzī. (His fifth postulate is Euclid's parallel lines postulate.)

The demonstration of al-Shīrāzī's postulate is already present in the Greek transmission in the commentary of Proclus. But he placed this demonstration at the end of his discussion of proposition I, 4 (Morrow 1970, 186–187). The demonstration is also found in the early Arabic commentary of al-Nayrīzī with an attribution to the Greek author Simplicius.[41] Ibn al-Haytham reports that he found this principle listed as the last of the axioms presented by Euclid (Sude 1974, 78). This is not the proper place for this principle, he says, because it is not self-evident and clear, and because it is susceptible of proof. Hence he has moved it to the last place in the list of postulates, following the parallel lines postulate. His proof, as is typical, centers on the motion of line segments and relies on a contradiction argument (Sude 1970, 79). The Pseudo-Ṭūsī, on the other hand, has removed the principle and its demonstration from the traditional list of postulates and placed it following the definition of the circle (Pseudo-Ṭūsī 1594, 5).[42]

Al-Shīrāzī's Arabic and Persian versions of the demonstration begin with a lemma: "A diameter bisects the circumference of a circle."[43] Proclus, following his discussion of the definition of the diameter of the circle in his Greek commentary,

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[40] De Young (2007, 36 n. 48) has identified the first as identical to Euclid's proposition I, 13. It was also the first proposition in al-Samarqandī's widely read and frequently copied collection of extracts from the *Elements*, *Ashkāl al-Ta'sīs* (De Young 2001, 81–82).

[41] Simplicius had noted, according to the quotation of al-Nayrīzī, that the postulate was not found in the "ancient texts" (Besthorn and Heiberg 1897, I, 14; Arnzen 2002, 49; Curtze 1899, 35; Lo Bello 2003b, 97).

[42] Al-Samarqandī, in his *Ashkāl al-Ta'sīs* had also placed this postulate last in his list of postulates (Tehran, Majlis Shūrā, MS 3380, page 88). Qāḍīzāde, in his commentary on *Ashkāl al-Ta'sīs*, added the demonstration from Pseudo-Ṭūsī (Qāḍīzāde 1858, 11; Souissi 1984, 51–52).

[43] To al-Shīrāzī's Arabic (but not the Persian) versions of this lemma there is added a second premise that: "The two angles produced by the intersection of the circumference and the diameter

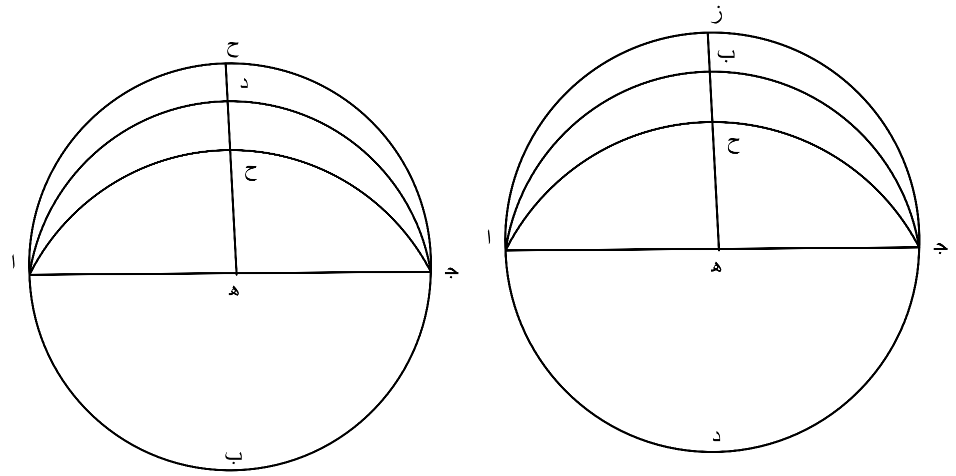


Figure 3: Diagrams for postulate 5 preliminary lemma. Left, al-Abharī *Iṣlāḥ*, edited from Dublin, Chester Beatty Library, Arabic MS 3424, f. 3a; right, edited from Pseudo-Ṭūsī, *Taḥrīr* (1594, 6). The diagram in the demonstration of al-Shīrāzī has the same form as that of al-Abharī.

credits Thales with being the first to prove that a diameter bisects the circumference of a circle (Morrow 1970, 124–125). The verbal "demonstration" of this premise that he gives, using a superposition argument, is presumably that of Thales. Al-Nayrīzī, at the beginning of the Arabic transmission, also included this principle in his statement of the definition of a circle. He then gave a geometrical argument, which he attributed to Simplicius, following his definition of the circle and its diameter (Arnzen 2002, 27–29; Curtze 1899, 20–21; Lo Bello 2009, 12–13).[44] Ibn al-Haytham describes the content of this principle, although in purely verbal form reminiscent of the presentation of Proclus, in his discussion of Euclid's definition of the circle and its diameter (Sude 1974, 44–46).[45] Al-Abharī appears to have been the first to move this geometrical proof from the definition of the circle and place it as an

of a circle are equal." This assertion concerning the equality of the angles can also be found in the Pseudo-Ṭūsī *Taḥrīr* (Pseudo-Ṭūsī 1594, 5) as a porism to his demonstration.

[44] The same geometrical demonstration appears also in the Latin commentary attributed to Albertus Magnus, where is it also placed in the discussion of the definition of the circle and its diagmeter (Tummer 1984, I, 18–20; Lo Bello 2003a, 17–19). The Latin version of Annaritius used two diagrams to represent the different cases, while Albertus used three diagrams to represent the same cases. These diagrams have the same labeling as the single composite diagram used by the Pseudo-Ṭūsī (1594, 6). (See Figure 3.)

[45] A geometrical demonstration is also found in the *Taḥrīr* of the Pseudo-Ṭūsī (1594, 5), where it is placed immediately following Euclid's definition of the circle and its center. This demonstration is summarized by Qāḍīzāde as a porism to al-Samarqandī's demonstration of his fifth postulate (Qāḍīzāde 1858, 11–12; Souissi 1984, 52).

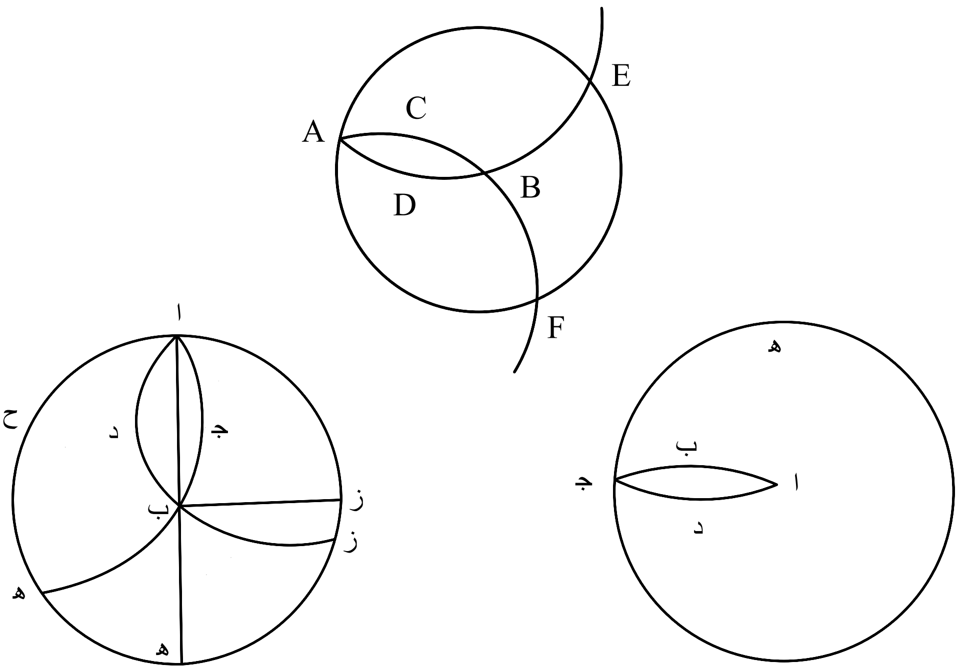


Figure 4: Diagram for postulate 5. Above, from Proclus's commentary, edited from Morrow (1970, 187). Below left, from al-Nayrīzī's Arabic commentary, edited from Leiden MS 399.1, f. 4a; below right, edited from the Pseudo-Ṭūsī *Taḥrīr* (1594, 6).

introductory lemma to the demonstration of the fifth postulate, just as it appears in al-Shīrāzī's demonstration.

Al-Shīrāzī's "demonstration" relies on an argument by contradiction, imagining the rotation of a half-circumference about its fixed diameter so that it comes to be superimposed on the opposite half-circumference. This can only happen if the arc connecting the two endpoints of the diameter is superimposed on the original circle. This "demonstration" of this postulate is summarized by Proclus in his discussion of Euclid's proposition I, 4, where he used only one diagraph (Figure 4) (Morrow 1970, 187). Similarly, al-Abharī used only one diagram in his *Iṣlāḥ*, which follows essentially the argument summarized by Proclus (Figure 4).[46] The demonstration is worked out in more detail by al-Shīrāzī, who needed four diagrams to explain the possible cases. Since this is the only fully worked out version of the demonstration, it is probable that it is the work of al-Shīrāzī. The detailed explication may have met a perceived need to provide pedagogical assistance to beginning readers of Euclid.

Following his "demonstration," al-Shīrāzī added a porism, namely that "one straight line cannot be continued rectilinearly by two straight lines not in line one with another." This porism had already been "demonstrated" by Proclus, but his

[46] The same argument is used in the Pseudo-Ṭūsī *Taḥrīr* (Figure 4), where the demonstration is placed immediately following the definition of the sector of a circle (Pseudo-Ṭūsī 1594, 5–6).

"demonstration" occured in conjunction with his demonstration of Euclid's proposition I, 4 (Morrow 1970, 169). As mentioned previously, a similar "proof" can be found in the commentary of al-Nayrīzī when demonstrating Euclid's second postulate (Besthorn and Heiberg 1897, 20; Arnzen 2002, 45; Curtze 1899, 31–32; Tummers 2004, 29; Lo Bello 2003b, 46–47).[47] The "demonstration" is also found in Ibn al-Haytham's discussion of Euclid's second postulate (Sude 1974, 78–79). Similarly, Pseudo-Ṭūsī placed this demonstration as a porism to the second postulate (Pseudo-Ṭūsī 1594, 6).[48]

V.4 Postulate 6

The sixth and last of the postulates discussed by al-Shīrāzī corresponds to Euclid's parallel lines postulate. The Arabic version is introduced with the claim that although the *muṣannif* had demonstrated the postulate, his demonstration had relied upon "many propositions" from the *Elements*. It was only al-Shīrāzī who has been able to put forward a demonstration that did not appeal to later propositions.[49]

This demonstration, al-Shīrāzī says, rests on the fundamental characteristic of parallel lines that they do not meet or intersect—part of one line cannot fall on one side of a line parallel to it and part on the other side. The Persian versions add an additional characteristic not mentioned in the Arabic version—that the distances between two given parallel lines can never differ.[50]

The demonstration begins with a lemma: when a straight line falls between two straight lines and this intermediate straight line is parallel to each one of the other lines, then the first two straight lines are parallel to one another.[51] This lemma is demonstrated using a contradiction argument, since the assumption of the al-

[47] In the Latin commentary of Albertus Magnus the demonstration of this principle is placed in the discussion of his first postulate (Tummers 1984, 19–20; Lo Bello 2003a, 24), which combined Euclid's postulates 1 and 2.

[48] Al-Samarqandī had placed this principle as his fifth (and last) postulate in his *Ashkāl al-Ta'sīs* (Tehran, Majlis Shūrā Library, MS 3380, p. 89). Qāḍīzāde, in his commentary on the *Ashkāl al-Ta'sīs*, has added a demonstration similar to that used by Pseudo-Ṭūsī to demonstrate his porism to his first postulate (Qāḍīzāde 1858, 12; Souissi 1984, 52–53).

[49] Al-Abharī did not include parallel lines in his collection of postulates because he believed it could be demonstrated. His demonstration follows the thirty-eighth proposition of book I. Readers interested in his demonstration may consult Jaouiche (1984, 116–118 and 247–249).

[50] The idea of equidistance between parallel lines was used already in the Arabic transmission by al-Nayrīzī in his "demonstration" of the parallel lines postulate (Hogendijk 2006).

[51] This principle is a specific case of the more general principle proved by Euclid in *Elements* I, 30 because it specifies that the third line lies between the original two lines. As pointed out by Heath (1926, I, 314-315) in his mathematical notes, De Morgan had recognized that this proposition as the logical equivalent of Playfair's Axiom. This axiom has taken a variety of forms over the past two

ternative hypothesis would be in violation of the fundamental characteristic of non-intersection of parallel lines. We can conclude from this, says al-Shīrāzī, that there is no point located anywhere between two intersecting straight lines in a plane surface that is not itself in that same plane surface.

The demonstration itself appears to be essentially the same as that attributed to Ptolemy by Proclus (Morrow 1970, 285–288; Heath 1926, I, 204–206; Vitrac 1990, I, 300–310). Al-Shīrāzī's version, however, is worked out in considerably more detail than that provided in the summary given by Proclus. Since today we know of Ptolemy's demonstration primarily from the report of Proclus, it would appear either that the commentary of Proclus may have been available to al-Shīrāzī, perhaps through an unknown translation or perhaps through extracts or paraphrases, or that the work attributed to Ptolemy may have survived into the early medieval period and may have been available to al-Shīrāzī.

VI Edition of the Arabic Text

The Arabic text has been edited using the four known copies of the treatise. We describe their bibliographic features here. Each of these copies is part of a collection of mathematical texts copied by a single copyist. These contents are analyzed in more detail in the Appendix.

ت Tunis, Bibliothèque nationale, MS 16167 (formerly known as Aḥmadiyya 5482). The entire codex, comprising 90 folios (13x21.5 cm, 23 lines each, written with *nastaʿlīq* script), has been copied by Darwīsh Aḥmad al-Karīmī in 869 AH / 1464 CE.[52] Apart from two corrections, the only marginalia is a gloss in the hand of the copyist discussing whether the parallel lines postulate should be placed among the propositions rather than as a postulate. The demonstrations of the Euclidean postulates are found on folios 71b–73a. The diagrams are drawn using red ink with black letter labels.

centuries, some of them quite different verbally from anything John Playfair wrote in his popular *Elements of Geometry* (Ackerberg-Hastings 2013).

[52] In his brief description of this codex, Rashed (2002, 736) stated that the manuscript was been copied before 971 AH (1563 CE). The text of the colophon: "Bi khayri dawāmihā" that, when read as Arabic alpha-numeric digits, gives: Ba-Kha-Ya-Ra-Dal-Waw-Alif-Mim-Ha-Alif (2+600+10+200+4+6+1+40+5+1), or 869 AH.

ف Istanbul, Millet (Il Halk) Kütübhanesı, Feyzullah 1359. The codex is complete in 256 folios. The volume appears to have been intended as a presentation copy (it is dedicated to Sultan Mehmet II), with generous use of gold ink / gold leaf on each page. There are no marginalia, only catch words to ease transition to the next folio. A colophon following the second treatise gives the date of copying as 869 AH, corresponding to 1464 CE. The copyist is not named. The demonstrations of the Euclidean postulates are found on folios 237b–239b. Diagrams are drawn in red ink with black letter labels.

م Munich, Bayerische StaatsBibliothek, cod. arab. 2697. The manuscript consists of 214 folios, written in a neat *nastaʿlīq* hand. The codex contains extensive marginal glosses in the hand of the copyist. Beside each of the first three postulates the copyist has placed in the margin a slightly edited version of the postulate as found in al-Samarqandī's *Ashkāl al-Ta'sīs*. Beside the fifth postulate the copyist has placed a Persian quotation (from section 5-2 in the Persian edition, below). He notes its source as the "commentary of Shīrāzī."[53] An internal colophon (folio 194a) gives the date of copying as 1142 AH, corresponding to 1729 CE. The name of the copyist is not mentioned. The demonstrations of the Euclidean postulates are found on folios 183b–189b. The diagrams are drawn using red ink with black letter labels.

د Dublin, Chester Beatty Library, Arabic manuscript 3460. The manuscript consists of 147 folios. It appears to be a pastiche of several fragments dealing with astronomical and geometrical topics, written in different hands, none of them containing a dated colophon. The demonstrations of the Euclidean postulates, part of a group of four short treatises copied in the same hand, are found on folios 137a–138a. Its diagrams appear to be drawn using black ink with black letter labels. There is only one marginal gloss, apparently in the hand of the copyist, inverted with respect to the main text of the page.

The transcription of the text was made from Tunis, Bibliothèque nationale 16167. In most cases, variant readings from the other manuscripts are of little significance

[53] These marginalia reflect the scholarly practice of *taḥqīq*, which is borrowed from the intellectual disciplines of philosophy and kalām (Brentjes 2019, 9–11).

in understanding the text. Since all currently known manuscripts date from at least a century and a half after the text was originally written, no attempt has been made to establish an *urtext*. In almost all cases of variation, we have retained the reading in the Tunis manuscript while noting variant readings from the other manuscripts.

The diagrams, however, have been edited from Tehran, Majlis Shūrā, Sinā 226 since these diagrams are larger and often clearer than those in Tunis, Bibliothèque nationale, 16167. There are few significant differences among the diagrams of the various Arabic and Persian manuscripts consulted for these editions.

Section numbers have been added in square brackets in order to facilitate comparisons between Arabic and Persian versions. Each "proof" is given a separate number. Thus if the demonstration of proposition X has an initial lemma. the reference number for the lemma would be X – 1 and the demonstration of the postulate would be numbered X – 2, etc.

بسم الله الرحمن الرحيم.

اعلم أن الأصول الموضوعة التي نقلها المصنف رحمه الله عن الأصل وذلك قوله: ''لنا أن نصل خطًا مستقيمًا بين نقطتين'' إلى قوله: ''فإنهما يلتقيان في تلك الجهة إن أخرجا.''

وإن كانت بحيث يجزم بها كلّ سليم الفطرة صحيح العقل. لكنّه ربما لم يحلّ باطن بعضهم عن انكار ما خصوصًا في القضية الأخيرة. فلنذكر هنا تنبيهات إذا وقف المتعلم عليها زال الترود من باطنه وارتفع الانكار.

[١] أما الأوّل، فنتخيّل نقطة ثالثة منطبقة على إحدى النقطتين اللتين نريد أن نصل بينهما بخط. ونفرض تلك النقطة تحركت من النقطة التي انطبقت عليها إلى الأخرى على سمت واحد. فلا محالة تكون مسافة تلك الحركة خطًا مستقيمًا لأنّه طول بلا عرض متحاذية نقطة.

[٢] وأمّا الثاني، فلنفرض نقطة في جهة طرف الخط المفروض كيف اتفق. ونصل بينها وبينه بخط مستقيم.

فإن لم يحدث من الخطين زاوية، كان كلّ منهما على استقامة الآخر واتحدا خطًا واحدًا.

وإن حدثت حرّكنا الخط حتى يبطل الزاوية. ويتمّ المطلوب.

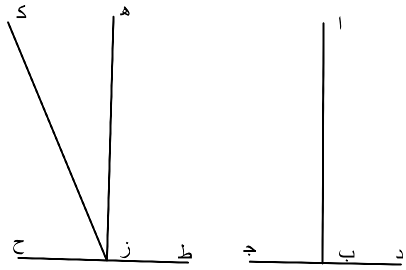
ويمكن بهذا الطريق اخراج الخط المستقيم إلى غير النهاية.

[٣] وأمّا الثالث، فلنفرض على طرف ذلك البعد الذي نريد أن نرسم ببعده دائرة نقطة. ونصل بينها وبين النقطة التي نريد أن نجعلها مركز الدائرة بخط مستقيم.

ثمّ نتوهّم الطرف المركزي ثابتًا والخط متحرّكًا حتى يصل إلى وضعه الأوّل. فإن الطرف المتحرّك منه يرسم محيط الدائرة.

---

٣ إن] وإن [م ، ف]   ٥ وإن كانت] – [م]   ٥ بحيث] حيث [ت]   ٨ [١] أما] – [م]   ٨ اللتين] – [ف]   ١٠ تكون] يكون [د]   ١٠ خطًا] – [د]   ١٢ جهة طرف] طرف جهة [م]   ١٤ واتحدا] واتحد [م]   ١٥ واحدًا] – [د]   ١٩ وبين] + ~~تلك~~ [د]   ٢٠ نتوهّم] يتوهّم [د ، م]



الشكل الأول — وهي تحقيقة من الورقة ٦و في المخطوطة
الموجودة في تهران، مكتبة مجلس شورى، رقم ٢٢٦ س.

[٤-١] وأمّا الرابع ، فبأن نفرض زوايا ا ب ج ، ا ب د ، هـ ز ط ، هـ ز ح قوائم. ونتوهّم انطباق ب على ز و د ج على ط ح. فينطبق ب ا على ز هـ (الشكل الأوّل).

وإلاّ فليكن مثل ك ز. فزاوية ك ز ح أعني ا ب ج [د - ١٣٤ ظ] مساوية لزاوية ك ز ط أعني ا ب د.

لكن زاوية هـ ز ح أعظم من زاوية ك ز ح. فيكون مساويها أعني زاوية هـ ز ط أعظم من زاوية ك ز ط المساوية لزاوية ك ز ح. هذا محال.

وبمثل هذا البيّان — أعني التطبيق المذكور — يعلّم أن الزاوية المساوية للقائمة قائمة [ف - ٢٣٨ و].

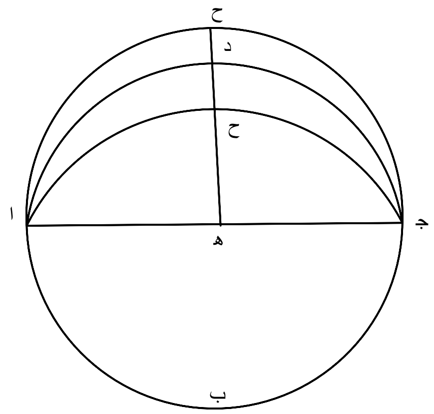
[٤-٢] وتبيّنه من وقوع هـ ز ، ك ز على خط ط ح أنّ كلّ خط مستقيم إذا قام على مثله [م - ١٨٤ و] فالزاويتان الحادثتان عن جنبتي الخط امّا قائمتان أو مساويتان لهما ضرورة أنّ سطحًا معيّنًا يصير تارة قائمتين وتارة حادة ومنفرجة.

ويتبيّن هذا المعنى بالتطبيق أيضًا.

فإنّه [ب - ٧٢ و] إن انطبق ا ب على خط هـ ز فزاويتا ز قائمتان.

---

٢ ونتوهّم] ويتوهّم [م]   ٩ وتبيّنه] تبيّنه [ت]   ٩ ط ح] + على [د ، م]   ١٢ ويتبيّن] ويبيّن [م]



الشكل الثاني — للمقدّمة القضية الخامسة وهي تحقيقة من الورقة ٦ظ في المخطوطة الموجودة في تهران، مكتبة مجلس شورى، رقم ٢٢٦ س. النقطة هـ ناقصة في المخطوطة الموجودة في اسطانبول، مكتبة فيض الله، رقم ١٣٥٩.

وإلّا فمساويتان لهما، لأنّ هـ ز قسم زاوية كـ ز ط إلى زاويتين أحدهما قائمة والأخرى إذا انضمّت إلى زاوية كـ ز ح حصلت قائمة أخرى.

[١-٥] وأمّا الخامس، فيتوقف بيانه على بيان مقدّمة وهي أنّ القطر منصّف للمحيط وأنّ الزاويتين الحادثتين من تقاطع القطر والمحيط متساويتان.

فنقول في بيان ذلك لنفرض قطر ا ج ثابتًا وقوس ا ب ج متحركًا حتى يصل إلى سطح الدائرة في جهة ا د ج.

فإمّا أن يقع خارج الدائرة أو داخلها أو بعضه خارجًا وبعضه داخلًا أو ينطبق على النصف الآخر من المحيط.

لكن الأقسام كلّها باطلة سوى القسم الأخير وهو يستلزم المطلوب. أمّا الأوّل، فلأنّا نخرج هـ ح د ويلزم تساوي هـ ح وهـ د لمساواتهما لـهـ ج (الشكل الثاني). ويعلم منه بطلان ما إذا وقع البعض داخلاً والآخر خارجًا.

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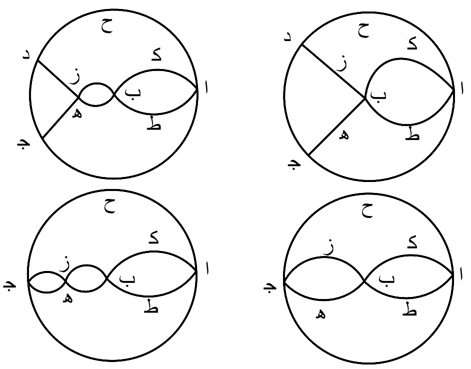
١ أحدهما] إحديهما [م] ٥ ا ب ج] ا ج [ت]

وأمّا الثاني، فظاهر إذ لزم منه تساوي قطعتي المحيط لكان التطابق وتساوي زاويتي اج ب ، اج د وزاويتي جا ب ، جا د.

[٥-٢] إذا تحصلت هذه المقدّمة، فنقول لو أحاط خطان كـا كـ ب ، ا ط ب بسطح كـا ب فلنرسم على مركز ب ببعد ا ب دائرة ا ح ج. ونخرج ا كـ ب ، ا ط ب في جهة ب (الشكل الثالث).



الشكل الثالث — القسم الأول من القضية الخامسة وهي تحقيقة من الورقة ٢٤٥و في المخطوطة الموجودة في استانبول، مكتبة سليمانية،مجموعة الحميدية، رقم ٧٩٠.

وحينئذ سواء تلاقيا قبل الوصول إلى المحيط [ف - ٢٣٨ ظ] أو لم يتلاقيا. فأمّا إن لم يتلاقيا عند المحيط كما في الصورتين الأوليتين أو تلاقيا كما في الصورتين الأخيرتين.

فإن كان الأوّل، لزم مساواة قوس ا ح د ج الأعظم لقوس ا ح د الأصغر لأنّهما نصفا محيط الدائرة [م - ١٨٤ ظ] واحدة. وذلك محال.

وإن كان الثاني، لزم أن يكون الزاوية التي يُحيط بها نصف المحيط مع أحد القطرين أعظم من التي يحيط بها النصف الآخر من المحيط مع القطر الآخر. هذا خلف.

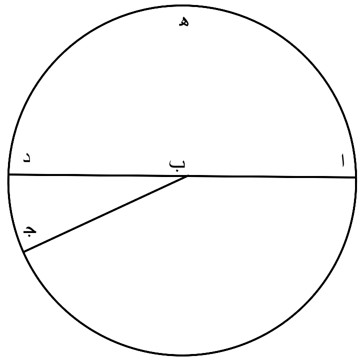
١ لكان] لمكان [د ، م] ٧ الأوليتين ... الصورتين] - [د] ٧ الأخيرتين] الأخيرين [م] ٩ الدائرة] دائرة [د] ١٠ كان] - [م] ١٠ يكون] - [م]

[٥ – ٣] ويعلم من ذلك أنّه لا يجوز أن يتصل خط مستقيم بخطين مستقيمين غير مسامتين على استقامتهما.

وإلاّ فلنفرض اب على استقامة بد، بج.

ثمّ نرسم على مركز ب ببعد أحد الخطوط إن تساوت وببعد الأقصر إن تفاوتت دائرة اهجد (الشكل الرابع).

ويلزم من ذلك أن يكون [ب - ٧٢ ظ] قوس اهدج الأعظم مساوية لقوس اهد الأصغر. هذا خلف.



الشكل الرابع — القسم الثاني من القضية الخامسة وهي تحقيقة من الورقة ٧و في المخطوطة الموجودة في تهران، مكتبة مجلس شورى، رقم ٢٢٦ س.

[٦-١-١] وأمّا السادس، وإن كان المصنّف، رحمه الله، بيّنه بطريق يتوقف على أشكال كثيرة من الكتاب. فإنّا يمكننا أن نبيّنه بطريق آخر لا نستعين فيه شيئ من الأشكال.

[٦-١-٢] فذلك أن نقول إنّه يُعلم من تعريف الخطوط المتوازية أنّه لا يقع بعض أحد المتوازيين في جانب من موازيه والبعض الآخر في جانبه الآخر.

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٢ مسامتين] متساوتين [م] ٥ اهجد] اهج [د ، م] ٦ من ذلك] (تحت السطر) [م] ٦ اهد] اهج [م] ٧ الأصغر] الأقصر [د] ٨ يتوقف] يتو [م] ١١ فذلك] وذلك [م ، ف] ١١ أنّه] – [د]

وإلاّ لكانا متلاقيين لا متوازيين.

ويلزم من ذلك أنّ كل خط مستقيم وقع بين خطين مستقيمين ووازاهما فهما متوازيان.

وإلا لكانا متلاقيين. فيلزم وقوع بعض أحد المتوازيين في جانب من موازيه وبعضه الآخر في الجانب الآخر منه. هذا خلف.

ويلزم من ذلك أنّ كلّ خط مستقيم وقع بين خطين متلاقيين فلا بدّ من ملاقاته لأحدهما إن اخرجت الثلاثة إلى غير النهاية.

وألاّ لكان موازياً لهما فيلزم توازيهما. هذا خلف.

ويلزم منه أنّ كلّ خط كائن في سطح فيه خطان متلاقيان فلا بدّ من أنّ يلاقي احدهما إن اخرجت الثلاثة إلى غير نهاية لأنّه على أيّ وضع فرض فلا يمكن خروجه [م - ١٨٥ و] عن أن يكون بينهما.

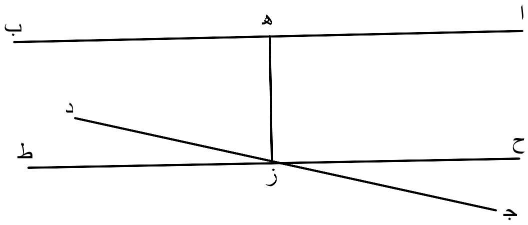
والسبب فيه أنّه لا يمكن خروج نقطة من ذلك السطح عن أن [ف - ٢٣٩ و] يكون بين خطين فيه متلاقيين مخرجين إلى غير نهاية في الجهتين.

[٢-٦] إذا تقرر ذلك فنقول كلّ خطين مستقيمين كـ اب، جد قام عليهما خط مستقيم كـ هز وصيّر الزاويتين الداخلتين في جهة واحدة كزاويتي ب هز، د زهـ أقلّ من قائمتين فإنهما إذا اخرجا في تلك الجهة تلتقيان (الشكل الخامس).

لأنّ زاوية ب هز مع زاوية هـ زد أصغر من قائمتين بالفرض ومع اهز مساوية لقائمتين كما تقدّم. فيكون اهز أعظم من هـ زد. فإذا توهّمنا تطبيق اهز على هـ زد وقع ها مثل ز طح ضرورة إن الزاوية أعظم.

---

٢ ويلزم] فيلزم [م]   ٤ من] – [م]   ٥ منه] – [د]   ٨ فيلزم توازيهما] (في الهامش) [ت]   ١٠ نهاية] النهاية [م]   ١٣ خطين] + بين [د]   ١٣ نهاية] النهاية [د ، م]   ١٥ كزاويتي] كزاويتين يتي [د]   ١٦ فإنهما] وإنّهما [م]   ١٦ تلتقيان] يلتقيان [م]   ١٩ وقع] ومع [م]



الشكل الخامس — القسم الأول من القضية السادسة وهي تحقيقة من الورقة ٧ظ في المخطوطة الموجودة في تهران، مكتبة مجلس شورى، رقم ٢٢٦ س.

[٦-٣] ونقول حينئذ أمّا أن يكون ح ط موازيًا لـ اهـ أو لا يكون (الشكل السادس).

فإن كان موازيًا و اب واقع بين ح ط ، جـ د فلا بدّ أن يلاقي احدهما عند الإخراج ولا يلاقي ح ط. فيلاقي جـ د في جهة ب ، د إذ لو لاقاه في الجهة الأخرى لزم تلاقي خط اب فيها. هذا خلف.

وإن لم يكن موازيًا فنقول زاوية اهـ ز [ب - ٧٣ و] مساوية لزاوية هـ ز ط. ويلزم منه تساوي زاويتي ب هـ ز ، هـ ز ح إذ كلّ من زاويتي هـ وزاويتي ز كقائمتين، كما مرّ.

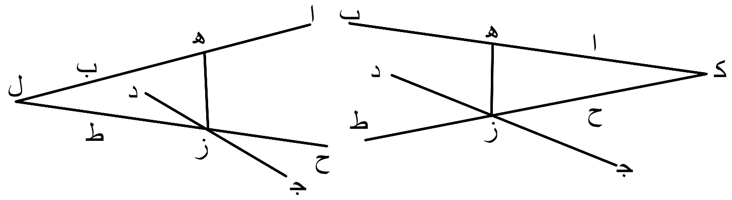
[٦-٤] وحينئذ أمّا ان تتلاقى اب ، ح ط في جهة ا ، ح أو في جهة ب ، ط.

فالأوّل محال.

وإلاّ فليتلاقيا على ك (الشكل السادس).

---

٥ خط] – [د] ٦ وإن] وأمّا [د] ٨ مرّ] + إنّه بيّانه على مستبيّن من مسائل الكتاب أحدهما بينه وإلاّ فليكن قوله ويلزم من ذلك أنّ كلّ خط مستقيم وقع بين خطين مستقيمين ووازاهما فهما متوازيان لكنّه بينهما بغير بيان الكتاب وذلك لا يوجب عدم استقامة بمسائل الكتاب نعم بيّانه اخرى بيّان المصنّف وأنّ يوفقا جميعا على مسائل الكتاب [في الهامش: ت] ١٠ فالأوّل] والأوّل [د ، م]



الشكل السادس — القسم الثاني من لاقضية السادسة وهي تحقيقة من الورقة ٨ظ
في المخطوطة الموجودة في تهران، مكتبة مجلس شورى، رقم ٢٢٦ س.

ونتوهّم انطباق ا هـ ز [د - ١٣٨ و] على هـ ز ط وه ز ح على ب ه ز ويلزم منه تلاقي ا ب ، ح ط في جهة ب ، ط. وقد كانا متلاقيين في جهة ا ، ح. فيلزم احاطة خطين مستقيمين بسطح. هذا محال.

[٦-٥] والثاني يستلزم المطلوب لأنّهما إذا تلاقيا في جهة ب ، ط وليكن تلاقيهما على ل وخط ز د قاطع لزاوية هـ ز ل فإذا اخرج قطع ه ل لأنّه لو قطع ز ل أو ه ز أحاط مستقيمان بسطح. فثبت المطلوب.

وإن شئنا قلنا لو لم يكن [م - ١٨٥ ظ] ح ط موازيًا لـ ا ب لتلاقيا في إحدى الجهتين. لكنّه لا يجوز التلاقي في [ف - ٢٣٩ ظ] شيئ من الجهتين بعين ما ذكرنا.

وان لم يشترط مشترط على نفسه أن لا يستعين بأشكال الكتاب كفاه أن يقول لا يجوز أن يلاقي ح ط ، ا ب.

وإلاّ لزم أن يكون خارجة ا هـ ز من مثلث هـ ل ز مساوية لداخلة هـ ز ط. وهو محال.

[٦-٦] هذا إن كانت الملاقاة في جهة ب ، ط. وإن كانت الملاقاة في الجهة الأخرى لزم أن تكون خارجة هـ ز ط من مثلث هـ ز ك مساوية لداخلة ا هـ ز. هذا محال.

وإذا لم يلاق ح ط فلا بدّ من أنّ تلاقي ج د لا في جهة ا ، ح.

---

١ على] ~~انطباق~~ على [د] ٢ متلاقيين] متلاقيان [ت] ٦ فثبت] فتبيّن [ت] ٩ مشترط] شرط [م] ٩ يقول] نقول [م] ١١ هـ ل ز] هـ ز ل [م] ١٣ الملاقاة] الملاقات [م] ١٣ الملاقاة] الملقات [م] ١٤ تكون] يكون [د ، م]

والاّ يلزم أن يكون خارجة ب‌هـ‌ز المساوية لـهـ‌زح التي هي أصغر من هـ‌زج أعظم من هـ‌زج. هذا محال. بل في جهة ب، د. وهو المطلوب.

والسلام على من اتبع الهدى.

[٢] أعظم من هـ‌زج] (في الهامش) ت

VII Edition of the Persian Text

Unlike the Arabic, which exists only as an independent treatise (usually copied into collections of similar mathematical treatises), the Persian version exists in three different forms, as outlined in Section III. These forms do not usually differ substantially in terms of mathematical content, but have differences in terms of diction and grammar. They are also dedicated to different rulers. In this edition, we have combined all three Persian forms. The text of the edition follows al-Shīrāzī's *Durrat al-Tāj*. Differences between the Persian versions are indicated in the variant notes. The most significant of these differences are also indicated in notes to the translations of the Arabic and Persian versions.

Although it is common practice to assign manuscript sigla on the basis of collection name, this is not practical in this case because in several instances we use multiple copies of the treatise from the same library. For this reason, we have opted instead to assign each manuscript an arbitrary sigla following the *abjad* (alphanumeric) order of the Arabic alphabet.

ا British Library, Additum 7695. The manuscript consists of 148 folios. It is an extract from *Durrat al-Tāj* containing only the geometrical section (Rieu 1881, II, 435). The Euclidean diagrams appear to be drawn in red lines with black letter labels. Diagrams for the demonstrations appear to be red lines with red lettering because they appear lighter than the surrounding text in the available black and white images. The demonstrations of the Euclidean postulates are found on folios 2a–5a.

ب Columbia University Library, Plimpton Or. 282. Al-Shīrāzī's Persian translation of al-Ṭūsī's *Taḥrīr* of the *Elements*. The manuscript was copied in 780 AH (1378 CE) in a hurried *nastaʿlīq* and is complete in 75 folios. The manuscript contains several lacunae and there are several errors in rebinding (corrected without notice in the images posted online). Diagrams are drawn in black ink with red letter labels. The demonstrations of the Euclidean postulates are found on folios 2b–3b.

ج Istanbul, Aya Sofya 2405. Al-Shīrāzī's *Durrat al-Tāj*. Euclidean diagrams are rendered in red lines with black letter labels. All other diagrams are rendered in black lines with red letter labels. The geometry section occupies folios 77a–126a. The demonstrations of Euclid's postulates are found on folios 77a–78a.

د Istanbul, Damad Ibrahim Paşa 815. Al-Shīrāzī's *Durrat al-Tāj*. Euclidean diagrams are rendered in red lines with black letter labels. All other diagrams are rendered in black lines with red letter labels. The geometry section occupies folios 144a–232a, The demonstrations of Euclid's postulates are found on folios 145b–147a.

هـ Istanbul, Damad Ibrahim Paşa 816. Al-Shīrāzī's *Durrat al-Tāj*. Euclidean diagrams are rendered in red lines with black letter labels. All other diagrams are rendered in black lines with red letter labels. The geometry section occupies folios 59a–102a. The demonstrations of Euclid's postulates are found on folios 59a–102a.

و Istanbul, Fazıl Ahmed Paşa 867. Al-Shīrāzī's *Durrat al-Tāj*. Euclidean diagrams are rendered in red lines with black letter labels. All other diagrams are rendered in black lines with red letter labels. The geometry section occupies folios 75a–129a. The demonstrations of Euclid's postulates are found on folios 75b–76b.

ز Istanbul, Hamidiye 790. Al-Shīrāzī's *Durrat al-Tāj*. Euclidean diagrams are rendered in red lines with black letter labels. All other diagrams are rendered with black lines and red letter labels. The geometry section occupies folios 242b–402a. The demonstrations of Euclid's postulates are found on folios 244a–247a.

ح Istanbul, Lala Ismail 288M. Al-Shīrāzī's *Durrat al-Tāj*. The condensation of the *Almagest* fills the margins. Dated 813 AH / 1410 CE. The geometry section occupies folios 34a–91a. Diagrams are rendered using gold / brown ink, with black letter labels. The demonstrations of Euclid's postulates are found on folios 34a–35a.

ط Istanbul, Ragip Paşa 838. Al Shīrāzī's *Durrat al-Tāj*. Euclidean diagrams are rendered using red ink, with black letter labels. All others are rendered with black ink, and red letter labels. The geometry section occupies folios 59b–103b. The demonstrations of Eiuclid's postulates are found on folios 60a–60b.

ى Tehran, Majlis Shūrā Library, Sinā 226. Al-Shīrāzī's translation of al-Ṭūsī's *Taḥrīr* of the *Elements*, complete in 260 folios. Euclidean diagrams are rendered with red ink and black letter labels. All other diagrams are rendered with black ink and red letter labels. The demonstrations of the postulates are found on folios 5a–9a.

ك Tehran, Majlis Shūrā Library 698. Al-Shīrāzī's *Durrat al-Tāj*. The colophon reports that it was completed in 698 AH / 1298–1299 CE. Euclidean diagrams are rendered in red ink with black letter labels. All other diagrams are rendered in black ink with red letter labels. The Persian edition of the *Elements* occupies folios 126b–201b. The demonstrations of the postulates are found on folios 127a–128b.

ل Tehran, Majlis Shūrā Library 1828. Al-Shīrāzī's *Durrat al-Tāj*. Euclidean diagrams are rendered in red ink with black letter labels. All other diagrams are rendered in black ink with red letter labels. The geometry section occupies pages 184–319. The demonstrations of Euclid's postulates are found on pages 185–188.

م Tehran, Majlis Shūrā Library 4720. Al-Shīrāzī's *Durrat al-Tāj*. Euclidean diagrams are rendered in red ink with black letter labels. All other diagrams are rendered in black ink with red letter labels. The geometry section occupies pages 140–248. The demonstrations of Euclid's postulates are found on pages 141–143.

ن Tehran, Majlis Shūrā Library 5142. Al-Shīrāzī's *Durrat al-Tāj*. The manuscript is incomplete at both ends and the codex is unfoliated. Diagrams are rendered in red ink. Euclidean diagrams usually have black letter labels. Other diagrams have red letter labels. The surviving geometry section occupies folios 2a–107b. The demonstrations of Euclid's postulates are found on folios 2b–5a.

س Tehran, Majlis Shūrā Library 4345/2. Independent treatise containing al-Shīrāzī's demonstrations of Euclid's postulates. Bound into a codex containing fourteen treatises, the majority written in Persian, although a few are in Arabic. All are copied in the same small, elegant *nastaʿlīq* scribal hand. The diagrams in this treatise are placed in the margin, drawn in black ink with red letter labels. A colophon following the treatise gives the date of copying as 1224 AH / 1809 CE . The demonstrations of Euclid's postulates are found on folios 27b–29b.

ع Tehran, Milli 28211. An independent treatise containing al-Shīrāzī's demonstrations of Euclid's postulates. This treatise appears to be the ninth in the codex. The accompanying discussion of al-Shīrāzī's summary diagram for book I appears to be classed as the tenth treatise. The catalog record indicates that this treatise occupies folios 21b–23a, although there are no folio numbers visible in the images we received from the library. Diagrams are rendered in black ink with red letter labels. There is no dated colophon.

ف Tehran, Dānishgāh Ilāhiyāt 764. An independent treatise containing al-Shīrāzī's demonstrations of Euclid's postulates. It is cataloged under the title: *Risāla dar uṣūl mawḍūʿa Uqlīdis* (Ghassemlou 2011, 51). The treatise is bound into a collection of disparate texts. Al-Shīrāzī's demonstrations of Euclid's postulates are found on folios 122b–126b (Ghassemlou 2011, 51). The entire codex was copied by Muḥammad Ḥassan b. Muḥammad ʿAlī in 1279 AH / 1862 CE.

چنین گوید مولانا قدوه الحکما و العلما قطب الملة و الحق و الدین الشیرازی متّع الله المسلمین بطول بقائه که قضایایی که اقلیدس در اصول ذکر کرده‌است یعنی اصول موضوعه که در صدر مقالهٔ اول اورده‌است، و من می‌گویم که اکثر این قضایا چنان است که متعلم سلیم الفطره هر چند به حکم [هـ - ٥٩ پ] صحت فطرت و ذکاء فطنت بر آن تصدیق کند، اما در باطن از انکاری خالی نباشد و او را خارخار طلب بیانی باشد سیما بر قضیهٔ [ی - ٥ پ] اخیره.

و از این جهت استادان صناعت مواخذت [ن - ٣ ر] کرده‌اند بر اقلیدس که آن را در عداد مسائل یاد کردن اولیتر از آنکه در مصادرات چه آن را در غیر علم هندسه بیان نتوان کرد. و هیچ کس از اهل صناعت بیان آن بی معاونت بعضی از اشکال کتاب نکرده. و از این جهت در اثناء مسائل [ا - ٣ پ] یاد کنند.

پس از جهت ازالت خارخار متعلمان سلیم الفطره لایق نمود اشارتی خفیف و ایمائی لطیف به بیان هر یکی کردن بی‌استعانت به مسائل کتاب.

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١ چنین] بسم الله الرحمن الرحیم چنین [س ، ع ، ف] ١ الحکما و العلما] العلما و الحکما المتحققین [ع ، ف] ١ الملة] الله [س] ٣ اول] اولی [ف] ٣ که] - [ب ، ی] ٣ این] - [ز] ٤ به حکم] حکم [هـ ، ح] ٤ فطرت] فطره [ح] ٤ فطنت] بصیرت [ب ، ی ، ن ، س ،ع ، ف] ٥ انکاری] انکاسال [ز] ٥ نباشد] نبود [س ، ع ، ف] ٥ خارخار] دغدغه [س ، ع ، ف] ٥ بیانی] بیاری [و] ، بیای [ح ، ن ، س] ٦ باشد] بود [س ، ع ، ف] ٧ مواخذت] مواخذه [د ، ز] ٨ عداد] عدد [هـ ، ح] ، اعداد [ل] ٨ مسائل] حسایل [ح] ، - [ف] ٨ آنکه] آنکه که [ا ، ج ، م ، ن ، س] ٩ کس] یك [د ، م] ٩ آن] از [ز ، ح] ١٠ نکرده] + است [ا ، ن] ، + اند [ب ، ی ، س ، ع ، ف] ١٠ یاد] + ~~کرده~~ [ك] ١٠ کنند] کند [ا] ١١ ازالت] - [هـ ، ح] ١١ خارخار] دغدغه [س ، ع ، ف] ١١ اشارتی] اشارت [ف] ١٢ ایمائی] اتمامی [ا ، ك] ١٢ به بیان] میان [ف] ، بر آن [ا] ١٢ هر یکی] آن [س] ١٢ بی‌استعانت] استعانت [ز]

[۱] اما بیان اول به آن باشد که نقطهٔ ثالث تخیل کنیم منطبق بر یکی از آن دو نقطه و آن را در وهم متحرّك فرض کنیم بر یك سمت تا دیگر نقطه، که مسافت آن حرکت خطی باشد مستقیم چه طولی باشد بی عرض و جمله نقطه‌ای که بر او فرض کنند بر محاذات یکدیگر باشند [ك - ۱۲۷ پ].

[۲] و بیان دوم به آنکه نقطه‌ای فرض کنیم در جهت طرف خط مفروض چنانکه اتفاق افتد و میان او و میان طرف خط خطی مستقیم وصل کنیم.

پس اگر از اتصال ایشان زاویه‌ای حاصل نشده [ج - ۷۷ پ] باشد ایشان بر استقامت یکدیگر باشند.

و اگر حاصل شده باشد تحریك کنیم خط را تا زاویه [ن - ۲۸ ر] [ف - ۱۲۳ ر] باطل شود و مقصود حاصل.

و به این طریق ممکن‌است که خط را اخراج کنند إلی غیر النهایه.

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۱ اما بیان اول به آن باشد که] قضیه اول: ما را بود که وصل کنیم میان هر دو نقطه که باشد به خطی مسنقیم. بیانش [س ، ف] ، قضیه اول: ما را هست که وصل کنیم میان هر دو نقطه که باشد به خطی مستقیم. بیانش [ع] ۱ ثالث] (در حاشیه) [ك] ، ثابت (و در حاشیه: ثالث) [ن] ۲-۳ تا دیگر ... نقطه‌ای که] - [ا] ، (در حاشیه) [ن] ۲ مسافت] مسافبت [ل] ۳ حرکت] ~~افتد~~ (و در حاشیه: مستقیم چه طولی باشد) [ب] ۳ باشد مستقیم چه طولی] بود مستقیم چه طولی [س ،ع ، ف] ، - [هـ ، ز] ۴ کنند] کند [م] ۴ باشند] باشد [ا] ، - [ب ، ی ، ن ، س ، ع ، ف] ۵ و بیان دوم] قضیه دوم: اخراج کنیم هر خطی مستقیم [ف - ۱۲۳ ر] محدود که بود بر استقامت او. بیانش [س ، ف ، ع] [ا ، ن] ۵ و بیان] بیان [ا] ۵ به آنکه] آنکه [ا] ، با آنکه [س ،ف ، ع] ۵ طرف] - [ز] ۶ افتد] آند [ل] ۶ و میان] میان [ح] ۶ خطی] (در حاشیه) [ك] ۷ نشده ... باشد] نشود [ن ، س] ۷ ایشان] ایشان را [د] ۸ باشند] باشد [ك] ۹ حاصل شده] - [ب ، ی ، س ، ع ، ف] ۹ باشد] نباشند [ب] ۱۰ حاصل] + شود [ل] ۱۱ و به این] به این [د] ۱۱ را] (زیر خط) [ب] ، + تا (و در حاشیه: را) [ك] ۱۱ النهایه] نهایه [ج ، ز ، ی ، ن ، س ، ف]

[۳] وبیان سوم به آنکه نقطه‌ای فرض کنیم در آن بعد که می‌خواهیم که دایره را به آن بعد بکشیم و میان [ی - ۶ ر] او و میان آن نقطه که به جای مرکز است به خطی مستقیم وصل کنیم.

و طرف مرکز را ثابت توهم کنیم و خط را متحرك تا به جای خویش آید که از طرف متحرك او محیط دایره‌ای حاصل شود.

[۴ - ۱] و بیان چهارم به آنکه زوایا ا ب ج، ا ب د، هـ ز ط، هـ ز ح [ز - ۲۴۴ پ] قوائم فرض کنیم و در توهم ب بر ز تطبیق کنیم و د ج بر ط ح چه به ضرورت ب ا بر ز هـ افتد.

و الا فرض کنیم که چون ز ک افتد. پس زاویهٔ ک ز ح اعنی ا ب ج مساوی ک ز ط باشد اعنی ا ب د.

پس هـ ز ح به جهت آنکه بزرگتر از ک ز ح است بزرگتر از ک ز ط باشد که مساوی ک ز ح است (شکل اول).

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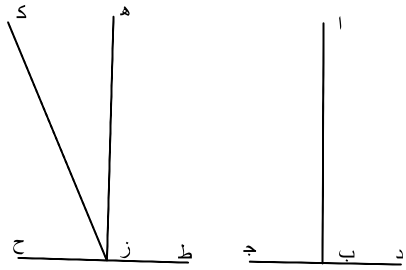
۱ وبیان سوم به آنکه] قضیه سوم: رسم کنیم بر هر نقطه و نه هر بعدی دایره. بیانش [س ، ف ، ع] ۱ به] - [ح] ۱ می‌خواهیم] می‌خوانم [ز] ۲ بکشیم] کشیم [ط] ۲ و میان] میان [ا] ٤ توهم] فرض [ی] ٤ خویش آید] خویش با زاید [ز] ، خود آند [ن ، س ، ع ، ف] ٦ و بیان چهارم به آنکه] جمله زوایا قایمه متساوی باشند. بیانش [س] ٦ به] - [ح] ٧ قوائم] - [د ، ز] ٧ فرض] + فرض [د ، ك] ٧ زَ] دَ [ا ، ج ، ك ، م] ٧ تطبیق] تطبق [د ، م] ٩ که] - [د ، و ، م] ٩ افتد] (بالای خط) [ك] ٩ ک ز ح] که ز ح [د] ١٠ باشد] بود [س ، ع ، ف] ، - [ح] ١١ از ک ز ح است] است از ک ز ح [ب] ١١-١٢ بزرگتر از ک ز ط باشد که ... است] در حاشیه [ك] ١١ از ک ز ط باشد که] باشد از ک ز ط که [ب] ١١ باشد که] (بالای خط) [ن] ، بود که [سُ ، ع ، ف] ١٢ مساوی] متساوی [س ، ع ، ف]



شکل اول — بر اساس نمودار موجود در نسخهٔ خطی شمارهٔ
۲۲۶ س کتابخانهٔ مجلس شوری اسلامی تهران (صفحهٔ ۶ ر).

و چون هـ زح از ك زط بزرگتر باشد لازم آید که هـ زط که مساوی هـ زح فرض کرده بودیم بزرگتر باشد از کـ زط. و این محال‌است. پس حکم ثابت باشد.

و به مثل این بیان [ح - ۳۴ پ] معلوم شود که زاویهٔ مساوی قایمه، قایمه باشد.

[۴ - ۲] و از وقوع هـ ز، کـ ز بر طـ ح ظاهراست که چون خطی مستقیم بر خطی مستقیم [ا - ۴ ر] [ن - ۳ پ] افتد دو زاویه که از دو جانب او حاصل شود یا دو قائمه باشد [ل - ۱۸۶] یا مساوی [د - ۱۴۶ ر] دو قائمه. چه یك سطح معین است که باعتباری او را دو قائمه می‌گویند و باعتباری دیگر حاده [ی - ۶ پ] و منفرجه.

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١ ك زط] ك زج [ز] ١ باشد] است (زیر خط: باشد) [ب] ، بود [س ، ع ، ف] ١-٢ لازم ... بزرگتر باشد] - [ا] ١ که] - [هـ] ٢ کرده بودیم] کردیم [س ، ف] ٢ بزرگتر باشد] بزرگتر بود [س ، ع ، ف] ، - [ز] ٢ باشد] شد [ب ، ز ، ح] ، بود [س ، ع ، ف] ٣ مساوی] مساو [ك] ٣ قایمه] (در حاشیه) [ك ، ل] ، (بالای خط) [ج] ٣ باشد] بود [س ، ع ، ف] ٤ کـ ز] ك [ز] ٤ ظاهراست] ظاهرت [ا] ٤-٥ بر خطی مستقیم] - [ا ، و] ، (در حاشیه) [م] ٥ دو] در [ك] ٥ از] آن از [ح] ٥ حاصل] حادث [ب ، ی ، س ، ع ، ف] ٦ دو] د [هـ] ٦ یك] - [س ، ع ، ف] ٧ باعتباری] یکبار [ب ، ی ، س ، ع ، ف] ٧ دو] د [ح] ٧ و باعتباری دیگر] و یکبار [ی ، س ، ع ، ف] ، و یکبار او را [ب] ٧ حاده ... و منفرجه] منفرجه و حاده ~~زاویه~~ [ب] ٧ و منفرجه] منفرجه [ا]

[٥ – ١] وبیان پنجم به آنکه اول بیان کنیم که چه را قطر منصف محیط دایره است.

بر این وجه که قطر اج را ثابت توهم کنیم و قوس ابج را متحرك تا به سطح دایره رسد از جهت ادج چه ناچار [ب - ٣ ر] بر او منطبق شود و مقصود حاصل.

و الا فرض کنیم که چون احج افتد خواه بیرون و خواه اندرون. و این محال‌است [س - ٢٨ پ].

چه اگر هـحد اخراج کنند لازم آید که هـح [ز - ٢۴٥ ر] و هـد به سبب آنکه هر دو مساوی هـج اند متساوی باشند. و این باطل‌است. پس حکم حق باشد.

و اگر بعضی بیرون افتد و بعضی اندرون همین محال لازم آید. و از اینجا روشن شد که زاویهٔ اجب مساوی زاویهٔ اجد است وهمچنین جاب مساوی جاد (شکل دوم).

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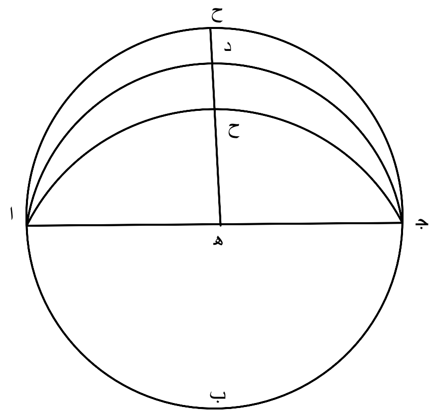
١ وبیان پنجم به آنکه] قضایه پنجم: دو خط مستقیم بسطحی محیط نشوند. بیانش [س] ١ اول] – [ا] ، (بالای خط) [ن] ١ چه را قطر] قطر چه را [هـ ، ح] ١ منصف] متصف [ح] ٣ که] + که [ب] ٣ قطر] (در حاشیه) [ك] ٣ را] (بالای خط) [ك] ٣ را] ر [ن] ، – [ب ، ی ، س ، ع ، ف] ٤ دایره] (بالای خط) [ل] ٤ ادج] اح [ح] ٤ چه] – [ف] ٤ و مقصود] و مطلوب [ب ، ی ، س ، ع ، ف] ٤ حاصل] + شود [ب ، (بالای خط) ل] ٥ و الا] و لا [ح] ٥ چون] – [ز] ٥ بیرون] برون [ب ، د ، و ، ز ، م] ، [(بالای خط) [ج] ٥ و خواه] خواه [ز] ٥ و این محال‌است] (در حاشیه) [ل] ٧ کنند] کند [م] ٧ آید] این [ج] ٨ مساوی] متساوی [س ، ع] ، تاوی [ف] ٨ اند] آید [ح] ٨ باشند] باشد [ج] ٨ پس ... باشد] – [س ، ع ، ف] ٩ بیرون] برون [ب ، د ، و ، ز ، م] ٩ و بعضی] و هر بعضی [ب] ٩ همین] – [ز] ٩ آید] + تا معلوم باشد [ب ، ط ، ی ، (رد خاشیه) ج] ، + تا معلوم بود [س ، ع ، ف] ١٠ اجب ... زاویهٔ] – [ح] ١٠ زاویهٔ] – [ا ، ب ، ی ، ن] ١٠ است] – [ی] ١٠ جاب مساوی] جا مساوی [و ، ح] ، – [ا] ١٠ جاد] + است [و ، ف]



شکل دوم — بر اساس نمودار موجود در نسخهٔ خطی شمارهٔ
۲۲۶ س کتابخانهٔ مجلس شوری اسلامی تهران (صفحهٔ ۶ پ).

[۵ - ۲] و چون این [ع - ۲۲ ر] مقدمه معلوم شد می‌گویم اگر اکب، اطب دو خط مستقیم باشند محیط به سطح اب بر مرکز ب به بعد آ دایرهٔ احج بکشیم و اکب، اطب در جهت ب اخراج کنیم (شکل سوم).

چه حال [ك - ۱۲۸ ر] از دو بیرون نباشد یا متلاقی نشوند در محیط چنانکه در صورت اول و دوم باشوند چنانکه در سوم و چهارم. و علی التقدیرین [و - ۷۶ ر] [م - ص ۱۴۱] خواه [ا - ۴ پ] بیش از رسیدن به محیط متلاقی [ی - ۷ ر] شوند و خواه نشوند.

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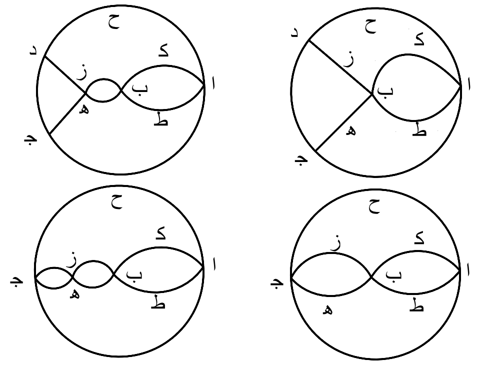
١ و چون] چون [ح] ١ می‌گویم] + که [هـ] ، گویم [ی ، ن ، س ، ع ، ف] ١ اگر] - [ا ، ن] ، + اگر [ع ، ف] ٢ باشند] باشد [د ، ز ، ط] ٢ محیط] محیطین [ب] ، + به [ی] ٢ به سطح] بر سطح [ب ، ی ، س ، ع ، ف] ٢ ب] با [ح] ٢ آ دایرهٔ] (در حاشیه) [ج] ، ای ایرهٔ [ح] ، دایرهٔ [د] ، اب دایرهٔ [ف] ٢ احج] احد [ع] ٣ اطب] (بالای خط) [ج ، ل] = (در حاشیه) [م] ٤ حال] محال [هـ ، ح] ٤ دو] هر دو [س] ٤ نباشد] نبود [س ، ع ، ف] ٤ یا] با [ا ، ن] ٤ نشوند] شوند [هـ] ٤ چنانکه] + چنانکه [د] ٥ اول] - [هـ ، ح] ٥ باشوند] یا شوند[ا ، د ، و ، ی] ٦ به محیط] محیط [م ، ن] ٦ متلاقی] ملاقی [ج ، ط] ، ومتلاقی [د] ، با متلاقی [ع]



شکل سوم — بر اساس نمودار موجود در نسخهٔ خطی شمارهٔ ۷۹۰ ، مجموعه حمیدیه، کتابخانهٔ سلیمانیه استانبول (صفحهٔ ۲۴۵ ر).

اگر نشوند لازم آید که قوس اح دج که اعظم است از قوس [ع - ۱۲۳ ر] اح د مساوی او باشد به سبب آنکه [ط - ۶۰ پ] هر دو نصف محیط یك دایره اند.

و اگر شوند لازم آید که زاویه‌ای که نیمهٔ محیط و یك قطر به او محیط باشد اعظم باشد از زاویه‌ای که آن نیمهٔ دیگر با آن قطر دیگر به او محیط. و این هر دو لازم محال است. پس حکم ثابت باشد.

و بدآنکه ما را بر این مطلوب که دو خط مستقیم محیط نشوند به سطحی دلیلی به غایت خوب روی نمود و آن است که اگر دو خط مستقیم به سطحی محیط شوند

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۱ نشوند] در محیط ملاقی شود [ع] ۱ آید] اند [ع] ۱ از قوس] (در حاشیه) [ی] ۱ اح د] قوس اج د [ع] ۲ باشد] باشند [ح] ، بود [س ، ع ، ف] ۳ و اگر] + در محیط ملاقی [ع] ٤ باشد] بود [س ، ع ، ف] ٤ با آن] به آن [س] ، و آن [ع] ٤ محیط] + باشد [ا ، ن ، س ، ع ، ف] ٤ هر دو] - [هـ ، ح] ٤ لازم] + ابد [س] ٥ محال] (بالای خط) [ج] ٥ باشد] شد [ب] ، بود [س ، ع ، ف] ٦-٤٥.٣ بدآنکه ... باشد] - [ب ، ی ، س ، ع ، ف] ٦ دو] در [ا] ٦ به سطحی] به سطی [ج] ٦-٧ به غایت] بیست [ی ، م] ٧ روی] رو [د] ٧ نمود] + و او [د ، ح ، ط ، ك ، ل ، م ، ن] ، + را [هـ ، و ، ز]

لازم آید که هر یك از ایشان اقصر باشد از آن [ز - ۲۴۵ پ] دیگر چه خط مستقیم اقصر خطی باشد که واصل باشد میان دو نقطه، چنان‌که ارشمیدس گفته است. و این لازم محال‌است. پس حکم ثابت باشد.

[۵ - ۳] و هم از آن مقدمه که تقدیم کردیم معلوم می‌شود [ن - ۴ ر] که نشاید که یك خط مستقیم به دو خط مستقیم بیوندد بر استقامت ایشان با آنکه آن دو خط مسامت یکدیگر نباشند.

و الا فرض کنیم که اب بر استقامت ب د، ب ج باشد. پس ب را مرکز سازیم و به بعد یکی از این خطوط اگر متساوی باشند و به بعد اقصر اگر مختلف باشند دایره بکشیم چون اهـ دج و لازم آید که اهـ دج که اعظم است از اهـ د مساوی [د - ۱۴۶ پ] او اد باشد به آن سبب که گفته شد. و آن باطل‌است. پس حکم حق باشد (شکل چهارم).

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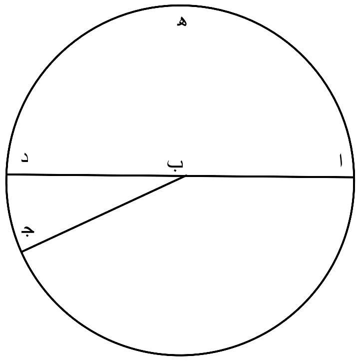
۱ لازم آید] - [ج ، ط] ۱ از] در [ا] ۱ خط] خطی [د] ۲ است] نیست [هـ] ۳ باشد] (بالای خط) [ن] ، شد [ح] ٤ از آن] از این [ب ، ی ، س ، ع ، ف] ٤ که ... کردیم] - [ب ، ی ، س ، ع ، ف] ٤ کردیم] یافت [د] ٤ می‌شود] شود [ح] ٤ نشاید] نمیۀ شاید [هـ ، ح] ٥ که یك خط ... با آنکه] (در خاشیه) [ك] ٥ که] (بالای خط) [م] ٥ بیوندد بر] بیدندد بر [ز ، م ، ف] ، شوند وبر [ا] ، بیدندر وبر [د] ، شوند بر [س] ، ببوندد بر [ع] ٥ آن] - [ز] ٦ مسامت] مسافت [ز] ٦ نباشند] نباشد [ز ، ح ، ط ، ك] ٧ که] - [ل] ٧ باشد] بود [س ، ع ، ف] ٧ را] + که [ل] ، - [ی ، س ، ع] ٧ سازیم] - (ودر حاشیة: کنیم) [ن] ٨ این] آن [ع] ٨ باشند] باشد [ح ، ط ، م] ٨ باشند] باشد [ج ، و ، ط ، ك ، م] ، - [ب ، ی ، س ، ع ، ف] ٩ دایره] (در حاشیه) [ج] ، (بالای خط) [ط ، ل] ٩ و لازم] لازم [ا ، ج] ، بر او لازم [ح] ٩ آید] آند [ع] ٩ اهـ د] اهـ [ن] ١٠ او اد] به او [س] ، مساوی او [د] ١٠ باشد] بود [س ، ع ، ف] ١٠ و آن] و این [ی ، س ، ن ، ع ، ف] ١٠ حکم] - [هـ] ١١ باشد] بود [س ، ع ، ف]



شکل چهارم — بر اساس نمودار موجود در نسخهٔ خطی شمارهٔ ۲۲۶ س کتابخانهٔ مجلس شوری اسلامی تهران (صفحهٔ ۷ ر).

[۶ - ۱ - ۱] و اما بیان ششم که هیچ کس [ل - ص ۱۸۷] از اهل صناعت بی‌استعانت [ی - ۶ پ] به بعضی از مسائل کتاب تعرض آن نرسانیده‌اند و نتوانسته‌اند، [ج - ۷۸ ر] یا اگر رسانیده‌اند به ما نرسیده‌است ما را به توفیق باری عز اسمه. و

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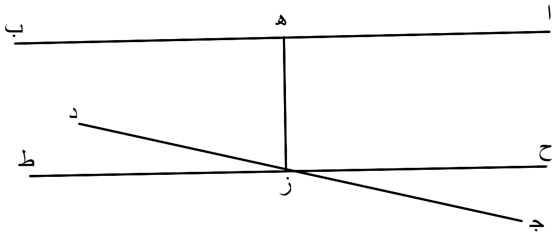
۱ و اما بیان ششم که] قضیه ششم: هر دو خط مستقیم که خطی مستقیم بر ایشان افتد و دو ززاویه داخله که از یک جهت باشند کم از دو قایمه باشند ایشان را چون در آن جهت اخراج کنند [س - ۲۹ ر] به هم رسند. بیانش [س ، ع] ، قضیه ششم: هر دو خط مستقیم که خط مستقیم بر ایشان افتد و دو زاویه داخله که از یک جهت باشد کم از دو قایمه باشد ایشان (بالای خط : را) چون ذر آن جهت اخراج کنند به هم رسند. بیانش: [ف]   ۱ صناعت] + را [د]   ۲ به بعضی از] به بعضی [هـ ، ح] ، بعضی [ب ، ی] ، بعضی از [م ، س ، ع ، ف]   ۲ مسائل] + از [ج ، هـ ، ح ، ط]   ۲ تعرض] به عرض [ز ، س]   ۲ آن] به آن [ج ، ط ، ن]   ۲ نتوانسته‌اند] نتوانسته [س ، ع ، ف]   ۳ یا اگر رسانیده‌اند] - [ا]   ۳-۴۷۰۲ و به یمن ... ملک اسلام سلطان سلاطین مازندران اعز الله انصاره وضاعف اقتداره] - [س ، ع ، ف]   ۳-۴۷۰۱ و به یمن] و به من [ج ، و] ، به مین [ز] ، و یمن [ا ، ب ، د ، ح ، ک ، م ، ن]

به یمن همت و حسن تربیت ملك اسلام سلطان سلاطین مازندران اعز الله انصاره وضاعف اقتداره، وجهی [ا - ۵ ر] روی نمود خوب و تام چون وجه بدر لیله التمام بی‌استعانت به مسائل کتاب.

[۶ - ۱ - ۲] و بیان آن بر سبیل اجمال این است که از مفهوم خطوط متوازی معلوم می‌شود که نشاید که بعضی از احد المتوازیین در یك جانب [هـ - ۶۰ ر] افتد از دیگر متوازی و بعضی در جانبی دیگر.

و الا [ف - ۱۲۴ پ] متلاقی باشند نه متوازی (شکل پنجم).



شکل پنجم — بر اساس نمودار موجود در نسخهٔ خطئ شمارهٔ ۲۲۶ س کتابخانهٔ مجلس شوری اسلامی تهران (صفحهٔ ۷ پ).

١-٢ ملك اسلام سلطان سلاطين مازندران اعز الله انصاره وضاعف اقتداره] مخدوم عز نصره [ی] ، عز نصره [ب] ١ ملك اسلام] - [ب ، ز ، ی] ٢ خوب ... التمام] تام [ب ، ی ، س ، ع ، ف] ٢ چون وجه] رجیون [ن] ٢ التمام] التام [هـ ، ح] ٣ به مسائل] + بل [ح] ٤ آن] این [ج ، ط ، ل] ٤ این است] آن است [ب ، ی ، س ، ع ، ف] ٥ از] + کما [ف] ٥ احد] احدی [ز] ٥ در] از [ح] ٦ دیگر] + می [ز] ٦ متوازی] موازی [ا ، ج ، هـ ، ز ، ح ، ط ، ك] ٦ در] از [ب ، ی ، س ، ع ، ف] ٦ جانبی] جانب [س ، ع ، ف] ٧ باشند] باشد [ج ، م ، ن]

و نه آنکه ابعاد میان ایشان مختلف شود چنانکه فطرت سلیم بر آن دلالت می‌کند.

پس هر دو خط مستقیم که خطی مستقیم میان ایشان افتد و موازی ایشان باشد ایشان متوازی باشند.

چه اگر متلاقی شوند لازم اید که بعضی از احد المتوازیین در یك جانب [ب - ۳ پ] افتد از آن دیگر و بعضی در جانبی دیگر [ز - ۲۴۶ ر] با یك خط موازی دو خط متلاقی باشد. و این نیز باطل‌است چه ابعاد او با یکی از ایشان به ضرورت مختلف گردد. پس موازی هر دو نبوده باشد.

پس هر دو خط مستقیم متلاقی که میان ایشان خطی مستقیم افتد [ح - ۳۵ ر] چون اخراج کنند الی غیر النهایه ناچار ملاقی یکی از ایشان گردد.

والا لازم آید که متلاقیان متوازی باشند.

پس هر خطی مستقیم که در سطحی باشد که در آن سطح دو خط متلاقی باشند ناچار آن خط ملاقی یکی از ایشان گردد، چون ایشان را اخراج [ن - ۴ پ] کنند

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۱ و نه … می‌کند] - [س ، ع ، ف] ۱ می‌کند] نمی‌کند [هـ] ۲ پس] پیس [ا] ۲ که … مستقیم] ملاقی که [ك] ، (در حاشیه) [ل] ، - [ا] ۲ ایشان] + خطی مستقیم میان ایشان [ك] ۲ باشد] بود [س ، ع ، ف] ۳ متوازی] - [ك] ۳ باشند] باشد [ك ، م] ٤ شوند] آند [ف] ٤ اید] آند [ع] ٥ در] از [ب ، ی] ٥ جانبی] جانب [ب ، س ، ف] ٥-٧ با یك … نبوده باشد] - [س ، ع ، ف] ٥ با یك] به آنکه [ا ، د ، هـ] ، به این که [ج] ، تا یك [م ، ن] ٦ نیز] - [ب ، ی] ٦ با] - [ح] ٦ به ضرورت] - [هـ ، ح] ٧ دو] + ~~نبوده~~ [ل] ٨ متلاقی] (در حاشیه) [ی] ٨-٩ ایشان خطی … ناچار] - [و] ، (در حاشیه) [م] ٩ چون] و چون [هـ ، ل] ٩ کنند] کند [ا ، ح ، ن] ٩ ملاقی … از ایشان] متلاقی [د ، ز ، (در حاشیه) م] ، - [و] ١٠ متلاقیان] متقابلان [د ، و ، م] ١٠ باشند] باشد [ا ، ز] ١١ مستقیم] باید که غیر متوازی باشند (در حاشیة) [و] ، (بالای خط) [س] ١١ سطحی] سطح [ا ، م ، ن] ١١ باشند] باشد [ح ، ی ، ك] ١٢ کنند] کند [ا ، د ، م ، ن]

الی غیر النهایه [ی - ۷ ر] چه او را بر هر وضع که فرض کنند از میان ایشان بیرون نباشد.

و سبب آن است که هیچ جزو از سطح از میان دو خط متقاطع بر آن سطح خارج نباشد.

[۶ - ۲] پس هر دو خط مستقیم چون اب، جد که خطی مستقیم چون هز بر ایشان افتد و دو زاویهٔ [ك - ۱۲۸ پ] داخله که از یك جهت باشند چون ب هز، د زه کمتر از دو قائمه باشند ایشان را چون اخراج کنند در آن جهت ملتقی شوند.

چه به سبب آنکه زاویهٔ [ا - ۵ پ] ب هز با هزد کمتر از دو قائمه است به فرض و با اهز چند دو قائمهٔ چنانکه معلوم شد. پس اهز از هزد بزرگتر باشد. و از این جهت چون زها در توهم بر هزد تطبیق کنیم ها چون زطح افتد [س - ۲۹ پ].

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۱ الی] - [ا] ۱ النهایه] نهایه [ا ، و ، ز ، ی ، ك ، ن] ۱ وضع] وضعی [د ، و ، م] ، موضع [هـ ، ح] ، دو وضع [ج] ، دو خط ضع [ك] ، ضع [ف] ۱ که] - [ب] ۱ کنند] کند [ا ، م] ۲ نباشد] نبود [س ، ع ، ف] ۳ است] آن است [ن] ۳ میان] + ایشان [ك] ۳ متقاطع] + بیرون (و بالای خط: بر دو سطح) [س ، ع ، ف] ، (و در حاشیه: بر هر وضع) [ج] ۳-۴ بر آن سطح خارج] - [س ، ن ، ع] ۵ خطی] خط [ن] ۵ مستقیم] + چون اب، جد که خطی مستقیم [ج ، ك] ۵ هز] هن [ح] ۶ و دو] و در [ز] ۶ باشند] باشد [ج ، ح ، ط ، ل] ۶ چون] + چون [ك] ۷ ب هز، د زه] هزب، دهز [ح] ، ب هز، دهز [هـ] ، ب هز، د زه [ل ، ن ، س] ۷ باشند] باشد [ا ، ب ، ح ، ی ، ك] ۷ ملتقی] ملقی [ب] ۹ با هزد] (در حاشیه) [ی] ۹ با] بام [ا] ۹-۱۰ به فرض] + زَ [ز] ۱۰ چند] جد [ز] ۱۱ چون] (زیر خط) [م] ۱۱ در] - [س] ۱۱ هزد] هد [ح] ۱۱ ها] + بَ (بالای خط) [و ، ی] ۱۱ زطح] اطح [س]

[۶ - ۳] پس گویم ح ط یا موازی [ف - ۱۲۵ ر] اب باشد یا نباشد.

اگر باشد اب به سبب آنکه میان ح ط، ج د افتاده‌است [ع - ۲۲ پ] چون اخراج کنند ملاقی یکی از ایشان شود وملاقی ح ط نمی‌تواند شد. پس ملاقی ج د شود. و هو المطلوب.

[۶ - ۴] و اگر نباشد یا ملاقی او شود در جهت ح، ا یا در جهت ب، ط [د - ۱۴۷ ر].

و اول باطل‌است چه به سبب آنکه زاویۀ اهز مساوی هزط است. لازم آید که زاویۀ [ز - ۲۴۶ پ] ب هز مساوی هزح باشد چه هر دو زاویۀ هـ و هر دو زاویۀ ز همچند دو قائمه‌اند آند.

و چون چنین باشد اگر ها، زح متلاقی شوند بر ک مثلاً (شکل ششم).

چون [ی - ۷ پ] اهز بر هزط تطبیق کنیم و هزح بر ب هز لازم آید که [م - ص ۱۴۳] هب، زط [و - ۷۶ پ] در جهت ب، ط متلاقی شوند. و ایشان بر استقامت در جهت ح، ا متلاقی شده‌اند. پس لازم آید که اب، ح ط با آنکه هر دو مستقیم اند به یک سطح محیط باشند. هذا خلف.

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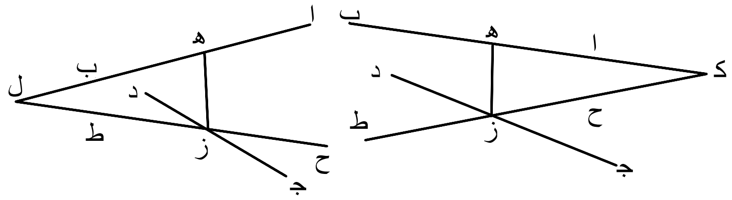
۱ گویم] + که [و ، م] ۱ ح ط] ط ز [و] ، خط [هـ ، ح] ۱ یا] (بالای خط) [ن] ۱ موازی] متوازی [ح] ۱ باشد یا نباشد] بود یا نبود [س ، ع ، ف] ۲ اگر] چون اگر [ی ، ك] ۲ باشد] – [ا] ، باشده [ی] ۲ اب] و اب [س ، ف] ، + ان [ف] ۲ افتاده] واقع [ب ، ی ، س ، ع ، ف] ۳ کنند] کنیم [د ، و ، م] ۳ ح ط] خط [هـ ، ح] ۳ نمی‌تواند] نتوان [ح] ، نمی‌توند [ن] ٤ المطلوب] + و ~~هو المط~~ [ب] ٥ یا] (بالای خط) [ك] ٥ ملاقی] + ملافی [هـ ، ح] ٥ در] دو [ا] ، در دو [ن] ۷ هزط] زطهـ [د] ۷ است] نیست [هـ] ۷ آید] نباید [هـ ، ز] ، آند [ب] ۸ ب هز] زهد [ا] ۸ باشد] – [ف] ۹ آند] – [ا] ، است [ب ، ی ، س ، ع ، ف] ۱۰ باشد] بود [س ، ع ، ف] ۱۰ مثلاً] متلاقی [هـ ، ح] ۱۱ هزط] هز [هـ ، ح] ۱۱ ب هز] ب هـ [س ، ف] ۱۱ آید] اند [ع] ۱۱ که] + که [ب] ۱۲ ب] ز [س] ۱۳ اب، ح ط] اح، ط ب [ح] ۱٤ دو] + خط [س] ۱٤ باشند] باشد [ح ، ك]



شکل ششم — بر اساس نمودار موجود در نسخهٔ خطی شمارهٔ ۲۲۶ س کتابخانهٔ مجلس شوری اسلامی تهران (صفحهٔ ۸ پ).

[۶ - ۵] و دوم مستلزم مطلوب‌است چه اگر در جهت ب، ط به هم رسند بر ل مثلاً خط زد به سبب آنکه زاویهٔ هـزل را قسمت کرده‌است چون اخراج کنند قاعدهٔ هـل را قطع کند چه اگر زهـ با زل قطع [ل - ص ۱۸۸] کند لازم آید که دو خط مستقیم به سطحی محیط شده باشند. و این باطل‌است. پس حکم ثابت باشد. و به وجهی دیگر می‌گویم اگر حـط موازی اب نباشد در یک جهت ملاقی او شود و لازم آید که در جهت دیگر هم ملاقی او شود. و هر دو باطل‌اند. و مستلزم مطلوب [ن - ۵ ر] چنانکه تقریر کرده شد.

۱ و دوم] (بالای خط) + شده‌اند بمن لازم [ل] ، دوم [س ، ع ، ف] ۱ است] - [ك] ۱ رسند] رسیده [ا ، ی] ، راسد [ج] ۲ هـزل] هـزك [ز] ، زل [ح ، ن] ۳ هـل] ل [ا ، هـ ، ح] ، هـك [ز] ۳ را] وا [ز] ۳ کند] کنند [ب] ۳ چه] - [د] ۳ با] یا [ج ، و ، ی] ۳ زل] زطل [ب ، ی ، ك ، ن ، س ، ع ، ف] ، زك [ز] ۳ آید] - [س] ، (بالای خط) [ع] ٤ باشند] باشد [ح ، ك ، ن] ٤ باشد] باشدد [ح] ، بود [س ، ع ، ف] ٥ و به وجهی] و جهی [ح] ٥ می‌گویم] گویم [ب ، ی ، س ، ع ، ف] ٥ حـط] خط [هـ ، ح] ٦ و لازم ... شود] (در حاشیه) [ف] ٦ آید] + که اب شاید که ملاقی حـط شود والا لازم آید [ح] ٦ که] + اب [ح] ، + اید [هـ] ٦ هم] - [ب ، س ، ع ، ف] ٦ او] - [ب ، ی ، س ، ع ، ف] ، + او [ا] ٦ شود] باشد [ح] ٦ اند] - [ف] ٦ و مستلزم] مستلزم [ب] ٧ کرده شد] رفت و الله اعلم بالصواب [س ، ع ، ف]

و اگر نه شرط رفته بودی که استعانت به مسائل کتاب نرود این قدر کافی بودی که گفتندی [ا - ۶ ر] که اب نشاید که ملاقی حط شود.

[۶ - ۶] و الا لازم آید که خارجهٔ اهز از مثلث هلز مساوی داخلهٔ هزط باشد.

اگر ملاقات در جهت ب،ط باشد یا خارجهٔ هزط از مثلث هکز مساوی داخلهٔ اهز باشد [ی - ۸ ر].

اگر ملاقات در جهت آ،ح باشد. هذا خلف.

و چون [ب - ۴ ر] ملاقی حط نشود به ضرورت ملاقی جد شود [ز - ۲۴۷ ر] نه در جهت آ،ح.

و الا لازم آید که خارجهٔ بهز که مساوی هزح است که کوچکتر است از هزج بزرگتر از هزج باشد. هذا خلف. بل در جهت ب،د. و هو المطلوب.

۱-۲ و اگر نه شرط ... شود] - [س ، ع ، ف] ۱ شرط] + و اگر نه [ا] ۱ بودی] بود [ب]
۱ که] + هم (بالای خط) [ب] ۱ نرود] برود [ز] ، نزند [هـ ، و] ۲ گفتندی] گفتدی [هـ]
۲ که] - [ب ، و ، ی] ۳-۵ و الا ... باشد] - [س ، ع ، ف] ۳ اهز] اهد [د ، ح] ، اهز را [ک ، ل] ، ازهد [ن] ۳ از] در [ب] ۳ هلز مساوی] - [ز] ۳ هزط] زط [د ، ک ، ن]
۵ اگر] + نه [ز] ۵ ملاقات] ملاقی [د ، هـ ، و ، م] ۵ در جهت] - [ی] ۵ ب،ط] آ،ح [ا ، د ، ن] ۵ باشد] بود [ب ، ی] ۵-۷ یا خارجهٔ هزط ... اگر ملاقات در جهت آ،ح باشد] - [ا ، س ، ع ، ف] ۵ یا خارجهٔ هزط] - [ن] ۵ یا] - [د ، و ، م ، ن] ۵-۶ از مثلث هکز مساوی داخلهٔ اهز باشد] (رد حاشیه) [ن] ۶ باشد] بود [ی] ۷ اگر ملاقات در جهت آ،ح باشد] - [ن] ۷ ملاقات] + کنند [ل] ۷ باشد] بود [ب ، ی] ، - [ح] ۷ هذا خلف] - [س ، ع ، ف]
۸-۹ و چون ... جهت آ،ح] - [ح ، س ، ع ، ف] ۸ نشود] نشوند [ل] ۹ در] از [(بالای خط) ی] ۱۰-۱۱ و الا ... هذا خلف] - [ح ، س ، ع ، ف] ۱۰ بهز] بزهـ [ن] ۱۰ است] - [ب ، ی] ۱۱ بزرگتر از] - [ل] ۱۱ هزج] هدج [ج] ۱۱ بل ... المطلوب] - [س ، ع ، ف]
۱۱ بل] بد [و] ۱۱ در] از [ی]

VIII English Translations and Textual Notes

In order to facilitate comparison of the Arabic and Persian versions, we place their English translations in parallel columns. The right column contains the translation of the Persian text and the left column contains that of the Arabic text. We have broken the English translations into sections paralleling the formatting of the two editions in Sections VI and VII. We have added reference numbers paralleling their placement in the Arabic and Persian editions to make it easier to locate parallel textual passages.

Although we try to remain true to the original text, it is sometimes necessary to introduce words not present in the original text in order to produce an understandable translation. These words are enclosed in pointed brackets < >. Explanatory notes to clarify, for example the reference of a pronoun in the text, are enclosed in parentheses ( ).

The general parallelism between the Arabic and Persian texts is immediately obvious—including much common technical vocabulary. This common technical vocabulary is scarcely surprising because Persian borrowed a great deal of its mathematical terminology from Arabic.

In the name of Allah, the merciful, the compassionate.

I want to make known the postulates of the subject <of geometry> that the author, may God have mercy on him, quoted (*naqala ʿan*) at the beginning, that is, his statement "We may connect a straight line between <any> two points" through his statement "the two of them meet on that side if extended."

In the name of Allah, the merciful, the compassionate.[54]

Now our master, a leader of religious authorities and sages of the two truths, the pole (or pillar) of the religious community and truth and religion, al-Shīrāzī, may Allah be generous to Muslims through the length of his enduring, has brought to perfection those principles (*qaḍāyāyi*) that Euclid mentioned in the *Elements*—I mean, the postulates (*uṣūl mawḍūʿa*) that he posited (*ṣadara*) in the first book.[55]

[54] This pious invocation is found only in the independent Persian version. In the other Persian versions, these demonstrations are embedded in a larger work, so the invocation is omitted. Or perhaps we could say that it has been absorbed into the general basmalah at the beginning of the treatise.

[55] This short preamble appears only in the independent Persian version.

These postulates should be evident to anyone who has sound instinct and a keen intellect. But perhaps he does not resolve (*yaḥallu*) some of them within <himself>—rejecting especially what is in the last postulate (Euclid's parallel lines postulate). So let us discuss here instruction (*tanbīhāt*) that may aid the student who is hesitant about them (the postulates) in order to remove <his> inner uneasiness and to suppress <any> objection.

I say that most of these postulates are such that, although a student having sound instinct and penetration of insight[56] may accept them, nevertheless his thought might not be without any objection and so he would be impelled (*khārkhār*) to seek a demonstration for these <postulates>,[57] especially in the case of the last.

And from this point of view, the skilled practitioners of the art have reproached Euclid that he would have done better to place them (the postulates) among the theorems, rather than among the things posited (*muṣādarāt*), seeing that they are not demonstrable outside the science of geometry. Yet not one of the practitioners <of this science> has been able to demonstrate those <postulates> without assistance from some of the propositions of <Euclid's> book. And for this reason they included them under the problems (*masāʾil*).

[56] The term in *Durrat al-Tāj* is *fiṭnat*, a term borrowed from Arabic whose root meaning is intelligence or cleverness. In the independent version and the *Taḥrīr* translation the term used is *baṣīrat*, also a term borrowed from Arabic, whose root meaning is discernment or mental perception.

[57] The independent version replaces *khākhār*, whose root meaning is scratching or scrubbing and, by extension, a desire or impulse of the heart, with *daghdagha*, a derivative of an Arabic verb meaning to tickle (someone) and, by extension, to have an inclination toward something.

[1] As for the first <postulate>, we imagine a third point superimposed on one of the two points between which we want to connect a <straight> line. We specify <that> this point moves from the point on which it is superimposed to the other along a single path (*samt wāid*). Thus it is indubitable that the distance of this motion is a straight line because it is length without breadth, each point facing (*mutaḥādhiya*) the other.

[2] As for the second <postulate>, let us specify a point in the direction of an extremity of the specified line, however it may fall. We connect between it (the point) and it (the line) by a straight line.

Then, if an angle is not produced (*ḥadatha*) from the two lines, each of the two of them is in a straight line with the other and the two together are joined <as> a single line.

But if <an angle> is produced, we may move the line until the angle ceases to exist (*yabṭala*) and that which was sought is completed.

By this method (*ṭarīq*) it is possible to extend a straight line without end.

Hence, for the sake of removing the anxiety[58] from those students of sound intellect it was appropriate to have a slight hint and a delicate intimation concerning the demonstration of each one without seeking the assistance of the theorems of <Euclid's> book.

As for the demonstration of the first <postulate>, it is that we imagine a third point superimposed upon one of two points and we specify that it is moved in imagination (*wahm*)[59] in a single direction until it reaches the other point. The path of the motion <of that point> will be a straight line, since it is length without breadth and all of its points will be facing (*muḥādhāt*) one to another.

And the demonstration of the second <postulate> is from the fact that we specify a point in the direction of an extremity of a specified line, however it may fall, and between it and the extremity of the line we connect a straight line.

Then, if from their connection an angle is not formed (*ḥāṣil*), they are in a straight line one with the other.

But if <an angle> is <formed>, we move the line until the angle ceases to exist (*bāṭil shūd*). And <that which was> intended has come about.

And by this method (*ṭarīq*) it is possible to extend the line without end.

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[58] The term *khārkhār* is replaced in the independent Persian version by *daghdagha*.

[59] The Arabic does not specify that the point is moved in the imagination.

[3] As for the third <postulate>, let us assume, at the endpoint of that distance with which we want to draw a circle, a point. We connect between it and the point that we wish to make the center of the circle by a straight line.

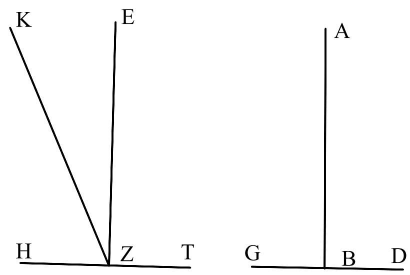
Then we imagine the central endpoint to be fixed and the line to be moved until it arrives at its initial position. Thus the moved endpoint from it (the straight line) draws (*yarsumu*) the circumference of a circle.

[4 – 1] As for the fourth <postulate>, it is by virtue of the fact that we specify angles ABG, ABD, EZT, EZH as right <angles>. And in imagination we superimpose <point> B on <point> Z and <line> DG on <line> TH. Thus <line> BA is <superimposed> upon <line> ZE.

The demonstration of the third <postulate> is that we assume a point at that distance with which we want to draw a circle and between it and that point that is designated as the center <of the circle> we connect a straight line.

We imagine the endpoint <that is> the center <to be> fixed and the line <to be> moved until it falls upon itself such that from the moved extremity there is formed (*ḥāṣil shūd*) the circumference of a circle.

The demonstration of the fourth <postulate> is that we specify angles ABG, ABD, EZT, EZH as right <angles>. In imagination we superimpose <point> B on <point> Z and <line> DG on <line> TH. Then, of necessity, <line> BA is <superimposed> on <line> ZE.



Edited from Tehran, Majlis Shūrā Islāmī, Sinā 226, f. 6a.[60]

But if not, let <line BA> be like <line> KZ. Thus angle KZH—I mean, <angle> ABG—is equal to angle KZT—I mean, <angle> ABD.

But if not, we specify that it (line BA) is like <line> ZK. Thus angle KZH—I mean, <angle> ABG—is equal to <angle> KZT—I mean, <angle> ABD.

Thus <angle> EZH, from the fact that it is greater than <angle> KZH, would be greater than <angle> KZT, which <is> equal to <angle> KZH.[61]

[60] The diagram in Feyzullah 1359 lacks line KZ, although point K is present in the diagram.

[61] This explanatory sentence is not present in the Arabic version.

But angle EZH is greater than angle KZH. Now it was equal to it—I mean, angle EZT is greater than angle KZT, the equal of angle KZH. This is impossible.

And since <angle> EZH is greater than <angle> KZT, it would be necessary that <angle> EZT, which was specified equal to <angle> EZH, at the same time be greater than <angle> KZT. This is impossible. Thus the proposition is established.

And on the example of this demonstration—I mean, the mentioned superposition <argument>[62]—one may know that an angle equal to a right <angle> is a right <angle>.

And on the example of this demonstration, it may be known that an angle equal to a right <angle> is a right <angle>.

[4 – 2] And from the falling of EZ, KZ on TH it may be demonstrated, that any straight line, if it stands upon its like (i.e., on another straight line), the two angles formed (*ḥadithatān*) on the two sides of the line are either two right angles or equal to the two of them (i.e., two right angles). <Thus>, it is necessary that a rhomboidal surface have either two right angles or an acute <angle> and an obtuse <angle>.

And one may also show this principle by superimposition.[64]

For if <line> AB be superimposed on line EZ, the two angles at Z are right <angles>.

But if not, the two of them (i.e., the two angles formed) are equal to the two of them (i.e., two right angles) because <line> EZ divides angle KZT into two angles, one of the two of them being right and the other, if combined with (*inḍammat*) angle KZH, produces another right angle.

And from the falling of EZ, KZ on TH it is evident (*ẓāhir*) that whenever a straight line is incident upon a straight line, the two angles produced (*ḥāṣil shūd*) on two sides of it are either two right angles or <angles> equal to two right angles. Thus on the one hand[63] one may say that a rhomboidal surface has two right angles and on the other hand <it has> an acute <angle> and an obtuse <angle>.

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[62] This explanatory phrase is not present in the Persian versions.

[63] Al-Shīrāzī apparently decided to replace the more Persian term *yakbār*, which was used in the Independent version and the *Taḥrīr* translation, with a term derived from the Arabic, *bi-iʿitabārī*.

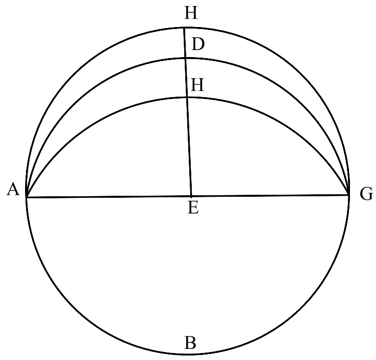
[64] This second demonstration is not found in the Persian versions.

[5 – 1] As for the fifth <postulate>, its demonstration is dependent on the demonstration of a lemma (*muqaddima*), namely that the diameter <of a circle> bisects the circumference and that the two angles produced from the intersection of the diameter and the circumference are equal to one another.[65]

And for the demonstration of the fifth <postulate>—we first demonstrate that any diameter is the bisector of the circumference of the circle.

We say with regard to the demonstration of that <lemma>: let us specify (*linufraḍu*) diameter AG as fixed and arc ABG as moved until it arrives at the plane of the circle on the side of ADG.

For this approach, we imagine (*tavahham*) diameter AG <to be> fixed and arc ABG <to be> moved until it reaches the surface of the circle on the side of ADG, since it is necessarily superimposed upon it and that which was sought (*maqṣūd*)[66] is produced.



Edited from Tehran, Majlis Shūrā Islāmī, Sinā 226, f. 6b. The label for point E has been omitted from the diagram in Feyzullah 1359.

Now, either it (arc AG) falls outside the circle or inside it or some of it outside and some of it inside or it is superimposed on the other half of the circumference.

But each of these cases is impossible except the last case. It is necessarily what was sought.

And if not, we assume that AHG falls either outside or inside <ADG>. This is impossible.

[65] The second part of this enunciation is not found in the Persian transmission.

[66] In al-Shīrāzī's *Taḥrīr* translation and in the independent Persian version, the term used is the more common *maṭlūb*.

As for the first <case>, it is because <when> we extend EHD, it is necessary that EH and ED be equal to one another on account of the equality of the two of them to EG. From it one may know the incorrectness of that which <occurs> when some <of EH> falls inside and the rest <falls> outside <ED>.

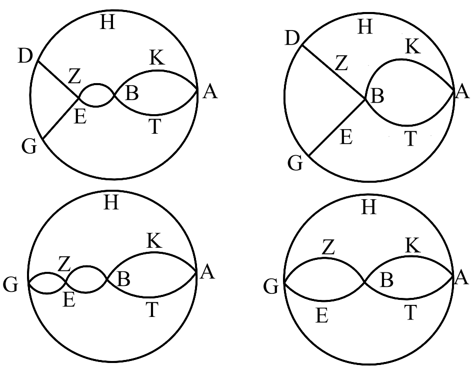
For if EHD is extended, it is necessary that EH and ED, on account of the fact that each of the two is equal to EG, should be equal to one another. This is impossible. Thus the principle is correct.[67]

As for the second <case>, it is clear (*ẓāhir*) since there necessarily follows from it the equality of the two portions of the circumference on account of being superimposed, as well as the equality of angles AGB and AGD and of angles GAB and GAD.

And if some lies outside and some inside, this <also> is impossible.[68] And hence it is clear (*roshān*) that angle AGB is equal to angle AGD and in the same way, <angle> GAB is equal to <angle> GAD.

[5 – 2] Since this lemma has been attained (*taḥaṣṣalat*), we say: If two lines, such as <lines> AKB and ATB bound a surface, such as AB, let us draw about center B with distance AB circle AHG. Let us extend AKB and ATB in the direction of B.

Since this lemma is known, we say: If AKB, ATB are two straight lines bounding surface AB, we draw about center B with distance A circle AHG. We extend AKB, ATB in the direction of B.



Edited from Istanbul, Süleymaniye Library, Hamidiye 790, f. 245a.[69]

[67] This concluding sentence is not in the independent Persian version.

[68] The *Taḥrīr* translation and the independent version add "as may be known" (*tā maʿlūm*).

[69] The diagram in Tehran, Majlis Shūrā Islāmī, Sinā 226, folio 7a is damaged and could not be used.

Then it makes no difference whether the two meet one another before reaching the circumference or they do not meet one another. As for <the case> when the two do not meet one another at the circumference, <it is> just as in the first two diagrams. Or <when> the two meet one another <at the circumference>, as in the last two diagrams.

Now, if it be the first <case>, the equality of arc AHDG, the greater, to arc AHD, the smaller,[70] is necessary because the two of them are half circumferences of a single circle. That is a impossible.

And if it be the second <case>, the angle that is bounded by half the circumference together with one of the two diameters is necessarily greater than that which is bounded by the other half of the circumference together with the other diameter. This is a contradiction.

For the situation is such that the two might not meet one another outside as in the first and second diagrams or meet one another at the circumference as in the third and fourth <diagrams>. And according to two <other> points of view, the two either meet one another before arriving at the circumference or do not <meet one another>.

If they do not <meet>, it is necessary that arc AHDG, which is greater than arc AHD, be equal to it by reason of the fact that each of the two is a half circumference of a circle.

And if they do <meet>, it is necessary that the angle that is bounded by half the circumference together with one diameter be greater than the angle that is bounded by half the circumference together with that other diameter. And each of these two must be impossible. Thus the principle is established.

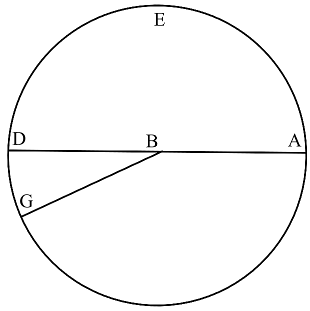
And since that which was sought from this is that two straight lines do not surround a surface, there remains a subtle argument to be made. And that is that if two straight lines should surround a surface, it would be necessary that each one of them be less than the other since a straight line is the shortest line that connects between two points, as Archimedes has said. And that necessity is impossible. Thus the principle is established.[71]

[70] Illustrated in the top two diagrams.

[71] This argument is omitted from the Arabic version. It is also omitted from the *Taḥrīr* translation and the independent Persian version.

[5 – 3] And it may be known from that <lemma> that a straight line is not continued rectilinearly by two straight lines, the two of them not lying opposite one another (*musāmatayn*).

Likewise from that lemma that we have presented[72] one may know that it is not the case that a straight line continues two straight lines rectilinearly unless those two lines be opposite (*musāmat*) to one another.



Edited from Tehran, Majlis Shūrā Islāmī, Sinā 226, folio 7a.

But if not, we assume <that> AB <is> in a straight line with BD, BG. Then we draw (*naruma*) a circle about center B with the distance of one of the lines, if they are equal to one another, and with the distance of the shorter <line> if they are not equal. It is necessary from that <that> arc AEDG, the greater, be equal to arc AED, the smaller. This is a contradiction.

If not, we assume that AB is in a straight line with BD, BG. Then we make B the center <and> draw (*bekashīm*) a circle like AEDG with the distance of one of these lines if they are equal to one another and with the distance of the shorter if they are unequal. It is necessary that AEDG, which is greater than AED, be equal to it for the reason that has been mentioned. That[73] is impossible. Thus the proposition is established.

[72] The phrase "that we have presented" is not present in the *Taḥrīr* translation or the independent version.

[73] The pronoun "this" is used in the independent version.

[6 – 1 – 1] As for the sixth <postulate>—indeed the author (*al-muṣannif*), may Allah have mercy upon him, demonstrated it by a method <that is> dependent on many propositions from the book.

As for the demonstration of the sixth <postulate>, no one from the community of practitioners, without having recourse to some propositions of the treatise coming before that, has been able to follow up on and get mastery over that <postulate>. And although it (the authorship of the demonstration) has been granted to us, it has not been attained except by the grace of the Creator—may his name be glorified.

Now, it is indeed possible for us to demonstrate it according to another method, without having recourse to any of the propositions <of the treatise>.

And there is no one from the community of practitioners who, without the assistance of some of the propositions, is able to enlarge upon it or to obtain it, <not even> someone of lofty ambition and excellent rank, a prince of Islam, a sultan of sultans, <one of> the Māzandarān—may Allah magnify his associates and multiply his excellence[74]—to whose mind there has occurred its full argument, thorough and complete, without recourse to the propositions of the treatise.[75]

[6 – 1 – 2] That is that we say: It may be known from the characterization (*taʿrīf*) of lines parallel to one another, that some <part> of one of the parallel <lines> does not fall on <one> side of its parallel and some on the other side.

The demonstration of that, by way of summarizing, is that from an understanding of parallel lines it is not proper that some <part> of one of the two parallel lines lies on one side of the other parallel <line> and some on the other side <of it>.

But if not, we let the two of them meet one another, not <being> parallel to one another.

But if not, they would meet one another, <they would> not be parallel to one another.

[74] The phrase "not even someone … multiply his excellence" does not occur in the Arabic version or in the independent Persian version. The *Taḥrīr* translation replaces this phrase with "a master with regard to his assistance."

[75] The phrase "without recourse …" is omitted the independent version and in the *Taḥrīr* translation.

And it is necessary from that that when any straight line falls between two straight lines and be parallel to the two of them, then the two of them are parallel to one another.

But if not, we let the two meet one another. Then it is necessary that some <part> of one of the two lines parallel to one another falls on one side of the <line> parallel to it and some on the other side of it. This is a contradiction.

And it is necessary from that that should any straight line fall between two lines meeting one another, it will inevitably meet one of the two of them if the three <lines> be extended indefinitely.

But if not, it would be parallel to the two of them. Thus it is necessary that the two be parallel to one another. This is a contradiction.

And it is not the case that the distances between them are different, as sound wisdom would suggest concerning that.[76]

Thus <if> any two straight lines be such that a straight line falls between them and is parallel to them, they are parallel to one another.

For if they should meet one another, it would be necessary that some <part> of one of the two parallel lines would fall on one side of that other <line> and some on the other[77] side of that line that is parallel to the two lines that meet one another.

This also is impossible since the distance it is from one of them would necessarily be different. Thus it cannot be parallel to each of the two of them.[78]

Thus any two straight lines meeting one another are such that a straight line falling between them, when extended indefinitely, would inevitably meet one of them.

But if not, it would be necessary that the two <lines> meeting one another would be parallel to one another.

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[76] The concept of unequal distances does not appear in the Arabic version.

[77] The remainder of this paragraph and the next two paragraphs are not found in the independent Persian version.

[78] This argument based on unequal distances is not found in the Arabic version.

And it is necessary from it that any line being in a surface in which two lines meet one another, would inevitably meet one of the two of them if the third be extended without end because, at whatever location it is specified, its extension is not possible except that it be between the two of them.

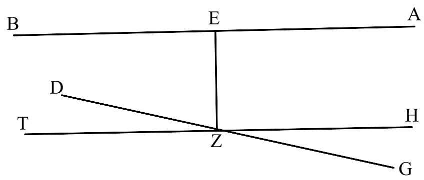
Thus any straight line being in a surface in which there are two lines meeting one another, that line inevitably meets one of them when extended indefinitely, since at each position that one may specify it is between them, not outside <them>.

The reason for it is that it is not possible that there exists a point from that surface not included in what is between two lines that meet one another in it, when extended without end in both directions.

The reason for that is that each part from a surface from between two intersecting lines is within, not outside, that surface.

[6 – 2] Since that is established, we say: Any two straight lines <such as> AB, GD upon which a straight line, such as EZ, falls, and the two interior angles on the same side, such as angles BEZ, DZE, being less than two right angles, the two of them, if extended on that side, meet one another.

Thus any two straight lines, such as AB, GD, upon which a straight line, such as EZ, falls, and two interior angles that are on one side, such as <angles> BEZ, DZE, are less than two right angles, the two of them also, when extended on that side, meet one another.



Edited from Tehran, Majlis Shūrā Islāmī, Sinā 226, f. 7b.

Because angle BEZ together with angle EZD is less than two right angles by assumption (*farḍ*) and, together with <angle> AEZ, is equal to two right angles, just as preceded. Thus <angle> AEZ is greater than <angle> EZD. For if we imagine the superposition of <angle> AEZ on <angle> EZD , <line> EA falls like <line> ZTH <and> necessarily the angle is greater.

Since on account of the fact that angle BEZ, together with <angle> EZD, is less than two right angles by assumption and together with <angle> AEZ <it> is like two right angles, as is known. Thus <angle> AEZ is greater than <angle> EZD. And from that perspective, since we may imagine the superimposition of <angle> ZEA upon <angle> EZD, <line> EA falls upon <line> ZTH.

[6 – 3] We say, therefore, that either <line> HT is parallel to <line> AE or it is not.[79]

For if it (line HT) be parallel <to line AE>, and <line> AB falls between <lines> HT <and> GD, then it is inevitable that it meets one of the two of them upon extension. Let it not meet HT. Thus it meets GD on the side of B, D since if it should meet it (line GD) on the other side, it would necessarily meet line AB on it (that side).[80] This is a contradiction.

[6 – 4] But if it (line HT) be not parallel <to AE>, we say: Angle AEZ is equal to angle EZT. And from this there follows necessarily the equality of angles BEZ, EZH since the entirety of the angles <at> E and the angles <at> Z are as two right angles, just as has occurred.[82]

But should AB, HT meet one another it would be either on the side of A, H or on the side of B, T.

Thus we say that <line> HT either is parallel to <line> AB or it is not.

For if it (line HT) be <parallel to AB>, then <line> AB, on account of the fact that it falls[81] between HT, GD, when extended, would meet one of the two of them. Let it not meet HT. Thus it must meet GD. This is what was sought.

And if it is not so, they should meet one another either on the side of H, A or on the side of B, T.

[79] The Arabic version labels the line AE, while the Persian versions label it AB.

[80] The argument is dependent on the diagram. Since line AB lies between HT and GD, and it is assumed that it does not meet HT, it must meet GD if extended. But if extended in the opposite direction, it cannot meet GD. Therefore HT must meet AB on that side. But AB and HT cannot intersect because they are assumed parallel to one another.

[81] In *Durrat al-Tāj*, *wāqiʿ* is replaced by *uftādeh*.

[82] This paragraph has been repositioned in the Persian transmission so that it follows the statement of the impossibility of the first case.

But the first <case> is impossible.

But if not, let them meet one another at K.

But we may imagine the superposition of AEZ on EZT and EZH on BEZ. And it is necessary from that (the superimposition of these angles) that AB, HT meet on the side of B, T.
But they were <assumed> to meet one another on the side of A, H. Thus it is necessary that two straight lines enclose a surface. This is impossible.

[6 − 5] And the second, of necessity, is what is sought, because the two of them, if they meet one another on the side of B, T—let them meet at L—and line ZD is cutting (*qāṭiʿ*) angle EZL, then if extended it cuts EL because if it cuts (*qaṭaʿa*) ZL or EZ then two straight lines would surround a surface. Thus what was sought is established.

But the first <case> is impossible on account of the fact that angle AEZ being equal to angle EZT, it is necessary that angle BEZ be equal to <angle> EZH since the two angles at E and the two angles at Z together are two right angles.[83]

And <that> being so, let EA, ZH meet at K, for example.

Then if we superimpose AEZ on EZT and EZH on BEZ, it is necessary that EB, ZT[84] would meet one another on the side of B, T.
And the two of them being straight, meet one another on the side of H, A. Thus it would be necessary that AB, HT, each being <a> straight <line>, bound a surface. This is a contradiction.

And the second is the desired requirement, since if on the side of B, T the two of them meet at <point> L, for example. Line ZD, on account of that, divides (*qasama*) angle EZL since, being extended, it cuts (*qaṭa*) base EL. For if ZE or ZL is cut, it is necessary that two straight lines enclose a surface. That is impossible. Thus the principle is established.

[83] The explanation for the impossibility has been repositioned to precede the previous paragraph in the Arabic version.

[84] EB, ZT are parts of lines AB, HT mentioned in the Arabic transmission.

And if we wish we may say <that> if HT be not parallel to <line> AB, they will meet one another on one of the two sides. But it is not permissible that they meet one another on either of the two sides, precisely according to what we have discussed.

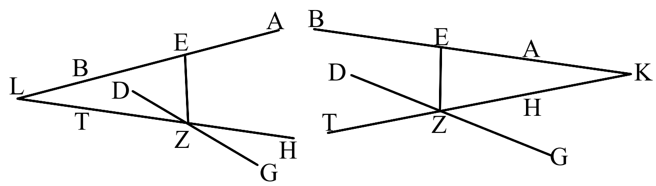
And if there is not stipulated a condition with respect to its assumption that one should not seek assistance from the propositions of the treatise, it is sufficient to state that it does not happen that HT, AB meet one another.

But if not, it is necessary that exterior <angle> AEZ from triangle ELZ is equal to interior <angle> EZT. That is impossible.

And by another approach, we say <that> if HT be not parallel to <line> AB, it would meet it on one side. And it is necessary that it also[85] meet it on the other side. And each of these two is impossible. And the necessity of what is sought is in this way asserted.[86]

And if there exists no stated condition that one should not seek assistance from propositions of the treatise, it would be sufficient that one states that AB does not meet HT.

But if not, it is necessary that exterior <angle> AEZ from triangle ELZ should be equal to interior <angle> EZT.



Edited from Tehran, Majlis Shūrā Islāmī, Sinā 226 (folio 8b).

[6 – 6] Now if they meet, <it is> on the side of B, T. But if they should meet on the other side, it is necessary that exterior <angle> EZT from triangle EZK be equal to interior <angle> AEZ. This is impossible.

But if it does not meet HT, it is inevitable that it meet GD, not on the side of A, H.

If they meet on the side B, T, exterior <angle> EZT from triangle EKZ is equal to interior <angle> AEZ <and> if they meet on the side A, H, this would be a contradiction.

And since it does not meet HT, by necessity it meets GD <but> not on the side of A, H.

[85] The word "also" is omitted from the independent Persian version.

[86] The independent version ends at this point with the phrase "And Allah is more knowing with regard to the difficulties."

But if not, it would be necessary that exterior <angle> BEZ, which is equal to <interior angle> EZH—that is, it is smaller than <angle> EZG—is greater than <angle> EZG. This is impossible. Rather, <they meet> on the side of B, D. This is what was sought.

Peace upon whoever follows the right path.[87]

But if not, it would be necessary that the exterior <angle> BEZ, which is equal to <angle> EZH—which is smaller than <angle> EZG—is greater than <angle> EZG. This is impossible. Rather <they meet> on the side of B, D. That is what was sought.

[87] A common pious prayer offered at the conclusion of a treatise. There is no comparable conclusion to either the Persian translation of the *Taḥrīr* or the insertion in *Durrat al-Tāj* since these are an insertion into a larger treatise.

IX Concluding Thoughts: The Pursuit of Patronage

Our study of al-Shīrāzī's Arabic and Persian versions shows that there are few substantial differences between them. We cannot be certain in which language al-Shīrāzī first chose to write his proofs, but since Arabic was still the lingua franca of the mathematical sciences, we consider it probable that these proofs were first written in that language.

The proofs of the first four postulates can be traced back to the Greek commentators. They entered the Arabic transmission through the commentary of al-Nayrīzī, were discussed by Ibn al-Haytham in his commentaries on the *Elements*, made more systematic in the *Iṣlāḥ* of al-Abharī, and given their final form as a self-contained unit by al-Shīrāzī. It was the Arabic and Persian versions of al-Shīrāzī's text that continued to be copied and circulated in the succeeding centuries.[88]

We hypothesize that al-Shīrāzī's Arabic demonstrations were translated into Persian and inserted into his Persian edition / translation of al-Ṭūsī's *Taḥrīr* of the *Elements*. This treatise was dedicated to the local political ruler, apparently as part of a strategy to curry official favor and gain patronage that would support al-Shīrāzī as he continued to research and teach.[89] In this effort, it appears that he was initially successful. But humans, even political authorities, are fickle and one scholar's benefit often made him the object of jealousy and intrigue from those who did not enjoy the same patronage position. When a new vizier intrigued at the court to get his pension reduced, al-Shīrāzī felt forced to parlay his magnum opus, his encyclopedic summary of Aristotelian thought, to induce a minor ruler to provide him the stable income needed to continue his scholarly writing and teaching. He incorporated his translation of the *Taḥrīr*, with only minor editing, into this overview of Aristotelian knowledge that we now know as *Durrat al-Tāj*.

Since we rarely encounter examples of mathematical works being repurposed for reuse with a different patron, these Persian demonstations offer a fascinating case study of how scholars might go about doing so, offering particularly poignent and

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88 The Arabic edition of the *Elements* ascribed to Pseudo-Ṭūsī, although nearly contemporaneous with that of al-Shīrāzī, did not provide a significant improvement on al-Shīrāzī's work. It left some of the needed lemmas needed for the demonstrations among the definitions of book I, so that its demonstrations are less self-contained. And the demonstration of the parallel lines postulate is entirely removed and placed following Euclid's proposition 29.

89 Dedication of books to worldly authorities was one of the more common ways for scholars to provide service to their patron. It is not always easy to determine whether the dedication was part of a request to enter a patronage relationship or gratitude for a relationship that had already been established (Brentjes 2008, 308). Being commissioned to undertake political negotiations, such as al-Shīrāzī's mission to the Mamluk court in Cairo, might be another kind of service that scholars were sometimes asked to perform within the patronage relationship (Brentjes 2008, 312 and 315).

revealing insight into the practicalities of survival by a scholar who lived in perilous and tumultuous times. Although not primarily a mathematician, he attempted to use mathematics as a tool for the advancement of his career, an effort that was ultimately unsuccessful. He died penniless because the promised payment for his recently completed revision of his commentary on Ibn Sīnā's *Canon of Medicine* had not yet been delivered, and one of his wealthier students paid for his funeral (Walbridge 1992, 24).

X Appendix: Arabic Medieval Geometrical Collections

Although al-Shīrāzī was recognized as an outstanding scholar in his day and rarely lacked students who wished to learn from him, he frequently had to contend with political events that were beyond his control. But despite his personal struggles for patronage, his few writings on geometry, whether in Arabic or in Persian, continued to be circulated and copied in the centuries after his death, suggesting that they had been found to be of value by later generations of students. In this appendix, we examine in greater detail the pedagogical use of al-Shīrāzī's demonstrations in their Arabic version.

The Arabic version of the demonstrations of the postulates is currently known in four untitled manuscript copies, none of which bears al-Shīrāzī's name. Each of these copies is part of a collection of mathematical treatises that were copied by a single copyist, suggesting that these collections were intended to be read and used as a unit. In this appendix we describe the content of these collections. Such collection had a similar structure, consistng of an initial larger and more comprehensive treatise, followed by several smaller and more focused discussions of Euclid's *Elements*.

X.1 Tunis, Bibliothéque nationale, 16167

This codex comprises ten treatises commenting on, or explaining all or specific parts of, Euclid's *Elements*. Its contents include:

- Ibn al-Haytham (died about 429 AH / 1038 CE), *Sharḥ muṣādarāt Uqlīdis l-Ibn al-Haytham* (Commentary on the Premises of Euclid's *Elements*), ff. 1b–59b.[90]

[90] Sezgin (2000) published a facsimile edition of two manuscripts—Bursa, Haraççıoğlu 1172/1 and Istanbul, Feyzullah 1359/2. Two partial editions of the Arabic text have been published. Barbara Hooper Sude (1974) edited the Arabic text of books I–VI using four manuscripts and made an English translation of these books; Aḥmed (2005) published an edition of the entire work based on three Arabic manuscripts.

For a summary of Ibn al-Haytham's biography and his contributions to mathematical sciences, see Sabra (1972). Based on variant forms of Ibn al-Haytham's name in copies of his works as

- Al-ʿAbbās ibn Saʿīd al-Jawharī (d. about 220 AH / 835 CE), *Ziyādāt al-ʿAbbās ibn Saʿīd fī al-maqāla al-khāmisa min Uqlīdis* (Additions to the fifth book of Euclid's *Elements*), ff. 60b–61a.[91]
- Al-Ahwāzī (d. about 329 AH / 941 CE), *Kalimāt min sharḥ al-maqāla al-ʿāshira min Kitāb Uqlīdis* (Extracts from his commentary on the tenth book of Euclid's *Elements*), ff. 61b–65a.[92]
- Abū Jaʿfar al-Khāzin (d. between 350 and 360 AH / 961 and 971 CE), *Tafsīr ṣadr al-maqāla al-ʿāshira min Kitāb Uqlīdis* (Commentary on the premises of tenth book of Euclid's *Elements*), ff. 65b–71a.[93]
- Quṭb al-Dīn al-Shīrāzī, Untitled (Discussion of the proofs of Euclid's postulates), ff. 71b–73a.
- Kamāl al-Dīn al-Fārisī (d. 718 AH / 1319 CE), *Qāla (…) al-Ḥasan al-Fārisī inna mā qālahu (…) al-Ṭūsī fī akhīri al-maqāla al-thālitha ʿāshar* (Note on al-Ṭūsī's exposition of the last proposition of book XIII <of the *Elements*>), ff. 73a–74a.[94]

well as in biographical dictionaries, Rashed (1993, 8–19) suggested that there were two medieval scholars named Ibn al-Haytham—a position adopted also by Rosenfeld and Ihsanoğlu (2003, 130–138). Sabra (1998; 2002–2003) rejected this hypothesis. Thomann (2017, 931–932) has presented additional evidence suggesting that Sabra's interpretation may be incorrect. Sabra also disagreed with the conventional statement that Ibn al-Haytham died in 1038 AH. Based on a historical record of a manuscript in Ibn al-Haytham's hand dated 432 AH (between 11 September 1040 and 30 August 1041 CE), Sabra argues that he must have died some time after this date.

[91] Little is known of al-Jawharī's personal life (Sabra 1973; Brentjes 1997). He is mentioned several times in conjunction with astronomical observations made at the court of Caliph al-Maʾmūn (reigned 198 / 813 to 218 / 833). Of the writings on Euclidean geometry attributed to him, only a few excerpts are known from quotations in later works. His additions to book V exist in both Arabic and Persian versions. They have been edited and translated by De Young (1997; 2008–2009).

[92] Abū al-Ḥusayn Aḥmad ibn al-Ḥusayn al-Ahwāzī al-Kātib was apparently active during the 4th century AH (10th century CE), although almost nothing is known of his personal life. The eight short sections of his discussion of *Elements* X were copied into numerous collections of mathematical tracts (Sezgin 1974, 312–313; Rosenfeld and Ihsanoğlu 2003, 80). Al-Ahwāzī's Arabic commentary has been edited by Mohammed Rida Fatimi Dazfuli (1391 AH) and translated into modern Persian. The main themes of al-Ahwāzī's tract have been briefly described by Matviyevskaya (1967, 199–209; 1987).

[93] The few verifiable facts that we know about the life of al-Khāzin are summarized by Dold-Samplonius (1973) and by Rashed and el-Bizri (2011, 504–506). For information on surviving manuscripts, see Sezgin (1975, 298–299) and Rosenfeld and Ihsanoğlu (2003, 81–82). Farès (2009) has discussed the concept of irrationality embodied in al-Khāzin's explication of book X.

[94] Kamāl al-Dīn is usually described as a student of al-Shīrāzī. His best known work among modern historians is in optics and theory of the rainbow. For a summary of his scientific and mathemat-

- Kamāl al-Dīn al-Fārisī (d. 718 AH / 1319 CE), *Maqāla li-l-Fārisī yudhifu ʿalā taḥrīr al-Abharī fī-l-masʾala al-mashhūra min Kitāb Uqlīdis* (Treatise on al-Abharī's exposition of the well-known problem in Euclid's *Elements*), ff. 74a–75a.[95]
- Anonymous author, *Ḥadd Uqlīdis taʾlīf al-nisba* (Euclid's definition of compounding of ratios), f. 75b.[96]
- Abū Dāwūd Sulaymān ibn ʿIṣma al-Samarqandī, *Kitāb fī dhawāt al-ismayni wa-l-munfaṣilāti fī al-maqāla al-ʿāshira min Kitāb Uqlīdis* (Treatise on binomials and apotomes from the tenth book of Euclid's *Elements*), ff. 76b–86b.[97]
- Thābit b. Qurra (d. 288 AH / 901 CE), *Fī al-ʿillati al-latī lahā rattaba Uqlīdis ashkāl kitābihi dhālika al-tartīb* (Treatise on the cause of why Euclid disposed the propositions of his book in such an order), ff. 86b–90b.[98]

X.2 Istanbul, Feyzullah Library, MS 1359

Codex Istanbul, Feyzullah 1359 comprises nine treatises explaining or commenting on all or parts of Euclid's *Elements*. Its contents include the following:

- Naṣīr al-Dīn al-Ṭūsī (597–672 AH / 1201–1274 CE), *Taḥrīr Kitāb Uqlīdis fī al-Uṣūl* (Redaction or edition of the *Elements*), ff. 1b–150a.[99]

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ical ouevre see Rashed (1973, 212–219). Abdeljaouad (2014–2015) has edited the Arabic text and translated it into English.

[95] Abdeljaouad (2018–2019) has edited the Arabic text and translated it into English.

[96] There appears to be at least one additional copy of this treatise: Tehran, Dānishgāh 284/3. See Ghassemlou (1387 SH, 277).

[97] Little reliable biographical information is available. Sezgin (1975, 337–338) says that he was active during the 4th century AH (10th century CE) but Rosenfeld and Ihsanoğlu (2003, 78), citing al-Bīrūnī, report that Abū Dāwūd particpated in making observations on the obliquity of the ecliptic in Balkh between 270 and 275 AH (883 and 888 CE). His work in astronomy has been quoted by several later authors. His extant writings on geometry remain unstudied.

[98] This treatise has several alternative titles: *Kitāb fī al-taʾattī li-istikhkrāj al-aʿmal al-handasiyya* (Treatise on how to solve geometric problems) or *Risāla fī kayf yanbaghī an yuslaka li nayl al-maṭlūb fī al-maʿānī al-handasiyya* (Treatise on the way one must proceed to obtain desirable geometric truths). For a quick overview of older studies, see the summary by Rosenfeld and Grigorian (1976). Rashed (1996, 735–765) has edited this treatise and translated it into French. For other editions and translations of several treatises mentioned here, see also Rashed (2009).

[99] This initial treatise has been published in a modern full-color facsimile edition (Fazlıoğlu 2012). Al-Ṭūsī's frequently copied redaction of the *Elements* has not been edited or translated into modern vernaculars in its entirety, although some sections, such as the demonstration of Euclid's parallel lines postulate, have been translated and studied (Jaouiche 1986, 99–112; 201–226). The editorial

- Ibn al-Haytham (d. about 429 AH /1038 CE), *Sharḥ musādarāt Uqlīdis l-ibn al-Haytham* (Commentary on the premises of Euclid's *Elements*), ff. 150b–237a.[100]
- Quṭb al-Dīn al-Shīrāzī, Untitled (Discussion of the proofs of Euclid's postulates), ff. 237b–239b.[101]
- Al-ʿAbbās ibn Saʿīd al-Jawharī (d. about 220 AH / 835 CE), *Ziyādāt al-ʿAbbās ibn Saʿīd fī al-maqāla al-khāmisa min Uqlīdis* (Additions to the fifth book of Euclid's *Elements*), ff. 239b–240b.[102]
- Al-Ahwāzī (d. about 329 AH / 941 CE), *Kalimāt min sharḥ al-maqāla al-ʿāshira min Kitāb Uqlīdis* (Excerpts from his commentary on the tenth book of Euclid's *Elements*), ff. 241a–245a.[103]
- Abū Jaʿfar al-Khāzin (d. between 350 and 360 AH / 961 and 971 CE), *Tafsīr ṣadr al-maqāla al-ʿāshira min Kitāb Uqlīdis* (Commentary on the premises of tenth book of Euclid's *Elements*), ff. 245a–2252a.[104]
- Anonymous author, *Ḥadd Uqlīdis ta'līf al-nisba fī'l-uṣūl* (Definition of composition of ratios in the *Elements*), f. 252b.[105]
- Anonymous author, *Al-qawl fī iqāmat al-burhān ʿalā al-ḥukm al-madhkūr fī al-shakl al-khāmis ʿashara min al-maqāla al-thāniyya ʿashra min hādhihi al-kitāb* (A discussion concerning the demonstration of the famous principle in proposition fifteen of book twelve), ff. 253a–254b.[106]
- Kamāl al-Dīn al-Fārisī, (d. 718 AH / 1319 CE), *Qāla (…) al-Ḥasan al-Fārisī inna mā qālahu (…) al-Ṭūsī fī akhīri al-maqāla al-thālitha ʿāshra* (Note on al-Ṭūsī's exposition of the last proposition of book XIII <of the *Elements*>), ff. 254b–255b.[107]

notes of al-Ṭūsī preserve some evidence concerning the characteristics of the Arabic translation attributed to al-Ḥajjāj ibn Yūsuf ibn Maṭar (De Young 2003). Many of the manthematical notes describing alternative demonstrations of the Euclidean propositions were drawn from the *Kitāb Ḥall Shukūk Kitāb Uqlīdis*, usually without an explicit attribution (De Young 2009). The treatise has been often confused with another *Taḥrīr* whose text was printed in Rome in 1594 (De Young 2012a).

[100] This treatise has been published in a black and white facsimile edition by Sezgin (2000). The treatise is also included in Tunis, Bibliothèque nationale 16167. See footnote 88, above.

[101] This treatise is also present in Tunis 16167.

[102] Also included in Tunis 16167. See note 89, above.

[103] Also included in Tunis 16167. See note 90, above.

[104] Also included in Tunis 16167. See note 91, above.

[105] Also included in Tunis 16167. See note 94, above.

[106] Although this treatise has been frequently copied, its author has not yet been positively identified.

[107] Also included in Tunis 16167. See note 92, above.

- Anonymous author, *Wujida fī baʿḍ nusakh Uqlīdis baʿd tamām al-maqāla al-khāmisa ʿāshr* (There is found in some copies of Euclid after the completion of the fifteenth book...), f. 256a.[108]

X.3 Munich, Bayerische Staatsbibliothek, cod. arab. 2697

This manuscript comprises copies of eleven treatises, one of which is in Persian, all copied in the same hand. Its contents, devoted to discussions of Euclid's *Elements*, are as follows:

- Naṣīr al-Dīn al-Ṭūsī, *Taḥrīr uṣūl al-handasa li-Uqlīdis* (Edition / Redaction of the *Elements of Geometry* of Euclid), ff. 1b–145a.[109]
- Anonymous author, *Aghrāḍ maqālāt Uqlīdis* (Aims of the books of Euclid's [*Elements*]), ff. 146b–150a.
- Al-Ahwāzī, *Sharḥ al-maqāla al-ʿāshira min Kitāb Uqlīdis*, ff. 151b–166b.
- al-Ahwāzī, *Kalimāt min Sharḥ al-maqāla al-ʿashira min Kitāb Uqlīdis*, ff. 167a–171a.[110]
- Abū Jaʿfar al-Khāzin, *Tafsīr ṣadr al-maqāla al-ʿāshira min Kitāb Uqlīdis* (Commentary on the premises of book X of the *Elements*), ff. 171b–177b.[111]
- Anonymous author, Untitled (On the tenth book of the *Elements*), ff. 178a–179a.
- ʿAbd Allāh al-Khawwām, *Fuṣūl ʿalā fahm al-maqāla al-ʿāshira min Kitāb Uqlīdis* (Expositions for understanding the tenth book of Euclid's *Elements*), ff. 179b–180b.
- Abū Saʿīd al-Sijzī, *Al-Burhān min Istikhrāj* (The proof from <his> extract), f. 180b–183a.
- Quṭb al-Dīn al-Shīrāzī, Untitled (Discussion of the proofs of Euclid's postulates), ff. 183b–189b.
- Al-ʿAbbās ibn Saʿīd al-Jawharī, *Hādhihi ziyādāt li-ʿAbbās ibn Saʿīd fī al-maqāla al-khāmisa* (These are additions of ʿAbbās ibn Saʿīd to the fifth book of the *Elements*), ff. 191a–192a.[112]

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108 This treatise has been frequently copied. Its author has yet to be positively identified.

109 Also included in Feyzullah 1359. See note 97, above. There are extensive marginalia drawn from many sources. Many of these glosses are identical to glosses in Princeton University Library, Yahuda 4848 (358) (Mach 1977, 418). These glosses are important for including a set of alternative diagrams attributed to al-Ḥajjāj (De Young 2014).

110 Also in Tunis 16167 and Istanbul, Feyzullah 1359. See note 90, above.

111 Also in Tunis 16167 and Istanbul, Feyzullah 1359. See note 91, above.

112 Also in Tunis 16167 and Istanbul, Feyzullah 1359. See note 89, above.

- Jamshīd al-Kāshī, *Risāla dar sharḥ Ālāt raṣad* (<Persian> Commentary on observational instruments), ff. 192b–194a.
- Banū Mūsā, *Kitāb Maʿarifat misāḥāt al-ashkāl al-basīta wa-l-kurīya* (Treatise on measuring plain and spherical figures), ff. 195b–205b.

X.4 Dublin, Chester Beatty Library 3640

This manuscript is not as unified as the previous three. The original cataloging mentioned only two astronomical treatises (folios 1–126). The remaining 20 folios contain a number of short treatises on various mathematical topics written in a variety of hands. Beginning on folio 135b, we find four or five very short treatises or extracts from treatises, all of them copied in the same hand, ending at folio 136a. They are largely illegible in the microfilm but appear to be in a hand similar to the initial astronomical treatises.

Folios 136b–139a also contain four mathematical treatises. They are all copied in the same hand, but it is not certain that it is the same hand as the initial astronomical treatises. These treatises include:

- Jamāl al-Dīn, *Fā'ida min mukhtaṣar mawlanā Jamāl al-Dīn fī qawlihi fī al-ḥisāb min misāḥa saṭḥ al-kura* (A teaching from the summary of our master Jamāl al-Dīn concerning his discussion about measurement of a spherical surface), f. 136b.
- Al-ʿAbbās b al-Saʿīd al-Jawharī (active during the first half of the 9th century CE), *Hādhihi ziyādāt li-l-ʿAbbās bin Saʿīd fī al-maqāla al-khāmisa min Kitāb Uqlīdis* (These are the additions of al-ʿAbbās b. Saʿīd in the fifth book of Euclid's treatise), ff. 136b–137a.[113]
- Quṭb al-Dīn al-Shīrāzī, Untitled (Discussion containing proofs of Euclid's postulates), ff. 137a–138a.
- Al-Ḥassan ibn al-Ḥassan ibn al-Haytham (d. about 429 AH /1038 CE), *Maqāla al-ulā fī al-raṣad wa-l-tanbīh ʿala mā fīhi min al-ghalaṭ* (The first book of ʿAlī al-Ḥassan b. al-Ḥassan b. al-Haytham concerning observation and caution concerning its errors), ff. 138a–139a.

The fact that each these collections of treatises were copied by a single scribe suggests that they were considered to be related conceptually to one another or to belong together thematically. In this case, the common thread is clear—the treatises are all discussions of Euclid's *Elements*. The collections described in the previous section are not unique. A number of similar compilations devoted to Euclidean geometry are known from the 8th–9th centuries AH (15th–16th centuries CE). For example, an

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[113] Also in Tunis 16167, Feyzullah 1359 and BSB Arab 2697. See note 89, above.

earlier compilation (Princeton Univerity Library, Yahuda 358) also contains several of the treatises found in these compilations under study here.[114]

- Naṣīr al-Dīn al-Ṭūsī, *Taḥrīr Uṣūl al-handasah li-Uqlīdis*, folios 1–75b.[115]
- Al-ʿAbbās b. Saʿīd al-Jawharī, *Ziyādāt fī'l-maqālah al-khāmisah min kitāb Uqlīdis* (Additions to the fifth book of Euclid). folios 80b–81a.[116]
- Al-Ahwāzī, *Kalimāt min Sharḥ al-Maqālah al-ʿāshirah min Kitāb Uqlīdis* (Extracts from the commentary on the tenth book of Euclid), folios 81b–82b.[117]
- Abū Jaʿfar al-Khāzin, *Tafsīr ṣadr al-Maqālah al-ʿāshirah min kitāb Uqlīdis* (Explication of the premises of book X of Euclid), folios 82b–86b.[118]
- Anonymous author, *Fā'idah ʿalā'l-maqālah al-sābīʿah wa'l-thāminah wa'l-tāsiʿah* (Highlights (extracts) from books VII–IX of Ibn al-Haytham's commentary *Sharḥ muṣādarāt*), folios 87b–89a.[119]

A later compilation, Leiden University Library manuscript Or. 14, also includes three of the treatises found in the collections described above. The compilation is dated 1036 AH (1626 CE) and the name of the copyist was Darwīsh Aḥmad b. al-Ḥajj Ḥussam al-ʿAkalshānī (Witkam 2007, 19–20). This codex is a much more general collection of mathematical works. Most of the treatises included deal either with higher level mathematics or cosmography, but there are three treatises found in the earlier collections that are included (Witkam 2007, 22): Kamāl al-Dīn al-Fārisī, *Qāla (...) al-Ḥasan al-Fārisī inna mā qālahu (...) al-Ṭūsī fī akhīri al-maqāla al-thālitha ʿashara* (Note on al-Ṭūsī's exposition of the last proposition of book XIII <of the Elements>) pages 298–300;[120] Abū Jaʿafar al-Khāzin, *Tafsīr Ṣadr al-Maqāla al-ʿāshirah* (Explication of the premises of book X), pp. 327–340;[121] al-Ahwāzī, *Kalimāt min Sharḥ al-Maqāla al-ʿāshira* (Extracts from his commentary on book X), pp. 341–349.[122] Rashed (1996, 736) has argued that several treatises in Leiden Or. 14 were modeled

[114] This compilation was completed in Mashhad in 736 AH (1336 CE) by "M.b.S.b.A. al-Asadī" (Mach 1977, 418).

[115] See note 90, above.

[116] See note 89, above.

[117] See note 90, above.

[118] See note 91, above.

[119] Two additional copies are known: Istanbul, Carullah 2060, ff. 156b–160b; Tehran, Majlis Shūrā Library, 34, pp. 202–209 (page 202 is incorrectly numbered 204).

[120] See note 86, above.

[121] See note 85, above.

[122] See note 84, above.

on treatises in Tunis, Bibliothèque nationale 16167. Other similar collections may still be waiting to be identified in poorly cataloged manuscript libraries.

The fact that copies of several of these treatises appear in multiple collections suggests these treatises were circulating within the mathematical community of the time. Such compilations well may have played a pedagogical role in preparing students to teach the mathematical sciences. Although some of these collections of treatises carry few marginal or interlinear annotations that would suggest extensive use by students or readers, this fact does not itself necessarily militate against ascribing to them a pedagogical role (Brentjes 2018, 230). Aside from the work of Abdeljaouad (2014–2015; 2018–2019) and Rashed (1996; 2011), little scholarly attention has until recently been directed toward such collections of treatises. Further investigation of this unexplored genre may reveal more details about how ideas circulated within the mathematical community and how students were prepared for participation in the life of the mathematical community.[123]

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Istanbul, Süleymaniye Kütübhanesı, Damad Ibrahim Paşa 815.
Istanbul, Süleymaniye Kütübhanesı, Damad Ibrahim Paşa 816.
Istanbul, Süleymaniye Kütübhanesı, Fazıl Ibrahim Paşa 867.
Istanbul, Süleymaniye Kütübhanesı, Hamidiye 790.
Istanbul, Süleymaniye Kütübhanesı, Lala Ismail 288M.

[123] See the recent detailed discussion of MS Munich, Bayerische Staatsbibliothek, codex arab 2697 by Brentjes (2019) for an example of how such comprehensive analyses might be carried out and the kinds of information about the mathematical community that such analyses can provide.

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